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Solution of the translation equation on extensions of semigroup with zero-multiplication

1. Let $(S, \cdot)$ be a semigroup with zero-multiplication $(S \cdot S = \{0\})$,
$(G, \cdot)$ a semigroup with identity element 1, such that
$S \cap G = \emptyset$ and $\mathrm{card}(S) > 1$.

We introduce

$$\Sigma := S \cup G$$

and for $x, y \in \Sigma$

$$x \circ y := \begin{cases} x \cdot y, & \text{when } x, y \in S \text{ or } x, y \in G, \\ 0, & \text{when } x \in G,\ y \in S \text{ or } x \in S,\ y \in G. \end{cases}$$

Under above assumptions $(\Sigma, \circ)$ is the ideal extension of the semigroup S by G^0 (G with adjoined zero element) obtained using the homomorphism $\varphi : G \to S$ such, that $\varphi(x) = 0$ for $x \in G$ (see [1], p.167).

R e m a r k 1. The zero-homomorphism is the unique homomorphism from $(G, \cdot)$ to $(S, \cdot)$, Indeed for $x \in G$ and a homomorphism $h : G \to S$ we have

$$h(x) = h(1 \cdot x) = h(1) \cdot h(x) = 0.$$

Consequently if $(G, \cdot)$ is a group, then $(\Sigma, \cdot)$ is isomorphic to every ideal extension of $(S, \cdot)$ by $(G^0, \cdot)$ constructed using homomorphisms from G to S.

2. Let Γ be an arbitrary nonempty set and

$$F: \Gamma \times \Sigma \to \Gamma$$

be a solution of the translation equation

(1) $F(F(\alpha, x), y) = F(\alpha, x \cdot y)$ for $\alpha \in \Gamma$, $x, y \in \Sigma$.

Introduce $F_1 := F \mid \Gamma \times G$,

$F_2 := F \mid \Gamma \times S$.

Evidently F_1, F_2 are solutions of the translation equation with the fibre Γ on $(G, \cdot)$ and $(S, \cdot)$ respectively.

Moreover there is

(2) $F_1(F_2(\alpha, x), y) = F_2(\alpha, 0)$, for $\alpha \in \Gamma$, $x \in S$, $y \in G$,

(3) $F_2(F_1(\alpha, x), y) = F_2(\alpha, 0)$, for $\alpha \in \Gamma$, $x \in G$, $y \in S$.

It is easily seen:

THEOREM 1. A mapping $F: \Gamma \times \Sigma \to \Gamma$ is a solution of the translation equation on the semigroup $(\Sigma, \cdot)$ with the fibre Γ iff there exist solutions F_1, F_2 of the translation equation with the fibre Γ on G and S respecively, satisfying (3), (4) and $F_1 = F \mid \Gamma \times G$, $F_2 = F \mid \Gamma \times S$.

3. We know the general solution of the translation equation on a semigroup with zero-multiplication (see [2])

and now we shall investigate if every solution F_1 of the translation equation on G can be extended to the solution of this equation on Σ and the analogous problem for solutions of the translation equation on S.

First we shall prove the following:

THEOREM 2. For every solution F_2 of the translation equation on S with the fibre Γ there exists an extension F which is a solution of the translation equation on Σ, with the same fibre Γ; under additional condition $F(\Gamma\times\Sigma) = F_2(\Gamma \times S)$ this extension is uniquely determined.

P r o o f. Let $F_2: \Gamma \times S \to \Gamma$ be a solution of the translation equation. Then (see [2]) there are determined

$$\{M_i\}_{i\in I}, \{P_i\}_{i\in I}, \{\alpha_i\}_{i\in I} \text{ such that}$$

$$\alpha_i \in P_i \subset M_i \qquad \text{for} \quad i\in I$$

$\{M_i\}_{i\in I}$ is a partition of Γ, where $M_i \cap M_j = \emptyset$ for $i \neq j$ and moreover

$$F_2(M_i \times S) = P_i, \quad F_2(\alpha, x) = \alpha_i$$

for $\alpha \in P_i$, $x \in S$ or $\alpha \in M_i$, $x = 0$.

Suppose that $F: \Gamma \times \Sigma \to \Gamma$ satisfies the translation equation on Σ and $F_2 = F | \Gamma \times S$.

For $i \in I$ and $\alpha \in P_i$ we find $\beta \in M_i$ and $x_0 \in S$ such that $F(\beta, x_0) = F_2(\beta, x_0) = \alpha$.

If for some $y \in G$ there is $F(\alpha, y) \neq \alpha_i$, then

$F(F(\beta, x_0), y) = F(\alpha, y) \neq \alpha_i$ and on the other hand

$F(F(\beta, x_0), y) = F(\beta, x_0 \cdot y) = F(\beta, 0) = F_2(\beta, 0) = \alpha_i.$

This contradiction proves, that

$$F(P_i \times G) = \{\alpha_i\} \quad \text{for} \quad i \in I.$$

Let now $i \in I$ and $\alpha \in M_i$. If for some $x \in G$ we have $F(\alpha, x) \in M_j$ and $i \neq j$, then

$$F(F(\alpha, x), 0) = F_2(F(\alpha, x), 0) \in F_2(M_j \times S) = P_j$$

and

$$F(F(\alpha, x), 0) = F_2(\alpha, 0) = \alpha_i \notin P_j.$$

Whence $F(M_i \times G) \subset M_i$ for $i \in I$.

Assuming additionally $F(\Gamma \times \Sigma) = F_2(\Gamma \times S)$ we obtain $F(M_i \times G) \subset M_i \cap F_2(\Gamma \times S) = P_i$. In this case for every $\alpha \in M_i$, $x \in G$ there is

$$F(\alpha, x) = F(F(\alpha, 1), x) \in F(P_i \times G) = \{\alpha_i\}$$

and consequently

$$F(M_i \times G) = \{\alpha_i\} \quad \text{for} \quad i \in I. \tag{4}$$

If we denote $F_1 := F \mid \Gamma \times G$, under condition $F_2(\Gamma \times S) = F(\Gamma \times \Sigma)$ we have thus for $i \in I$

$$F_1(M_i \times G) = \{\alpha_i\}. \tag{5}$$

The mapping F_1 is by (5) uniquely determined and it is a solution of the translation equation on G.

In fact if F_1 is defined by (5), $i \in I$, $\alpha \in M_i$, $x, y \in G$, then $F_1(F_1(\alpha, x), y) = F_1(\alpha_i, y) = \alpha_i = F_1(\alpha, x \cdot y)$.

We shall examine that F_1 defined by (5) satisfies (2) and (3). If $i \in I$, $\alpha \in M_i$, $x \in S$, $y \in G$ we have for such F_1

$$F_1(F_2(\alpha,x),y) \in F_1(P_i \times G) = \{\alpha_i\},$$

so that

$$F_1(F_2(\alpha,x),y) = \alpha_i = F_2(\alpha,0).$$

Similarly if $i \in I$, $\alpha \in M_i$, $x \in G$, $y \in S$ we obtain

$$F_2(F_1(\alpha,x),y) = F_2(\alpha_i,y) = \alpha_i = F_2(\alpha,0).$$

In result the mapping F_1 determined by (5) is a solution of the translation equation on G extending F_2 to the solution F of this equation on Σ; when $F_2(\Gamma \times S) = F(\Gamma \times \Sigma)$ it is the unique mapping with these properties.

R e m a r k 2. The mapping $F_1: \Gamma \times G \to \Gamma$ given by (5) is a trivial solution of the translation equation on G in the sense that $F_1(\alpha,x) = F_1(\alpha,1) = f(\alpha)$ when $\alpha \in \Gamma$, $x \in G$ and

$$f(f(\alpha)) = f(\alpha) \quad \text{for every} \quad \alpha \in \Gamma.$$

COROLLARY. A solution F_1 of the translation equation on G can be extended to solution of this equation on Σ with the same fibre Γ and property $F(\Gamma \times \Sigma) = F(\Gamma \times S)$ iff F_1 is trivial.

P r o o f. (a). If F_1 is not trivial and $F_1 = F|\Gamma \times G$, $F(\Gamma \times \Sigma) = F(\Gamma \times S)$ then $F_2 := F|\Gamma \times S$ is a solution of translation equation on S which is extended to F by F_1. This is in contradiction to Theorem 2; (b). Let $f: \Gamma \to \Gamma$, $ff = f$, $F_1(\alpha,x) = f(\alpha)$ for every $\alpha \in \Gamma$, $x \in G$. Assume that $F_1(\Gamma \times G) = \{\alpha_i\}_{i \in I}$, where $\alpha_i \neq \alpha_j$ for $i \neq j$.

Next denote $M_i := f^{-1}(\{\alpha_i\})$ and choice $P_i \subset M_i$ such that $\alpha_i \in P_i$. Let further f_i be a surjective mapping $f_i: M_i \times S \to P_i$ with condition $f_i(\alpha,x) = \alpha_i$ when $\alpha \in P_i$, $x \in S$ or $\alpha \in M_i$, $x = 0$. The mapping $F_2: \Gamma \times S \to \Gamma$ given by formula

$$F_2 := \bigcup_{i \in I} f_i$$

is a solution of the translation equation on S and (2), (3) are obviously fulfilled.

So

$$F(\alpha,x) := \begin{cases} F_1(\alpha,x) & \text{when} \quad \alpha \in \Gamma,\ x \in G \\ F_2(\alpha,x) & \text{when} \quad \alpha \in \Gamma,\ x \in S \end{cases}$$

is an extension of F_1 on Σ.

It is evident, that this extension must not be unique in general.

References

[1] Clifford A.H., Extensions of semigroups, Trans. Amer. Math. Soc., 68, p.165-173.

[2] Piechowicz L., Serafin S., Solution of the translation equation on some structures, Zeszyty Nauk.UJ, Prace Mat. 21, Kraków 1979, p.109-114.