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The Green function for some polyharmonic problems in the unbounded sector

1. In the paper [1] is given the construction of the Green function for p-harmonic Dirichlet problem in m dimensional half-space in case of $m > 2p$ and in case of $m < 2p$ when m is odd. The analogous problem in case of $m = 2p$ was investigated in [4].

Let $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in E_m$, E_m being m dimensional Euclidean space. Let

$$E_{12}^{+} = \{x: x_1 > 0,\ x_2 > 0,\ |x_i| < \infty,\ i = 3, \dots, m\}.$$

In the present paper we shall construct the Green functions $G(x,y)$ for some p-harmonic problems in the domain E_{12}^{+}.

2. Let

$$r_1^2 = \sum_{i=1}^{m}(x_i - y_i)^2,\quad r_2^2 = (x_1-y_1)^2 + (x_2+y_2)^2 + \sum_{i=3}^{m}(x_i-y_i)^2,$$

$$r_3^2 = (x_1+y_1)^2 + \sum_{i=2}^{m}(x_i-y_i)^2,\quad r_4^2 = (x_1+y_1)^2 + (x_2+y_2)^2 + \sum_{i=3}^{m}(x_i-y_i)^2,$$

$$r_5^2 = (x_1+y_1+v)^2 + \sum_{i=2}^{m}(x_i-y_i)^2,\quad r_6^2 = (x_1+y_1+v)^2 + (x_2+y_2)^2 + \sum_{i=3}^{m}(x_i-y_i)^2,$$

where v is an arbitrary real number.

We put now

$$A_n^1 = 1 \quad \text{for} \quad n = 0,1,\ldots,$$

$$A_n^k = \frac{n(n-1)\ldots(n-2k+3)}{(2k-2)!!} \quad \text{for } n=2,3,\ldots,\ k=2,3,\ldots,\left[\frac{n}{2}\right]+1.$$

We need some lemmas

LEMMA 1. ([1]). If q is an arbitrary real number, then

$$D_{y_2}^n r_j^q =$$

$$= (-1)^{nj} \sum_{k=1}^{\left[\frac{n}{2}\right]+1} A_n^k q(q-2)\ldots(q-2n+2k) r_j^{q-2n+2k-2} (x_2+(-1)^j y_2)^{n-2k+2}$$

for $j = 1-6$, $n = 1,2,\ldots$.

LEMMA 2.([4]).

$$D_{y_2}^n \ln r_j =$$

$$= \sum_{k=1}^{\left[\frac{n}{2}\right]+1} (-1)^k A_n^k (2n-2k)!! (-1)^{(j+1)n} r_j^{-2n+2k-2} (x_2+(-1)^j y_2)^{n-2k+2}$$

for $j = 1-6$, $n = 1,2,\ldots$.

LEMMA 3.([1]).

$$\sum_{k=2s-2}^{i-1} A_k^s \binom{i}{k} 2^{i-k} (-1)^k = \begin{cases} 0 & \text{for } i \text{ even}, \\ 2\, A_i^s & \text{for } i \text{ odd} \end{cases}$$

and for $s = 1,2,\ldots,\left[\frac{i+1}{2}\right]$.

LEMMA 4. ([4]). Let $a_{s,k}$ be a sequence of numbers,

then

$$\sum_{s=1}^{i} \sum_{k=1}^{\left[\frac{i-s}{2}\right]+1} a_{s,k} = \sum_{k=1}^{\left[\frac{i+1}{2}\right]} \sum_{s=2k-2}^{i-1} a_{i-s,k}$$

for $i = 1,2,\ldots$.

3. Let $q = 2p-m$, $j = 1,2,3,4,5,6$ and let

$$V_o(r_j) = \begin{cases} r_j^{\,q} & \text{for } m > 2p, \\ r_j^{\,q} & \text{for } m < 2p, \text{ where } m \text{ is odd}, \\ \ln r_j & \text{for } m = 2p, \end{cases}$$

$$V_k(r_j) = r_j^{\,q-2k} \quad \text{for } k = 1,2,\ldots.$$

We put now

$$C_k = \frac{2^k q(q-2)\ldots(q-2k+2)}{k!}.$$

We shall prove

THEOREM 1. Let h be a negative real constant. The function

$$G_1(x,y) = V_o(r_1) + V_o(r_3) - V_o(r_2) - V_o(r_4) + \sum_{k=1}^{p-1} C_k(x_2y_2)^k(V_k(r_1) + V_k(r_3)) + 2h\int_0^\infty e^{hv}(V_o(r_5) - V_o(r_6))\,dv + 2h\int_0^\infty e^{hv}\sum_{k=1}^{p-1} C_k(x_2y_2)^k\, V_k(r_5)\,dv$$

is the Green function for the problem

(1) $\quad \Delta_y^p\, G_1(x,y) = 0$ for $x \neq y$, $x,y \in E_{12}^+$,

(2) $\quad (D_{y_1} + h)\,\Delta_y^i\, G_1(x,y)\Big|_{y_1=0} = 0$, $i = 0,1,\ldots,p-1$,

(3) $\quad D_{y_2}^i\, G_1(x,y)\Big|_{y_2=0} = 0$, $i = 0,1,\ldots,p-1$.

P r o o f. By [3] (p.134) we remark that $\Delta_y^p V_o(r_j) = 0$, $j = 1$–6. Moreover, if the function $u(y)$ is harmonic, then the function $y_2u(y)$ is biharmonic (see [2], p.194) and consequently $\Delta_y^p\,(y_2^{\,k}V_k(r_j)) = 0$ for $k = 1,2,\ldots,p-1$ and $j = 1$–6.

It is easy to show that the integrals

$$\int_0^\infty e^{hv} D_y^{|\alpha|}(V_o(r_5) - V_o(r_6))\, dv, \qquad \int_0^\infty e^{hv} D_y^{|\alpha|}(V_k(r_5) y_2{}^k) dv,$$

$k = 1,2,\dots,p-1$ are locally uniformly convergent in E_{12}^+ for arbitrary multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$.

Hence

$$(4) \quad D_y^{|\alpha|} \int_0^\infty e^{hv}(V_o(r_5) - V_o(r_6))\, dv = \int_0^\infty e^{hv} D_y^{|\alpha|}(V_o(r_5) - V_o(r_6))\, dv$$

and

$$(5) \quad D_y^{|\alpha|} \int_0^\infty e^{hv} V_k(r_5) y_2{}^k dv = \int_0^\infty e^{hv} D_y^{|\alpha|}(V_k(r_5) y_2{}^k) dv.$$

Consequently $\Delta_y^p G_1(x,y) = 0$ for $x \neq y$, $x,y \in E_{12}^+$.

Now we prove (2). We can remark thak

$$(6) \quad \begin{aligned} & D_{y_1} \Delta_y^i (V_o(r_1) + V_o(r_3)) \Big|_{y_1=0} = 0, \\ & D_{y_1} \Delta_y^i (V_o(r_2) + V_o(r_4)) \Big|_{y_1=0} = 0, \end{aligned}$$

$$(7) \quad D_{y_1} \Delta_y^i \sum_{k=1}^{p-1} C_k (x_2 y_2)^k (V_k(r_1) + V_k(r_3)) \Big|_{y_1=0} = 0,$$

$$(8) \quad D_{y_1} \Delta_y^i\, 2h \int_0^\infty e^{hv}(V_o(r_5) - V_o(r_6))\, dv =$$

$$= \Delta_y^i\, 2h e^{hv} (V_o(r_5) - V_o(r_6)) \Big|_{v=0}^{v=\infty} +$$

$$- 2h^2 \int_0^\infty e^{hv} \Delta_y^i (V_o(r_5) - V_o(r_6))\, dv =$$

$$= -2h \Delta_y^i (V_o(r_3) - V_o(r_4)) - 2h^2 \Delta_y^i \int_0^\infty e^{hv}(V_o(r_5) - V_o(r_6))\, dv,$$

$$(9) \quad D_{y_1} \Delta_y^i\, 2h \sum_{k=1}^{p-1} C_k (x_2 y_2)^k \int_0^\infty e^{hv} V_k(r_5) dv =$$

$$= - \Delta_y^i \, 2h \sum_{k=1}^{p-1} c_k (x_2 y_2)^k \, V_k(r_3) -$$

$$- 2h^2 \Delta_y^i \sum_{k=1}^{p-1} c_k (x_2 y_2)^k \int_0^{\infty} e^{hv} \, V_k(r_5) \, dv.$$

By (6) - (9) we get (2).

Now we shall verify the condition (3). If $i = 0$, then $r_1 = r_2$, $r_3 = r_4$, $r_5 = r_6$ for $y_2 = 0$ and consequently $G_1(x,y)\Big|_{y_2 = 0} = 0.$

Let $1 \leqslant i \leqslant p-1$. By Lemma 1 we have

(10) $$D^i_{y_2}(V_o(r_1) - V_o(r_2))\Big|_{y_2=0} =$$

$$=((-1)^i - 1) \sum_{k=1}^{\left[\frac{i}{2}\right]+1} A_i^k q(q-2)\ldots(q-2i+2k) x_2^{\,i-2k+2} V_{i-k+1}(r_1)\Big|_{y_2=0}.$$

In virtue of Leibniz formula and Lemma 1, 4 we get

(11) $$D^i_{y_2} \sum_{k=1}^{p-1} c_k (x_2 y_2)^k \, V_k(r_1)\Big|_{y_2=0} =$$

$$= \sum_{k=1}^{p-1} c_k \, x_2^{\,k} \sum_{j=0}^{i} \binom{i}{j} D^j_{y_2}(y_2^{\,k}) D^{i-j}_{y_2} V_k(r_1)\Big|_{y_2=0} =$$

$$= \sum_{k=1}^{i} c_k x_2^{\,k} \, k! \binom{i}{k} D^{i-k}_{y_2} \, V_k(r_1)\Big|_{y_2=0} =$$

$$= \sum_{k=1}^{i} \sum_{s=1}^{\left[\frac{i-k}{2}\right]+1} c_k x_2^{\,k} k! \binom{i}{k} A^s_{i-k} (-1)^{i-k} (q-2k)\ldots(q-2i+2s) \times$$

$$\times V_{i-s+1}(r_1)\Big|_{y_2=0} x_2^{\,i-2s+2} =$$

$$= \sum_{s=1}^{\left[\frac{i+1}{2}\right]} \sum_{k=2s-2}^{i-1} 2^{i-k} \binom{i}{i-k} A^s_k (-1)^k q(q-2)\ldots(q-2i+2s) \times$$

$$\times V_{i-s+1}(r_1)\Big|_{y_2=0} x_2^{\,i-2s+2} =$$