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On the Dirichlet problem for the exterior of the circle and the summability of Fourier series

1. In the paper [1] F.Barański and E.Wachnicki solved the Dirichlet problem for the equation $(\Delta - c^2)u = 0$ in the inner of the circle. This solution gives rise to a method of summability of Fourier series. A comparison of this method with the Abel-Poisson method was made in the paper [3].

In the present paper we deal with the similar problem for the exterior of the circle.

2. Let $L^p_{2\pi}$ $(1 \leq p < \infty)$ denote the set of functions defined in the real line which are Lebesgue integrable to the p-th power over $(-\pi, \pi)$ and 2π-periodic with the norm

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}.$$

We shall consider the equation

(1) $$(\Delta - c^2)u = 0, \quad (c > 0),$$

for $u = u(x,y)$ in the exterior of a circle, i.e. in the region $D = \{(x,y): x^2 + y^2 > R^2\}$. For simplicity we shall write $u(x,y) = u(r\,e^{it})$; then the boundary condition will be given in $L^p_{2\pi}$ metric, i.e.

$$(2)\quad \lim_{r\to R^+}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r\,e^{it}) - \varphi(t)|^p\,dt\right)^{1/p} = 0 \quad (\varphi\in L^p_{2\pi})$$

and in infinity will take the form

$$(3)\qquad \lim_{r\to\infty} |u(r\,e^{it})| = 0 \quad \text{for every } t.$$

Using the polar coordinates $x = r\cos t$, $y = r\sin t$, $r\in(R,\infty)$, $t\in[0,2\pi]$ the equation (1) is the form

$$(4)\qquad r^{-1} D_r(r\,D_r\,u) + r^{-2} D_t^2\,u - c^2\,u = 0,$$

where

$u(r\,e^{it}) = u(r\cos t, r\sin t)$, and $D_r = \dfrac{\partial}{\partial r}$, $D_t^2 = \dfrac{\partial^2}{\partial t^2}$.

Applying the separation of variables we get

$$(5)\quad u(re^{it}) = \frac{a_0\,K_0(cr)}{2\,K_0(cR)} + \sum_{n=1}^{\infty}\frac{K_n(cr)}{K_n(cR)}(a_n\cos nt + b_n\sin nt),$$

where K_n is the MacDonald function ([5]) and a_n, b_n are the Fourier coefficients of the function φ.

We shall prove that the function (5) satisfies the conditions (1), (2), (3).

First we prove some lemmas.

LEMMA 1. For any positive integer n

$$(6)\quad K_n(cR) \geqslant \left(\frac{2}{cR}\right)^{1/2}(2cR)^{1-n}\left((n-1)!\right)^{-1}\Gamma\left(2n-\tfrac{1}{2}\right)\exp(-cR).$$

P r o o f. By formulae

$$(7)\quad (1+\xi)^{\mu} = \sum_{k=0}^{n-1}\frac{\Gamma(1+\mu)}{k!\,\Gamma(1+\mu-k)}\xi^k + \frac{\Gamma(1+\mu)}{(n-1)!\,\Gamma(\mu+1-n)}\xi^n \times$$
$$\times\int_0^1 (1-t)^{n-1}(1+\xi t)^{\mu-n}\,dt \quad (\text{s. [5] p.129}),$$

$$(8)\quad K_n(cR) = \left(\frac{\pi}{2R}\right)^{1/2} \left(\Gamma(n+\tfrac{1}{2})\right)^{-1} \exp(-cR)c^n \times$$

$$\times \int_0^\infty e^{-cs} s^{n-1/2} \left(1+\frac{s}{2R}\right)^{n-1/2} ds \quad (\text{s.[4] p.366})$$

we get

$$(9)\quad \left(1+\frac{s}{2R}\right)^{n-1/2} = \sum_{k=0}^{n-1} \frac{\Gamma(n+1/2)}{k!\,\Gamma(n+1/2-k)} \left(\frac{s}{2R}\right)^k +$$

$$+ \frac{\Gamma(n+1/2)}{(n-1)!\,\Gamma(1/2)} \left(\frac{s}{2R}\right)^n \int_0^1 (1-t)^{n-1} \left(1+\frac{st}{2R}\right)^{-1/2} dt$$

and

(10)

$$\int_0^\infty e^{-cs} \left(s\left(1+\frac{s}{2R}\right)\right)^{n-1/2} ds = \sum_{k=0}^{n-1} \frac{\Gamma(n+1/2)}{k!\,\Gamma(n+1/2-k)} (2R)^{-k} \int_0^\infty e^{-cs} s^{n+k-1/2} ds +$$

$$+ \frac{\Gamma(n+1/2)}{(n-1)!\,\Gamma(1/2)} (2R)^{-n} \int_0^\infty \left(e^{-cs} s^{2n-1/2} \int_0^1 (1-t)^{n-1} \left(1+\frac{st}{2R}\right)^{-1/2} dt\right) ds.$$

Since $\Gamma(n+1/2) > 0$ for any positive integer n, and $\Gamma(1/2) = \sqrt{\pi}$,

$$(11)\quad \int_0^\infty e^{-s} s^{n-1/2} \left(1+\frac{s}{2R}\right)^{n-1/2} ds \geqslant \sum_{k=0}^{n-1} \frac{\Gamma(n+1/2)}{k!\,\Gamma(n+1/2-k)} (2R)^{-k} \times$$

$$\times \int_0^\infty e^{-cs} s^{n+k-1/2} ds.$$

From the formula

$$\int_0^\infty e^{-qx} x^{p-1} dx = (q)^{-p} \Gamma(p) \quad (p>0,\ q>0) \ (\text{s.[4] p.174})$$

and the inequality (11) we have

(12)

$$\int_0^\infty e^{-cs} s^{n-1/2} \left(1+\frac{s}{2R}\right)^{n-1/2} ds \geqslant \sum_{k=0}^{n-1} \frac{\Gamma(n+1/2)}{k!\,\Gamma(n+1/2-k)} (2R)^{-k} \times$$

$$\times \Gamma(n+k+1/2)c^{-n-k-1/2} \geq \frac{\Gamma(n+1/2)}{(n-1)!\Gamma(3/2)}(2R)^{-n+1}\Gamma(2n-1/2) \times$$
$$\times c^{-2n+1/2}.$$

By (8), (9), (10) and (12) we get (6) which ends the proof of Lemma 1.

Let $\gamma(r,n) = (r^n K_n(cr) - R^n K_n(cR))(r^n K_n(cR))^{-1}$.

LEMMA 2. There exists a constant $A = A(R,c)$, such that

(13) $$|\gamma(r,n)| \leq A(r-R)n^{-1/2}\left(\frac{R}{r}\right)^n$$

for an arbitrary positive integer n and $r \in (R,\infty)$.

P r o o f. By formula

$$K_n(cr) = \frac{\Gamma(n+1/2)(2c)^n}{r^n\,\Gamma(1/2)}\int_0^\infty \frac{\cos rt}{(t^2+c^2)^{n+1/2}}\,dt$$

(s. [4] p.365)

and the Lagrange theorem we obtain

(14) $$|r^n K_n(cr) - R^n K_n(cR)| = \Gamma(n+1/2)(2c)^n \pi^{-1/2} \times$$
$$\times \left|\int_0^\infty \frac{\cos rt - \cos Rt}{(t^2+c^2)^{n+1/2}}\,dt\right| \leq$$
$$\leq \Gamma(n+1/2)(2c)^n \pi^{-1/2}\int_0^\infty \frac{(r-R)t}{(t^2+c^2)^{n+1/2}}\,dt, \quad \text{for } r \in (R,\infty).$$

Using the formula

$$\int_0^\infty \frac{t}{(t^2+c^2)^{n+1/2}}\,dt = 2(n-1/2)^{-1}\,\frac{c^2}{2}\left(\frac{c^2}{2}+1\right)c^{-2n-1}$$

(s. [4] p.169) and by (14) we obtain

(15) $$|r^n K_n(cr) - R^n K_n(cR)| \leq A_1\,\Gamma(n+1/2)(2c)^n c^{-2n-1} \times$$
$$\times (n-1/2)^{-1}(r-R) \quad \text{for } r \in (R,\infty),$$

where

$$A_1 = \pi^{-1/2}\, 2\, \frac{c^2}{2} \left(\frac{c^2}{2} + 1 \right).$$

In view of Lemma 1 and (15) we have

$$|\gamma(r,n)| \leqslant (r - R)\left(\frac{R}{r}\right)^n 2^{2n-1}\, \Gamma(n)\ \Gamma(n+1/2)\ \times$$
$$\times\ ((n-1/2)\,\Gamma(2n-1/2))^{-1} A_2 =$$
$$= A_2 (r - R)\left(\frac{R}{r}\right)^n \pi^{1/2}\, \Gamma(2n)\ ((n-1/2)\ \Gamma(2n-1/2))^{-1} =$$
$$= A_2 (r - R)\left(\frac{R}{r}\right)^n 2^{4n-2}\ ((2n-1)\,!)^2\ ((4n-3)\,!\,(n-1/2))^{-1},$$

where

$$A_2 = A_1 \exp(cR)\left(\frac{2}{cR}\right)^{1/2}.$$

Hence, by the Stirling formula we get

$$(16)\qquad |\gamma(r,n)| \leqslant A(R,c)\,(r - R)\left(\frac{R}{r}\right)^n n^{-1/2} \quad \text{for} \quad r \in (R,\infty),$$

where $A(R,c)$ is a positive constant, which ends the proof of Lemma 2.

LEMMA 3. If $\varphi \in L^p_{2\pi}$, then the function u defined by (5) belongs to the class C^2 in D.

P r o o f. We remark that

$$(17)\qquad \frac{K_n(cr)}{K_n(cR)} = \left(\frac{R}{r}\right)^n + \gamma(r,n).$$

By (16), (17) the series $\sum_{n=1}^{\infty} \left(\left(\frac{R}{r}\right)^n + |\gamma(r,n)|\right)$ is a majorant of the series (5), whence $u \in C^0$ in D.

We shall prove that $u \in C^1$ in D.

By ([5] p.117) and ([4] p.365,366) we obtain

$$(18)\qquad \frac{d}{dz}\, z^{-\nu} K_\nu(z) = -\, z^{-\nu} K_{\nu+1}(z),$$
$$K_{n-1}(z) - K_{n+1}(z) = -\,\frac{2n}{z}\, K_n(z),$$

(19) $K_n(cr) =$

$$= \pi^{1/2}\left(\frac{c}{2r}\right)^n \left(\Gamma(n+1/2)\right)^{-1} \int_0^\infty \exp\left(-c(t^2+r^2)^{1/2}\right)(t^2+r^2)^{-1/2} t^{2n} dt,$$

(20) $K_n(cr) = \int_0^\infty \exp(-\,cr\,\cosh\,t\,)\cosh\,nt\,dt.$

By (18) we have

(21) $$D_r\,K_n(cr) = \frac{n}{r}\,K_n(cr) - K_{n+1}(cr).$$

Since the function $\frac{\exp(-x)}{x}$ is increasing in $(0,\infty)$, from (19) we get

(22) $$\frac{n}{r}\frac{K_n(cr)}{K_n(cR)} = \frac{n}{r}\left(\frac{R}{r}\right)^n \int_0^\infty \exp\left(-c(t^2+r^2)^{1/2}\right)(t^2+r^2)^{-1/2} t^{2n} dt \;\times$$
$$\times\left(\int_0^\infty \exp\left(-c(t^2+R^2)^{1/2}\right)(t^2+R^2)^{-1/2} t^{2n} dt\right)^{-1} \leqslant \frac{n}{r}\left(\frac{R}{r}\right)^n.$$

In view of (18), (20) we have

$$\frac{K_{n+1}(cR)}{K_n(cR)} = \left(\frac{2n}{cR}\,K_n(cR) + K_{n-1}(cR)\right)\left(K_n(cR)\right)^{-1} \leqslant \frac{2n}{cR} + 1.$$

Hence, by (21) and (22) we obtain

$$\left|D_r\,\frac{K_n(cr)}{K_n(cR)}\right| = \left|\frac{n}{r}\,\frac{K_n(cr)}{K_n(cR)} - \frac{K_{n+1}(cr)}{K_n(cR)}\right| \leqslant$$
$$\leqslant \frac{n}{r}\left(\frac{R}{r}\right)^n + \frac{K_{n+1}(cR)}{K_n(cR)}\;\frac{K_{n+1}(cr)}{K_{n+1}(cR)} \leqslant$$
$$\leqslant \frac{n}{r}\left(\frac{R}{r}\right)^n + \left(\frac{2n}{cR} + 1\right)\left(\frac{R}{r}\right)^{n+1}.$$

This estimation proves that

$$D_r u(re^{it}) = D_r\,\frac{a_0}{2}\,\frac{K_0(cr)}{K_0(cR)} + \sum_{n=1}^{\infty} D_r\,\frac{K_n(cr)}{K_n(cR)}\left(a_n\cos\,nt + b_n\sin\,nt\right)$$

and $D_r\,u(r\,e^{it}) \in C^0$ in D.

From (22) we obtain

$$D_t\, u(re^{it}) = \sum_{n=1}^{\infty} \frac{K_n(cr)}{K_n(cR)} D_t(a_n \cos nt + b_n \sin nt)$$

and $D_t\, u(r\, e^{it}) \in C^0$ in D.

Similarly we can prove that the function u belongs to the C^2 in D.

LEMMA 4. ([2] p.46) If $\varphi \in L^p_{2\pi}$ and $p_r(s)$ is the Abel-Poisson kernel, then

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(s)p_r(s-t)\,ds - \varphi(t)\right|^p dt\right)^{1/p} \to 0 \quad \text{as} \quad r \to 1^-.$$

3. Let a_n, b_n denote the Fourier coefficients of the function φ. By (5) and (17) we have

$$u(re^{it}) = u_1(re^{it}) + u_2(re^{it}) + u_3(re^{it}),$$

where

$$u_1(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(s)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \cos n(t-s)\right) ds,$$

$$u_2(re^{it}) = \sum_{n=1}^{\infty}\gamma(r,n)(a_n \cos nt + b_n \sin nt),$$

$$u_3(re^{it}) = \left(\frac{K_0(cr)}{K_0(cR)} - 1\right)\frac{a_0}{2}.$$

We shall prove the following

THEOREM 1. If $\varphi \in L^p_{2\pi}$ $(1 \leqslant p < \infty)$, then the function u, defined by (5), is a solution of the problems (1), (2), (3).

P r o o f. It follows from Lemma 3 that the function $u(re^{it})$ is a solution of the equation (1) in D.

We shall show that the condition (2) is fulfilled.

From Lemma 4 it follows that

(23) $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|u_1(re^{it}) - \varphi(t)|^p\,dt\right)^{1/p} \to 0$ as $r \to R^+$.

Now, we notice that

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|u_2(re^{it})|^p\,dt\right)^{1/p} =$$

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=1}^{\infty}\gamma(r,n)(a_n\cos nt + b_n\sin nt)\right|^p dt\right)^{1/p} \leqslant$$

$$\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(M\sum_{n=1}^{\infty}|\gamma(r,n)|\right)^p dt\right)^{1/p} = M\sum_{n=1}^{\infty}|\gamma(r,n)|,$$

where M is a positive constant.

By Lemma 2 and criterion of Dirichlet we get

(24) $$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|u_2(re^{it})|^p dt\right)^{1/p} \to 0 \quad \text{as} \quad r \to R^+.$$

Moreover we get

(25) $$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{K_n(cr)}{K_n(cR)} - 1\right|^p dt\right)^{1/p} \to 0 \quad \text{as} \quad r \to R^+.$$

From (23) - (25) it follows that the condition (2) is satisfied. From Lemma 3 it follows that the condition (3) is satisfied, too.

4. We shall compare the method of the summability of the Fourier series determined by (5) with the Abel-Poisson method. Let $\varphi \in L^p_{2\pi}$ $(1 \leqslant p < \infty)$ and a_n, b_n be the Fourier coefficients of the function φ.

THEOREM 2.

$$\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\varrho^n(a_n\cos nt + b_n\sin nt)\right) \to s(t) \quad \text{as} \quad \varrho \to 1^-$$

if and only if

$$\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{K_n(cr)}{K_n(cR)}(a_n\cos nt + b_n\sin nt)\right) \to s(t) \quad \text{as} \quad r \to R^+.$$

P r o o f. In view of (17) we have

$$\frac{K_n(cr)}{K_n(cR)} = \left(\frac{R}{r}\right)^n + \gamma(r,n). \tag{26}$$

Since $\sum_{n=1}^{\infty} \gamma(r,n)(a_n \cos nt + b_n \sin nt) \to 0$ as $r \to R^+$, then by (26) we get Theorem 2.

References

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