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# On the solvability of the Riquier type boundary value problem in the half-space

1. Let $\Omega = \{x \in R^{n+1};\ x_{n+1} > 0\}$; E - linear elliptic differential operator of order $2\cdot k$ with complex coefficients. We shall consider the boundary value problem:

$$(1)\quad \begin{cases} E^p u(x) = 0 & ;\ x \in \Omega, \\ E^j u(\bar{x}) = f_j(\bar{x}) ; & \bar{x} \in \partial\Omega \ ;\ j = 0,1,2,\ldots,p-1, \end{cases}$$

where $E^0 = I$ (identity operator), $E^j = E^{j-1} \circ E$, $j = 1,2,\ldots,p$.

In [1] Agmon, Douglis, Nirenberg have considered the so called complementing conditions. If these conditions are fulfilled we are able to use some estimates (near the boundary) for a solution of the considered boundary value problem and to solve the problem with some additional assumptions on $f_j$.

In our paper we shall prove that with some assumptions on the operator E the complementing conditions are satisfied. With some additional conditions on the functions $f_j$ we shall give also the solution of problem (1). Next we shall give an example of the operator $E$ for which the complementing conditions do not hold.

Moreover, we shall show that the complementing conditions may not hold for the problem:

$$(1') \begin{cases} E^p u(x) = 0 & ; \quad x \in \Omega, \\ F^j u \bar{x} = f_j \bar{x} & ; \quad \bar{x} \in \partial\Omega \; ; \quad j = 0,1,2,\ldots,p-1, \end{cases}$$

when $F \neq E$ ($F^o = I$, $F^j = F^{j-1} \circ F$, $j = 1,2,\ldots,p-1$).

2. Complementing conditions are connected with the system of operators appearing in the boundary value problem, so we shall use the phrase: complementing conditions for the system of operators.

Let $Q \subset R^{n+1}$ be a domain with boundary $\Gamma$ of $C^1$ class,

$$D = \left(\frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_{n+1}}\right).$$

Suppose that $L = L(x,D)$ is a linear differential operator with complex coefficients of order $2m$; $x \in Q$.

$L' = L'(x,D)$ is the leading part of $L$ (the part of highest order).

Let $B_j = B_j(x,D)$ ($j = 1,2,\ldots,m$) be linear differential operators with complex coefficients, defined for $x \in \Gamma$ ;

let $B_j' = B_j'(x,D)$ be the leading part of $B_j$ ($j=1,2,\ldots,m$).

For every point $x$ in the closure $\bar{Q}$ we assume the following <u>condition on $L$:</u>

for every pair of linearly independent real vectors $u,w \in R^{n+1}$ the polynomial in the variable $T$: $L'(x,u + Tw)$

has exactly $m$ roots with positive imaginary parts.

At any point $x$ of $\Gamma$ let $n$ denote the normal to $\Gamma$ and $u \neq 0$ any real vector parallel to the boundary.

DEFINITION. We require that the polynomials, in $T$, $B_j'(x,u + Tn)$, $j = 1,2,\dots,m$, be linearly independent modulo the polynomial $\prod_{k=1}^{m}(T - T_k^+(u))$, where $T_k^+(u)$ are the roots of $L'(x,u + Tn)$ with positive imaginary parts.

Then, by definition, the systems of operators $L$, $B_j$ $(j = 1,2,\dots,m)$ fulfiles the complementing conditions.

3. We shall formulate and prove the theorem on fulfilling complementing conditions by a system of operators for the problem (1).

THEOREM.

1) If $Q = \Omega = \{x \in R^{n+1};\ x_{n+1} > 0\}$,

2) $E$ is a linear elliptic operator with complex coefficients of order $2k$,

3) for every pair of linearly independent real vectors $u,w \in R^{n+1}$ the polynomial in the variable $T$ $E'(x,u + wT)$ has exactly $k$ roots with positive imaginary parts (for any $x \in \Omega$),

4) $\forall_{x\in\Gamma}\ \exists_{0\neq\xi\in R^n}$ $E'(x,(\xi,T))$ has as a polynomial in $T$ at least one root of 1-order of a root of k-order with a positive imaginary part

then the system of operators for the problem (1) fulfils the complementing conditions.

P r o o f. For simplicity assume that $E$ is equal to its leading part $E'$.

Then $L' = (E^p)' = E^p$.

Order of $L = E^p$ is $2pk$ and 2) and 3) imply that the polynomial (in $T$) $L'(x, u + Tw)$ has exactly $pk$ roots with positive imaginary parts (for any $x \in \Omega$ and every pair of linearly independent $u, w \in R^{n+1}$).

Fix $x \in \Gamma$. For simplicity we shall not write $x$ in the operators.

In our case $\Gamma = \{x \in R^{n+1} : x_{n+1} = 0\}$. If $n$ is normal to $\Gamma$ at point $x$ and $0 \neq u \in R^{n+1}$, $u \perp n$ then:

$u + T \cdot n = (\bar{u}, 0) + T \cdot (\bar{0}, 1) = (\bar{u}, T)$ where $\bar{0}, \bar{u} \in R^n$.

(We use the notation $x = (\bar{x}, x_{n+1})$ for $x \in R^{n+1}$: $\bar{x} \in R^n$, $x_{n+1} \in R$).

For polynomials $U$, $W$ ($W \neq 0$) by $r\left(\frac{U}{W}\right)$ shall denote the rest of $\frac{U}{W}$.

Let $T_n^+$ are the roots of $E((\bar{u}, T))$ with positive im. parts and

$T_n^-$ are the roots of $E((\bar{u}, T))$ with negative im. parts.

Assume for example, that $T_1$ is a single zero of polynomial $E((\bar{u}, T))$.

In the case of $T_1$ zero of k-order the proof is similar to the above one.

Let

(2) $\lambda = T - T_1^+$; $\beta_n' = T_1^+ - T_n^+$; $\beta_n'' = T_1^+ - T_n^-$

$n=2,3,\dots,k,$ $n=1,2,\dots,k.$

Let

$M_j(\lambda) = \alpha\lambda^{j-1}(\lambda + \beta_1'')^{j-1} \prod_{n=2}^{k} (\lambda + \beta_n')^{j-1}(\lambda + \beta_n'')^{j-1},$

$j = 1,2,\dots,p,$

$$M(\lambda) = \lambda^p \prod_{n=2}^{k} (\lambda + \beta_n')^p,$$

$$R_j(\lambda) = r\left(\frac{M_j(\lambda)}{M(\lambda)}\right).$$

Putting $\lambda = T - T_1^+$ into the polynomials

$$\alpha\left(\prod_{n=1}^{k} (T - T_n^+)(T - T_n^-)\right)^{j-1}$$

and $\left(\prod_{n=1}^{k} (T - T_n^+)\right)^p$ we have $M_j(\lambda)$ and $M(\lambda)$ accordingly.

The linear substitution $\lambda = T - T_1^+$ has no influence on linearly independent, so we have only to show, that polynomials $R_1, R_2, \dots, R_p$ are linearly independent.

Because $\deg M > \deg M_j$ for $j < \frac{p}{2} + 1$, then

(3) $R_j(\lambda) = M_j(\lambda) =$

$$= \alpha\lambda^{j-1}(\lambda + \beta_1'')^{j-1} \prod_{n=2}^{k} (\lambda + \beta_n')^{j-1}(\lambda + \beta_n'')^{j-1}; \quad j < \frac{p}{2} + 1.$$

For $j \geq \frac{p}{2} + 1$ the polynomials $M_j$ and $M$ have the common divisor

$$Q = \lambda^{j-1} \prod_{n=2}^{k} (\lambda + \beta_n')^{j-1}.$$

For any polynomials $W, V, Q$: if $V\,Q \neq 0$, then we have

$$r\left(\frac{W\ Q}{V\ Q}\right) = r\left(\frac{W}{V}\right)\cdot Q$$

then

(4) $R_j$ =

$$r\left(\frac{\alpha(\lambda+\beta_1'')^{j-1}\prod_{n=2}^{k}\left|(\lambda+\beta_n'')^{j-1}\right.}{\lambda^{p-(j-1)}\prod_{k=2}^{m}\left|(\lambda+\beta_n')^{p-(j-1)}\right.}\right)\lambda^{j-1}\prod_{n=2}^{k}(\lambda+\beta_n')^{j-1}\Bigg|_{j\geqslant\frac{p}{2}+1}.$$

Let

$$R_j(\lambda) = \sum_{l=0} a_{jl}\lambda^l.$$

By formulas (3) (4) we get

$$a_{jl} = \begin{cases} 0 & ; \quad l < j-1, \\ \alpha\,\beta_1''^{\,j-1}\prod_{n=2}^{k}\left|\beta_n'^{\,j-1}\,\beta_n''^{\,j-1}\right. & ;\ l = j-1;\quad j < \frac{p}{2}+1, \\ \alpha\,\beta_1''^{\,j-1}\prod_{n=2}^{k}\left|\beta_n''^{\,j-1}\prod_{n=2}^{k}\right|\beta_n'^{\,j-1} & ;\ l = j-1;\quad j \geqslant \frac{p}{2}+1. \end{cases}$$

We have assumed, that $T_1^+$ is the root of 1-order hence: $\beta_n' \neq 0$ $(n = 2,3,\ldots,k)$ moreover $\beta_n'' \neq 0$ $(n = 1,2,\ldots,k)$; $\alpha \neq 0$. Then $a_{jl} = 0$ for $l < j-1$ and $a_{jl} \neq 0$ for $l = j-1$, $l = 0,1,2,\ldots$ and the determinant of matrix $\Delta = [a_{jl}]$ is different from zero, because $\Delta$ is a triangular matrix without zeros on the diagonal. Hence the polynomials $R_1, R_2, \ldots, R_p$ are linearly independent ([1] p.633).

Because $x$ was fixed the proof is completed.

4. Assume now, that the functions $f_j$ in problem (1) are of the class $\mathcal{E}^{\infty}(R^n)$ with compact supports.

Let the assumptions of theorem be fulfilled.

Moreover, assume that $E = E'$ and $E$ is the operator with constant coefficients.

Then Th.2.1 [1] implies the existence of a solution (1) and

$$u(\bar{x},t) = \sum_{j=1}^{m} \int_{R^n} K_j(\bar{x} - \bar{y},t) \cdot f_j(\bar{y})\, d\bar{y} = \sum K_j * f_j,$$

when $\bar{x} \in R^n$, $t \in R$; $K_j$ are the Poisson kernels:

$K_j(\bar{x},t) =$

$j=1,2,\dots,m$

$$= \begin{cases} \dfrac{\beta_j}{2\pi i} \displaystyle\int_{|\xi|=1} dw_\xi \left[ \int_\gamma \frac{N_j(\xi,T)(\bar{x}\cdot\xi + tT)^{m_j - n}}{M^+(\xi,T)} \log \frac{\bar{x}\cdot\xi + tT}{i}\, dT \right], \\ \text{; for } m_j \geqslant n, \\ \dfrac{\beta_j}{2\pi i} \displaystyle\int_{|\xi|=1} dw_\xi \left[ \int_\gamma \frac{N_j(\xi,T)}{M^+(\xi,T)(\bar{x}\cdot\xi + tT)^{n-m_j}}\, dT \right], \text{ for } 0 \leqslant m_j < n. \end{cases}$$

Here the principal branch of the logarithm in the complex plane slit along the negative real axis is taken, $dw_\xi$ is the area element on the unit sphere $|\xi| = 1$, and the $\beta_j$ are absolute constans give by:

$$\beta_j = \begin{cases} -\dfrac{1}{(2\pi i)^n (m_j - n)!} & \text{if } m_j \geqslant n, \\ (-1)^{n-m_j} \dfrac{(n - m_j - 1)!}{(2\pi i)^n} & \text{if } 0 \leqslant m_j < n, \end{cases}$$

$\gamma$ is a Jordan contour in $\operatorname{Im} T > 0$ enclosing all the roots of $M^+(\xi, T) = \prod_{k=1}^{m} (T - T_k^+(\xi))$ for all $|\xi| = 1$ where $T_k^+(\xi)$ are the roots of $L'(x, \xi + Tn) = L((\xi, T)) = E^p((\xi, T))$ with positive imaginary parts, $M^+(\xi, T) = \prod_{k=1}^{m} (T - T_k^+(\xi)) = \sum_{p=0}^{m} a_p^+(\xi)\, T^{m-p}$, $B_j(\xi, T) = B_j(\xi, T)(\text{mod } M^+) = \sum_{k=1}^{m} b_{jk}(\xi)\, T^{k-1}$ where $B_j = E^{j-1}$ $(j = 1, 2, \ldots, p)$,

$$M_j^+(\xi, T) = \sum_{p=0}^{j} \alpha_p^+\, T^{j-p},$$

$N_k(\xi, T) = \sum_{i=1}^{m} b^{ik}(\xi)\, M_{m-i}^+(\xi, T)$ where $[b^{jk}(\xi)]$ is the inverse matrix for $[b_{jk}(\xi)]$, finally m-order of $B_j$, 2m-order of $L = E^p$.

In our case $m = p \cdot k$, $m_j = (j - 1)k$ $(j = 1, 2, \ldots, p)$.

5. Take in problem (1') $E = \Delta$ (the Laplace operator)

$$F = i\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2; \quad p = 3, \quad n = 1.$$

The system of operators in the problem is now the following:

$$\Delta^3,\ F^2,\ F,\ I \quad (I - \text{identity operator}).$$

For this system of operators the complementing conditions do not hold.

Indeed:

$$B_j'(\xi, T) = (i\xi - T)^{2j} \quad j = 2, 1, 0,$$

$$r\left(\frac{B_j'}{(T - i\xi)^3}\right) = \begin{cases} 0 & ; \ j = 2, \\ (i\xi - T)^2; & j = 1, \\ 1 & ; \ j = 0 \end{cases}$$

and these remaiders are linearly dependent.

## References

[1] Agmon S., Douglis A., Nirenberg L., Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, I Comm. on pure and appl. math. Vol. XII, (1959), p.623-727.