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On Maximal Subgroups in Semigroup of Relations

Introduction. A necessary and sufficient condition for a set of functions from X^X to be completed to a group with superposition of functions has been formulated for the first time by Z.Moszner in his paper [2]. S.Serafin in his paper [3] has shown the general construction of maximal subgroups of a semigroup of functions from subsets X in X with superposition of functions. The same problem can be seen for binary relations. We deal with this problem in the present paper.

The general construction of all maximal subgroups in any semigroup is described in [1] on pages 41-43.

In this paper Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ which are known in the theory of semigroups are used for describing the maximal subgroups in the semigroup of relations.

Let $(S;\cdot)$ be a semigroup; we define for $a,b \in S$ the following sets:

$a \cdot S := \{a \cdot x : x \in S\}$, and $S \cdot b := \{x \cdot b : x \in S\}$.

We quote after [1] (pages: 72-73, 87-88):

DEFINITION 1. Let $(S;\cdot)$ be a semigroup with a unit. Then

(a) $\bigwedge_{a,b\in S} \quad a\mathcal{L}b :\Longleftrightarrow S\cdot a = S\cdot b,$

(b) $\bigwedge_{a,b\in S} \quad a\mathcal{R}b :\Longleftrightarrow a\cdot S = b\cdot S,$

(c) $\mathcal{H} := \mathcal{L}\cap\mathcal{R}$.

DEFINITION 2. Let $(S;\cdot)$ be a semigroup with a unit. Subset $H\subset S$ is a subgroup if and only if $(H;\cdot|_{H\times H})$ is a group.

The subset $H\subset S$ is a maximal subgroup if and only if

(1) H is a subgroup and

(2) $\bigwedge_{K\subset S}$ K - subgroup and $H\subset K \Rightarrow H = K$.

THEOREM 1. If idempotent e $(e\cdot e = e)$ from a semigroup $(S;\cdot)$ with a unit belongs to an abstraction class of an equivalent relation $\mathcal{H}$, which is given in a semigroup $(S;\cdot)$ then this class is a subgroup of this semigroup. (The proof of Theorem 1 you may find in [1] on pages 87-88).

The following lemmas are obvious:

LEMMA 1. Let $(S;\cdot)$ be a semigroup with a unit. Then:

(a) $\bigwedge_{a,b\in S} \quad a\mathcal{L}b \Longleftrightarrow \bigvee_{c,d\in S} \quad [c\cdot a = b \wedge d\cdot b = a],$

(b) $\bigwedge_{a,b\in S} \quad a\mathcal{R}b \Longleftrightarrow \bigvee_{c,d\in S} \quad [a\cdot c = b \wedge b\cdot d = a].$

P r o o f. Let $(S;\cdot)$ be a semigroup with a unit. Then for elements $a,b\in S$ from $S\cdot a = S\cdot b$ we obtain that $a \in S \cdot b$ and $b \in S \cdot a$. Then we find elements $c,d \in S$ that $c \cdot a = b$ and $d \cdot b = a$. If $a = d \cdot b$ and $c \cdot a = b$ for any $c,d \in S$ we have $S \cdot a = S \cdot (d \cdot b) = (S \cdot d) \cdot b \subset S \cdot b$ and $S \cdot b = S\,(c \cdot a) = (S \cdot c) \cdot a \subset S \cdot a$. Thus we have (a). Analogously, we find (b).

LEMMA 2. Let $(S;\cdot)$ be a semigroup eith a unit. A class of relation $\mathcal{H}$ to which idempotent e from S belongs is a maximal subgroup with the unit e in this semigroup $(S;\cdot)$.

P r o o f. Let H_e be a class of relation $\mathcal{H}$ to which idempotent e from S belongs and H the maximal subgroup with unit e. From Theorem 1 H_e is a subgroup and $H_e \subset H$. Let $x \in H$. Then we have $S \cdot x = S \cdot (x \cdot e) = (S \cdot x) \cdot e \subset S \cdot e$ and $S \cdot e = S \cdot (x^{-1} \cdot x) = S \cdot (x^{-1} \cdot x) \subset S \cdot x$. Thus $S \cdot x = S \cdot e$. Analogously, $x \cdot S = e \cdot S$. Thus $x \,\mathcal{H}\, e$ and $H \subset H_e$.

§1. In this paragraph we describe Green's relations $\mathcal{L}$, $\mathcal{R}$ in a semigroup of relations on set X.

Let X be an arbitrary set. B denotes a class of all binary relations which are given in X; $B := 2^{X \times X}$. Let $\circ$ be relational superposition as a binary operation in B:

$$(x,y) \in \xi \circ \alpha \iff \bigvee_{z \in X} (x,z) \in \xi \wedge (z,y) \in \alpha .$$

Then $(B;\circ)$ is a semigroup in which relation $\{(x,x) : x \in X\}$ is a unit.

LEMMA 3. For the relation $\varrho \in B$, $\varrho \circ \varrho = \varrho$ is true if and only if:

(1) ϱ is transitive and

(2) $$\bigwedge_{x,y \in X} (x,y) \in \varrho \Rightarrow \bigvee_{z \in X} (x,z) \in \varrho \wedge (z,y) \in \varrho .$$

Transitivenes of relation ϱ means $\varrho \circ \varrho \subset \varrho$, whereas condition (2) means $\varrho \subset \varrho \circ \varrho$.

The following sets have been described for every relation $\alpha \in B$:

$$D_\alpha := \left\{ x \in X : \bigvee_{y \in X} (x,y) \in \alpha \right\},$$

$$\mathbb{C}_\alpha := \left\{ y \in X : \bigvee_{x \in X} (x,y) \in \alpha \right\},$$

$$\alpha(x) := \left\{ y \in X : (x,y) \in \alpha \right\},$$

$$\alpha^{-1} := \left\{ (x,y) \in X \times X : (y,x) \in \alpha \right\}.$$

Let the connection $\xi \circ \alpha = \beta$ be true for the relations $\alpha, \beta, \xi \in B$. Then obviously:

(3) $\mathbb{C}_\beta \subset \mathbb{C}_\alpha$.

Let $(x,a) \in \beta$. Then $(x,y) \in \xi$ and $(y,a) \in \alpha$ for any $y \in D_\alpha$. For $b \in \alpha(y)$, $(y,b) \in \alpha$ is true, and then $(x,b) \in \beta$ because $\xi \circ \alpha = \beta$. That means $b \in \beta(x)$. Thus:

(4) $$\bigwedge_{(x,a) \in \beta} \bigvee_{y \in D_\alpha} (y,a) \in \alpha \wedge \alpha(y) \subset \beta(x).$$

Now it is assumed that relations $\alpha, \beta \in B$ satisfy the connections (3) and (4). We define:

$$\bigwedge_{x,y \in X} (x,y) \in \xi :\Leftrightarrow [x \in D_\beta \wedge y \in D_\alpha \wedge \alpha(y) \subset \beta(x)].$$

Let $(u,v) \in \xi \circ \alpha$. Then we find an element $z \in X$ for which $(u,z) \in \xi$ and $(z,v) \in \alpha$. Thus $\alpha(z) \subset \beta(u)$ and because $v \in \alpha(z)$ it follows that $v \in \beta(u)$ that is $(u,v) \in \beta$. Thus $\xi \circ \alpha \subset \beta$. Let $(u,v) \in \beta$ now. According to (4) we find such a $t \in D_\alpha$ that $(t,v) \in \alpha$ and $\alpha(t) \subset \beta(u)$. Then $(u,t) \in \xi$ and because $(t,v) \in \alpha$ so $(u,v) \in \xi \circ \alpha$. It follows that $\beta \subset \xi \circ \alpha$.

In this way the following Lemma 4 has been proved.

LEMMA 4.

$$\bigwedge_{\alpha,\beta \in B} \bigvee_{\xi \in B} (\xi \circ \alpha = \beta) \Leftrightarrow \left[\mathbb{C}_\beta \subset \mathbb{C}_\alpha \wedge \bigwedge_{(x,a) \in \beta} \bigvee_{y \in D_\alpha} (y,a) \in \alpha \wedge \alpha(y) \subset \beta(x) \right]$$

According to Lemma 1 we obtain:

LEMMA 5.

$$\bigwedge_{\alpha,\beta \in B} \alpha \mathcal{L} \beta \Leftrightarrow \begin{cases} (5)\ \mathbb{C}_\beta = \mathbb{C}_\alpha , \\ (6) \bigwedge_{(x,a) \in \beta} \bigvee_{y \in D_\alpha} (y,a) \in \alpha \wedge \alpha(y) \subset \beta(x) , \\ (7) \bigwedge_{(z,b) \in \alpha} \bigvee_{u \in D_\beta} (u,b) \in \beta \wedge \beta(u) \subset \alpha(z) . \end{cases}$$

Now relations between conditions (5), (6), (7) will be examined. Let conditions (6) and (7) be satisfied for relations $\alpha, \beta \in B$. Let $t \in \mathbb{C}_\alpha$. Then we find such $s \in D_\alpha$ that $(s,t) \in \alpha$. As (7) is true for (s,t) there is $v \in D_\beta$ for which $(v,t) \in \beta$ and $\beta(v) \subset \alpha(s)$. According to (6) we can find $r \in D_\alpha$ for $(v,t) \in \beta$ when $(r,t) \in \alpha$ and $\alpha(r) \subset \beta(v)$. Because $t \in \alpha(r) \subset \beta(v) \subset \mathbb{C}_\beta$, then $\mathbb{C}_\alpha \subset \mathbb{C}_\beta$. Analogously, $\mathbb{C}_\beta \subset \mathbb{C}_\alpha$. Thus (5) follows from (6) and (7).

Let $X = \{1,2,3\}$, $\alpha := \{(1,2), (1,3), (3,2)\}$, $\beta := \{(1,2), (2,3), (3,2)\}$ be considered now. Then we have $\mathbb{C}_\beta = \{2,3\} = \mathbb{C}_\alpha$ and:

for a pair $(1,2) \in \alpha$ we find $y = 1 \in D_\beta$ such that $(1,2) \in \beta$ and $\beta(1) = \{2\} \subset \alpha(1) = \{2,3\}$,

for the second pair $(1,3) \in \alpha$ we find $z = 2 \in D_\beta$ for which $(2,3) \in \beta$ and $\beta(2) = \{3\} \subset \{2,3\} = \alpha(1)$,

and for the third pair $(3,2) \in \alpha$ we find $v = 3 \in D_\beta$ such that $(3,2) \in \beta$ and $\beta(3) = \{2\} \subset \alpha(3) = \{2\}$.

Thus conditions (5) and (7) take place while condition (6) is not satisfied because for $(2,3) \in \beta$ we find one and only one element $t = 1$ in D_α for which $(1,3) \in \alpha$, but $\alpha(1) = \{2,3\} \not\subset \{3\} = \beta(2)$.

In the presence of symmetry of conditions (6) and (7) we draw a conclusion about the independence of condition (7) from the conjunction (5) and (6). From the above example it follows that (6) does not depend on (7) and reciprocally. Obviously condition (5) does not depend on (7), because for relations $\gamma := \{(1,1)\}$, $\delta := \{(1,1), (2,2)\}$ on the set $X = \{1,2\}$ condition (7) is satisfied, but not (5) $(\mathbb{C}_\gamma \neq \mathbb{C}_\delta)$. Analogously, the conclusion is true for (5) and (6).

Conclusion 1.

$$\bigwedge_{\alpha,\beta \in B} [\alpha \mathcal{L} \beta \Leftrightarrow ((6) \text{ and } (7) \text{ for } \alpha \text{ and } \beta)]$$

It must be noticed that

$$\alpha \mathcal{R} \beta \Leftrightarrow \bigvee_{\xi,\eta\in B} (\alpha\circ\xi=\beta \wedge \beta\circ\eta=\alpha) \Leftrightarrow$$
$$\Leftrightarrow \bigvee_{\xi,\eta\in B} (\xi^{-1}\circ\alpha^{-1}=\beta^{-1} \wedge \eta^{-1}\circ\beta^{-1}=\alpha^{-1}) \Leftrightarrow \alpha^{-1}\mathcal{L}\beta^{-1}$$

is valid in a semigroup $(B; \circ)$

Then it is true that:

LEMMA 6.

$$\bigwedge_{\alpha,\beta\in B} \alpha\mathcal{R}\beta \Leftrightarrow \alpha^{-1}\mathcal{L}\beta^{-1}$$

§2. In this paragraph we will describe the maximal subgroups in a semigroup of relations $(B; \circ)$

According to the definition of the relation $\mathcal{H}$ and Lemmas 5 and 6 and the conclusion 1 we can form the following theorem:

THEOREM 2.

$$\bigwedge_{\alpha,\beta\in B} \alpha\mathcal{H}\beta \Leftrightarrow \begin{cases} \text{(i)}\ \bigwedge_{(x,a)\in\beta}\ \bigvee_{y\in D_\alpha} (y,a)\in\alpha \wedge \alpha(y)\subset\beta(x), \\ \text{(ii)}\ \bigwedge_{(z,b)\in\alpha}\ \bigvee_{v\in D_\beta} (v,b)\in\beta \wedge \beta(v)\subset\alpha(z), \\ \text{(iii)}\ \bigwedge_{(c,s)\in\beta}\ \bigvee_{t\in C_\alpha} (c,t)\in\alpha \wedge \alpha^{-1}(t)\subset\beta^{-1}(s), \\ \text{(iv)}\ \bigwedge_{(d,t)\in\alpha}\ \bigvee_{n\in C_\beta} (d,n)\in\beta \wedge \beta^{-1}(n)\subset\alpha^{-1}(t). \end{cases}$$

Conclusion 2. If $\alpha\mathcal{H}\beta$ then $D_\alpha = D_\beta$ and $C_\alpha = C_\beta$.

According to Lemmas 2, 3 and Theorems 1, 2 we can formulate a necessary and sufficient condition when any relation $\alpha\in B$ belongs to the maximal subgroup with unit

$\varrho \in B$ in semigroup $(B;\ \circ)$

THEOREM 3.

$$\bigwedge_{\alpha,\varrho\in B} [\alpha \mathcal{H} \varrho \wedge \varrho\circ\varrho = \varrho \Leftrightarrow \text{(i), (ii), (iii), (iv) for } \alpha \text{ and } \varrho \text{ and (1), (2) for } \varrho].$$

Conclusion 3. Relation $\alpha \in B$ belongs to a maximal subgroup of the relations contained in B with superposition as a binary operation if and only if there is such $\varrho \in B$ that $\varrho = \varrho\circ\varrho$ and α and ϱ satisfy connections (i) -(iv).

§ 3. Now we shall formulate two conclusions which result mainly from Theorems 2 and 3 when relation α is a function but idempotent ϱ is a relation, or α and idempotent ϱ are functions.

Let us assume that relation $f \in B$ is a function whereas $\varrho \in B$ and $\varrho\circ\varrho = \varrho$. Let also $f \mathcal{R} \varrho$. Let $(x,a) \in \varrho$. Then according to conclusion 2 $a \in \mathbb{C}_\varrho = \mathbb{C}_f$. We always find such $y \in D_f$ that $(y,a) \in f$ and $f(y) = \{a\} \subset \varrho(x)$. The condition (i) is trivially satisfied but conditions (ii) - (iv) take suitable forms (with the functional notation for f):

$$\text{(ii')} \quad \bigwedge_{z\in D_f} \bigvee_{v\in D_\varrho} (v, f(z)) \in \varrho \wedge \varrho(v) = \{f(z)\},$$

$$\text{(iii')} \quad \bigwedge_{(c,s)\in\varrho} c \in D_f \wedge f^{-1}(f(c)) \subset \varrho^{-1}(s),$$

$$\text{(iv')} \quad \bigwedge_{d\in D_f} \bigvee_{n\in \mathbb{C}_\varrho} (d,n) \in \varrho \wedge \varrho^{-1}(n) \subset f^{-1}(f(d)).$$

Therefore there is:

Conclusion 4. If f is a function and ϱ is an idempotent from B, then $f \mathcal{H} \varrho$ if and only if conditions (ii'), (iii'), (iv') and $\mathbb{C}_f = \mathbb{C}_\varrho$ are satisfied for f and ϱ.

Let f, g be functions. Let also $f \mathcal{H} g$ and $g \circ g = g$. Then according to conclusion 2, conditions (ii) and (i) are trivially satisfied. But (iii) and (iv) take the following forms:

(iii'') $$\bigwedge_{c \in D_g} c \in D_f \wedge f^{-1}(f(c)) \subset g^{-1}(g(c)),$$

(iv'') $$\bigwedge_{d \in D_f} d \in D_g \wedge g^{-1}(g(d)) \subset f^{-1}(f(d)).$$

Conditions (iii'')and (iv'')are obviously equivalent to the condition:

$$\bigwedge_{c \in D_f} c \in D_g \wedge g^{-1}(g(c)) = f^{-1}(f(c)).$$

Because g is an idempotent, then $g(g(x)) = g(x)$ and $\mathbb{C}_g \subset D_g$ and $g|_{\mathbb{C}_g}$ is the identity function on $\mathbb{C}_g$.

Let now $y_1, y_2 \in \mathbb{C}_g = \mathbb{C}_f \subset D_g = D_f$ and let $f(y_1) = f(y_2)$. Then

$$f^{-1}(f(y_1)) = f^{-1}(f(y_2)) = g^{-1}(g(y_1)) = g^{-1}(g(y_2))$$

and consequently $g(y_1) = g(y_2)$ and then $y_1 = y_2$. It has been proved that $f|_{\mathbb{C}_g = \mathbb{C}_f}$ is an injection.

Besides, if $f \mathcal{H} g$, then there is such a relation h in a maximal subgroup with unit g in the semigroup B for f that $g = h \circ f$.

Let now $y \in \mathbb{C}_g$. Then $(y,y) \in g = h \circ f$ and thus there is such an element $x_0 \in D_f$ that $(y,x_0) \in h$ and $(x_0,y) \in f$ that is $f(x_0) = y$. And so $f|_{\mathbb{C}_g = \mathbb{C}_f}$ is a surjection $\mathbb{C}_g$ on $\mathbb{C}_g$. Thus we have:

Conclusion 5. If f, g are functions from B and g is an idempotent then:

(a) $f \mathcal{R} g$ if and only if $D_f = D_g$ and $\mathbb{C}_f = \mathbb{C}_g$ and $f \circ f^{-1} = g \circ g^{-1}$ and $f|_{\mathbb{C}_g}$ is a permutation $\mathbb{C}_g$,

(b) $g|_{\mathbb{C}_g}$ is the identity function on $\mathbb{C}_g$.

R e m a r k. Conclusion 5 has been already formulated by S.Serafin in [3] and is based on [1], but he has arrived at it in a different way. He has formulated it for the semigroup of partial transformations, which is a subsemigroup of semigroup B.

References

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