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On the equivalence of Whitney (b)—regularity and (b_s) —regularity

Summary. C.T.C. Wall has conjectured in [7] that conditions (a_s) and(b_s) are equivalent to the Whitney conditions (a) and (b). The contents of this paper is the second part of [1]. In [1] it has been proved that Whitney (a) - regularity is equivalent to (a_s) - regularity. In this paper we will show that (b) - regularity is equivalent to (b_s) - regularity. The same results have been obtained in his thesis [6] by D.J.A. Trotman. However there are differences in adopted methods (our proof is based on the C^1 case of the Whitney extension theorem).

1. We begin by recalling the definitions of regularity conditions and Theorem 1 from [1]. Let M and N be two manifolds embeded in R^n such that $N \subset \bar{M} - M$ and let $x \in N$.

DEFINITION 1. We say thet M is (a) - regular over N at x ((M,N) satisfies the condition (a) at x) if the following holds: For every sequence $\{x_m\}$ of points of M tending to x such that $T_{x_m}M$ tends to τ in the Grassmannian manifold of k-dimensional subspaces of R^n ($k = \dim M$) we have $T_x N \subset \tau$.

DEFINITION 2. We say that M is (a_s) - regular over N at x ((M,N) satisfies the condition (a_s) at x) if for any local C^1 - retraction at x, $\pi \subset R^n \times N$, x has a neighbourhood U such that $\pi|_{M \cap U}$ is a submersion.

DEFINITION 3. We say that M is (b) - regular over N at x ((M,N) satisfies the condition (b) at x) if the following holds:

For every sequence $\{x_m\}$ of points of M and $\{y_m\}$ of N such that $x_m \to x$, $y_m \to x$, $R(x_m - y_m) \to l$ (in projective space P^{n-1}), and $T_{x_m}M \to \tau$ we have $l \subset \tau$.

DEFINITION 4. We say that M is (b_s) - regular over N at x ((M,N) satisfies the condition (b_s) at x) if for any local C^1 - tubular neighbourhood of N in M at x, x has a neighbourhood U such that $(\pi, \varrho)|_{U \cap N} \to N \times R$ is a submersion.

The conditions (a), and (b) were first defined by H. Whitney in [8] and [9]. R. Thom [5] introduced conditions (a_s) and (b_s) and showed that these are necessary for (a) and (b) - regularity.

THEOREM 1. Whitney (a) - regularity is equivalent to (a_s) - regularity.

We refer to [1] for a proof of the Theorem 1.

R e m a r k 1. It is easy to see that (a), (a_s), (b), (b_s) regularities are C^1 diffeomorphism invariants and that these are far from being a topological invariants.

2. LEMMA 1. (i) (b) - regularity implies (a) - regularity.

(ii) (b_s) - regularity implies (a_s) - regularity.

P r o o f. For (i), let $\{x_m\}$ be a sequence of points in M such that x_m tends to x and $T_{x_m}M$ tends to τ, for some $\tau \in G_k(n)$.
We must show that T_xN is a subset of τ. Suppose otherwise. Then there exists a line $l \subset R^n$ passing through the origin, such that $l \subset T_xN$ but $l \not\subset \tau$. Since $l \subset T_xN$ we can choose a sequence of points $\{y_m\} \subset N$ such that $R(x_m - y_m) \to l$. But since $l \not\subset \tau$, this contradicts (b).

Part (ii) follows at once from the Remark 1.

E x a m p l e 1. Lemma 1 is sharp i.e. (a) - regularity does not imply (b) - regularity. Let M be a logarithmic spiral in $\mathbb{C}$ given by $\{\varphi : t \to e^t e^{it}\}$ and let N be the origin. Then the pair (M,N) does not satisfy condition (b). For:

$$\frac{d}{dt}(e^t e^{it}) = e^t e^{it} + i\, e^t e^{it}$$

and

$$\left\langle \frac{e^t e^{it}}{|e^t e^{it}|}, \frac{e^t e^{it} + ie^t e^{it}}{|e^t e^{it} + ie^t e^{it}|} \right\rangle = \frac{1}{\sqrt{2}}$$

so the angle between the line $T_{\varphi(t)}M$ and the secant $Re^{t}e^{it}$ is independent of t.

R e m a r k 2. Recall that the Grassmannian manifold $G_k(n)$ admits a structure of an analytic manifold introduced by the following atlas of inverse charts:

$$\varphi_{EF}: L(E,F) \ni f \longrightarrow \hat{f} = \{u + f(u): u \in E\} \in G_k(n),$$

where E,F are linear subspaces of R^n such that $E \oplus F = R^n$, and for a base of E

$$\{e_i\}, \quad i = 1,\ldots,k \quad \text{and for} \quad f \in L(E,F)$$

the familly

$$p_i(f) = e_i + f(e_i), \quad i = 1,\ldots,k$$

is a base of $\hat{f}$.

For the proof of equivalence (b) - regularity and (b_s) regularity we shall use C^1 case of the Whitney extension theorem:

Let K be a compact subset of R^n, $f_o,\ldots,f_n$ a family of continuous functions to R^p. Then there exists $f \in C^1(R^n,R^p)$ such that:

$$f\Big|_K = f_o, \quad \frac{\partial f}{\partial x_1}\Big|_K = f_1, \quad \ldots, \quad \frac{\partial f}{\partial x_n}\Big|_K = f_n$$

if and only if the following condition is fulfilled:

(*) $$f_o(x) = f_o(y) + f_1(y)(x_1 - y_1) + \ldots + f_n(y)(x_n - y_n) + o|x - y|$$

for $x,y \in K$ and $|x - y|$ tending to the origin.

3. THEOREM 2. Whitney (b) - regularity is equivalent to (b_s) - regularity.

P r o o f. of the Theorem 2. First we will show that (b) - regularity implies (b_s) - regularity (cf [2]). Let us suppose that the condition (b_s) fails at x. In view of Remark 1 we may assume that N is an open neighbourhood of the origin in $R^p \times \{o\} \subset R^n$, x is the origin and (π, ϱ) is the standard tubular neighbourhood of $R^p \times \{o\}$ in R^n that is:

$$\pi : R^p \times R^{n-p} \longrightarrow R^p, \quad \pi(x_1,\ldots,x_p,\ldots,x_n) = (x_1,\ldots,x_p)$$

and

$$\varrho(x_1,\ldots,x_n) = \sum_{i=p+1}^{n} x_i^2.$$

The hypothesis that condition (b) is satisfied at x implies by Lemma 1 that condition (a_s) (cf Theorem 1) is fulfilled at x.

Thus we may assume that there exists $\{x_m\}$, a sequence of points of M tending to the origin such that:

$$(d_{x_m}\varrho)\Big|_{T_{x_m}M} \neq 0.$$

Let us choose $\{y_m\}$, a sequence of points of N such that $y_m = \pi(x_m)$, then:

$$R(x_m - y_m) \perp \ker d_{x_m}\varrho \supset T_{x_m}M.$$

Finally: $R(x_m - y_m) \longrightarrow l$, $T_{x_m}M \longrightarrow \tau$ and

$$l \perp \tau$$

hence we have shown that M fails to be (b) - regular over N at the origin.

Conversely if the condition (b) fails at $x = 0$, then there exists $\{x_m\}$, a sequence of points of M and $\{y_m\}$, a sequence of points of $N \subset R^p \times \{o\}$ such that:

$$x_m \to o, \quad y_m \to o$$

$$T_{x_m} M \to \tau, \quad R(x_m - y_m) \to l$$

and

$$l \not\subset \tau$$

The hypothesis that condition (b_s) is satisfied at x implies that condition (a_s) is fulfilled at x, thus we may assume (Theorem 1 and Remark 1) that condition (a) is fulfilled at x,

$$\tau = R^k \times \{o\} \supset T_o N = R^p \times \{o\} \supset N$$

and

$$l = Re_n .$$

Let (π, ρ) be standard tubular neighbourhood of $R^p \times \{o\}$ in R^n and

$$W = \bigcup_{m=1}^{\infty} \rho^{-1}(\rho(x_m)) .$$

Now we will construct $h \subset R^n \times R^n$, a C^1 - diffeomorphism of neighbourhoods of the origin, such that:

$$h(N) = N, \quad h(x_m) = x_m$$

and

$$d_{x_m} h(T_{x_m} W) \supset T_{x_m} M.$$

In this case we will have

$$(h \circ \pi \circ h^{-1}, \rho \circ h^{-1})$$

a local C^1 - tubular neighbourhood of $h(N) = N$ and

$$d_{x_m}(\rho \circ h^{-1})|_{T_{x_m} M} = 0$$

so submersion of

$$(h \circ \pi \circ h^{-1}, \varrho \circ h^{-1})|_M$$

will fail near the origin (cf Definition 4).

For the required construction we will use the C^1 case of the Whitney extension theorem and two following inverse charts on $G_{n-1}(n)$ and $G_k(n)$:

$$(1) \qquad L\left(\sum_{i=1}^{n-1} Re_i, Re_n\right) \longrightarrow G'(Re_n)$$

$$(2) \qquad L\left(\sum_{i=1}^{k} Re_i, \sum_{i=k+1}^{n} Re_i\right) \longrightarrow G'\left(\sum_{i=k+1}^{n} Re_i\right)$$

where $G'(Re_n)$, $G'\left(\sum_{i=k+1}^{n} Re_i\right)$ denote all algebraic suplements of vector spaces:

$$Re_n \quad \text{and} \quad \sum_{i=k+1}^{n} Re_i$$

and $\{e_i\}$, $i = 1,\ldots,n$ is the canonical base of R^n.

In these charts we have:

$$T_{x_m} W = \hat{f}_m, \quad T_{x_m} M = \hat{g}_m$$

and

$$\{p_i(f_m)\}, \quad i = 1,\ldots,n-1$$

$$\{p_i(g_m)\}, \quad i = 1,\ldots,k$$

the bases of these spaces (cf Remark 2).

Let us define a sequence of linear maps T_m in the following manner

$$T_m \begin{cases} p_i(f_m) \longrightarrow p_i(g_m), & \text{for } i = 1,\ldots,k \\ p_i(f_m) \longrightarrow e_i, & \text{for } i = k+1,\ldots,n-1 \\ e_n \longrightarrow e_n & \end{cases}$$

Now, let us introduce the following notations:

$$K = \bigcup_{m=1}^{\infty} \{x_m\} \cup X$$

(where X is a compact neighbourhood of the origin of $R^p \times \{o\}$);

$$f_o = id|_K$$

$$f_i(x_m) = T_m(e_i) \quad \text{for} \quad i = 1,\dots,n \quad \text{and} \quad x_m \in \bigcup_{p=1}^{\infty} \{x_p\}$$

$$f_i(x) = e_i \quad \text{for} \quad i = 1,\dots,n \quad \text{and} \quad x \in X.$$

It is easily seen that $f_o,\dots,f_n$ are continuous on K. In order to show that family $f_o,\dots,f_n$ fulfils the condition (*) of the Whitney extension theorem let us check two possibilities:

(1) $$y \in X \quad \text{and} \quad x \in \bigcup_{m=1}^{\infty} \{x_m\} \cup X$$

(2) $$y \in \bigcup_{m=1}^{\infty} \{x_m\} \quad \text{and} \quad x \in \bigcup_{m=1}^{\infty} \{x_m\} \cup X$$

Case (1) $x = y + e_1(x_1 - y_1) + \dots + e_n(x_n - y_n) + o|x - y|$

i.e. $$o = o|x - y|$$

Case (2)

$x = y + T_m(e_1)(x_1 - y_1) + \dots + T_m(e_n)(x_n - y_n) + o|x - y|$

i.e.

$$(e_1 - T_m(e_1))(x_1 - y_1) + \dots + (e_n - T_m(e_n))(x_n - y_n) = $$
$$= o|x - y|$$

and it is obvious because $T_m(e_i) \longrightarrow e_i$ when $m \longrightarrow \infty$. The assumptions of the Whitney extension theorem are fulfilled, so there exists f a C^1 extension of f_o, such that $d_o f$ is an isomorphism. Then let us take suitable restriction of f as the required C^1 diffeomorphism h, which completes the proof of the Theorem 2.

4. The most striking property of (b) - regularity in the theory of singularities is that a (b) - regular stratification is locally topologically trivial (cf [2]). Another interesting problem has been posed by O. Zariski (cf [10]) of whether the set $S_a(M,N)$ (resp. $S_b(M,N)$) of points $x \in N$ where condition (a) (resp. condition (b)) fails is a closed subset of N. There is a complex analytic counterexample (due to O. Zariski) for (a) regularity:

E x a m p l e 2.

Let $V = \{(x,y,u,v) \in R^4$ or $C^4 : x^2 = uvy^2 + y^3\}$,

let $N = \{(x,y,u,v) \in R^4$ or $C^4 : x = y = o\}$

and $M = V - N$.

Then $N = \operatorname{Sing} V$ (the set of points $x \in V$ where the algebraic set V is singular) and (M,N) is (a) regular for $x \in (N - \{uv = o\}) \cup \{o\}$. So the set $S_a(M,N)$ is not closed. However, for complex analytic spaces, B. Teissier (cf [4]) has shown that $S_b(M,N)$ is closed.

In the real sub - analytic case it is unknown.

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