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On commuting idempotent functions

This paper presents some fragment of dealing on sub-semilattices in the semigroup $\mathcal{T}(M)$ of all mappings from M to itself.

1. Let M be a set and $f \in \mathcal{T}(M)$ an idempotent element in the semigroup $\mathcal{T}(M)$ i.e.

(1) $$ff = f.$$

The equality (1) is equivalent to the following condition: the components of the partition $M/_{f^{-1}f}$ can be indexed using elements of the range $\mathcal{C}_f$ such that

(2) $$M/_{f^{-1}f} = \{M_i\}_{i \in \mathcal{C}_f} \quad \text{and for } i \in \mathcal{C}_f \text{ we have } i = f(i) \in M_i,$$
$$f(M_i) = \{i\}.$$

We shall construct all the mappings $g \in \mathcal{T}(M)$ fulfilling the conditions:

(3) $$gg = g,$$

(4) $$gf = fg.$$

2. Suppose first that $g \in \mathcal{T}(M)$ fulfills (3), (4).

Then by (4)

$$f(g(M_i)) = g(f(M_i)) = g(\{i\}) = \{g(i)\}.$$

Hence $g(M_i)$ is contained in a certain M_j.

Thus

(5) $$\bigwedge_{i \in \mathbb{C}_f} \bigvee_{j \in \mathbb{C}_f} \left(g(M_i) \subset M_j \wedge g(i) = j\right), \text{ and}$$

$$\tilde{g} := g \mid \mathbb{C}_f \in \mathcal{T}(\mathbb{C}_f).$$

For the sequel introduce the sets

$S := g(\mathbb{C}_f)$ and for $s \in S$ $I_s := \tilde{g}^{-1}(\{s\})$, $U_s := \bigcup_{i \in I_s} M_i$.

Evidently $\mathbb{C}_f = \bigcup_{s \in S} I_s$ and $M = \bigcup_{s \in S} U_s$.

Moreover $s \in I_s$, since $S \subset \mathbb{C}_g$ and from (3) $g(s) = s$.

Thus the familly $\{I_s\}_{s \in S}$ is a partition of $\mathbb{C}_f$ and $\{U_s\}_{s \in S}$ is a partition of M satisfying the relation:

$$g(U_s) \subset M_s \subset U_s \quad \text{for} \quad s \in S.$$

Let

$$g_s = g \mid U_s.$$

It is obvious that $g_s \in \mathcal{T}(U_s)$ and according to (3) for every $x \in U_s$ there is

$$g_s(g_s(x)) = g_s(g(x)) = g(g(x)) = g(x) = g_s(x).$$

Thus

$$g_s g_s = g_s \quad \text{for} \quad s \in S$$

and

$$g = \bigcup_{s \in S} g_s.$$

3. Conversely, we shall prove that every mapping $g \in \mathcal{T}(M)$ constructed by conditions found in 2. satisfies (3) and (4).

Let $\emptyset \neq S \subset \mathbb{C}_f$, $\{I_s\}_{s \in S}$ be a partition of $\mathbb{C}_f$ (if $s_1 \neq s_2$, then $I_{s_1} \cap I_{s_2} = \emptyset$) such that $s \in I_s$ for $s \in S$.

Next denote $U_s := \bigcup_{i \in I_s} M_i$. $\{U_s\}_{s \in S}$ is a partition of M and $M_s \subset U_s$ $(s \in S)$.

Moreover let $g_s \in \mathcal{T}(U_s)$ be an arbitrary idempotent mapping from U_s to itself, satisfying the conditions:

$$g_s(I_s) = \{s\}, \quad g_s(U_s) \subset M_s.$$

If $g := \bigcup_{s \in S} g_s$, then for $x \in M$ there exists some $s \in S$ such that $x \in U_s$ and $g_s(x) \in M_s$, and consequently

$$g(g(x)) = g(g_s(x)) = g_s(g_s(x)) = g_s(x) = g(x).$$

To verify the equality (4) for $x \in M$ we find the unique $i \in \mathbb{C}_f$ with $x \in M_i$ and next the $s \in S$ such that $i \in I_s$ and

$$x \in M_i \subset U_s.$$

Hence $f(g(x)) = f(g_s(x)) \in f(M_s) = \{s\}$ and

$$g(f(x)) = g(i) = g_s(i) = s.$$

Thus we have proved our statement.