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On an asymptotic property of solutions of an invariant curves equation

1. It was proved in [2] a theorem ([2], Th. 3) on an asymptotic property of continuous solutions of a fractional iterates equation. It is the aim of this paper to prove an analogous theorem on continuous solutions of the invariant curves equation

(1) $$\varphi^2(x) = G(x, \varphi(x)) .$$

We shall accept the assumptions which are usually assumed in the theory of continuous solutions of equation (1)(see [1], ch XIV).

(H) The continuous function G is defined on the set

$$\Omega := \{(x,y): x \in [0,b], \beta(x) \leqslant y \leqslant x\},$$

where $\beta : [0,b] \to [0,b]$ is a continuous and strictly increasing function in $[0,b]$ such that

$$G(x,\beta(x)) = \beta(x) \quad \text{for} \quad x \in [0,b]$$

and

$$G(x,y) < y \quad \text{for} \quad x \in (0,b), \ \beta(x) < y < x.$$

Moreover G is strictly increasing with respect to both variables and

$$G(0,0) = 0, \quad G(b,b) = b, \quad G(x,x) < x \quad \text{for} \quad x \in (0,b).$$

2. DEFINITION 1. We say that a continuous function $f\colon [0,b] \to R$ belongs to the class U^r $(r > 0)$ of functions if and only if there exists a continuous function $h\colon [0,b] \to R$, such that

$$f(x) = h(x)\, x^r \quad \text{for} \quad x \in [0,b], \quad h(0) > 0.$$

DEFINITION 2. Let a $\Omega \subset R^2$ be a domain, such that $(0,0) \in \Omega$ is a cluster point of the set Ω, We say that a continuous function $G\colon \Omega \to R$ belongs to the class $U^{v,w}$ $(v,w > 0)$ of functions if and only if there exists a continuous function $A\colon \Omega \to R$, such that

(2) $G(x,y) = A(x,y)\, x^v y^w$ for $(x,y) \in \Omega$, $A(0,0) > 0$.

3. The following lemma (see [2], Lemma 4) will be useful in the sequel

LEMMA 1. Let $y_{n,k}$ be a double sequence. If

$$\lim_{n \to \infty} y_{n,k} = y_k \quad \text{and} \quad \lim_{k \to \infty} y_k = y_0,$$

then there exist sequences $i_n \to \infty$, $j_n \to \infty$ of positive integers, such that

$$\lim_{n \to \infty} y_{i_n, j_n} = y_0.$$

THEOREM. Let a function $G \in U^{v,w}$ $(v,w > 0)$ fulfills hypothesis (H) and let continuous function φ be a solution of equation (1) in $[0,b]$. If there exists a function $f \in U^r$ $(r > 0)$, such that

(3) $$\varphi(x) \geq f(x) \quad \text{for} \quad x \in [0,b],$$

where

(4) $$r^2 = w\, r + v$$

and

(5) $$v > 1 + w,$$

then $\varphi \in U^r$. Moreover, putting

(6) $$\varphi(x) = h(x)\, x^r \quad \text{for} \quad x \in [0,b],$$

we have

(7) $$[h(0)]^{1+r-w} = d \;:= A(0,0)$$

(here $A(x,y)$ is given by (2)).

P r o o f. First let us notice that (4) is a necessary condition for the continuous solution φ to belong to U^r. For if $\varphi \in U^r$ and (6) holds, then we have

$$\varphi^2(x) = h(\varphi(x))\,[\varphi(x)]^r = h(\varphi(x))\,[h(x)]^r\, x^{r^2}.$$

On the other hand

$$\varphi^2(x) = G(x, \varphi(x)) = A(x, \varphi(x))\, x^v [\varphi(x)]^w =$$
$$= A(x, \varphi(x))\,[h(x)]^w\, x^v\, x^{rw}.$$

Hence r must satisfy (4) as functions h and A have positive limits in points $0 \in R$ and $(0,0) \in R^2$, respectively.

Now let the assumptions of theorem be fulfilled and let us put

(8) $$f(x) = q(x)\, x^r \quad (q(0) > 0).$$

Then (3) and (6) imply

(9) $$\liminf_{x \to 0^+} h(x) > 0.$$

Let a sequence $x_n \to 0^+$ as $n \to \infty$. We shall prove (an indirect proof) that the sequence $h(x_n)$ is bounded. If it is not true, then there exists an index sequence k_n, such that $h(x_{k_n}) \to \infty$ as $n \to \infty$. Let a function h be such that condition (6) holds. The conditions (6) and (4) imply

$$\varphi^2(x_{k_n}) = h(\varphi(x_{k_n}))\,[\varphi(x_{k_n})]^r = h(\varphi(x_{k_n}))\,[h(x_{k_n})]^r\, x_{k_n}^{r^2} =$$
$$= h(\varphi(x_{k_n}))\,[h(x_{k_n})]^r\, x_{k_n}^{v+wr}.$$

On the other hand (1), (2) and (6) imply

$$\varphi^2(x_{k_n}) = G(x_{k_n}, \varphi(x_{k_n})) = A(x_{k_n}, \varphi(x_{k_n}))\, x_{k_n}^v\,[\varphi(x_{k_n})]^w =$$
$$= A(x_{k_n}, \varphi(x_{k_n}))\, x_{k_n}^v\,[h(x_{k_n})]^w\, x_{k_n}^{rw} =$$
$$= A(x_{k_n}, \varphi(x_{k_n}))\,[h(x_{k_n})]^w\, x_{k_n}^{v+wr}.$$

Hence

$$h(\varphi(x_{k_n}))\,[h(x_{k_n})]^{r-w} = A(x_{k_n}, \varphi(x_{k_n}))$$

and as ever continuous solution of equation (1) has limit zero at zero (see [1], ch. XIV, § 3) and by virtue of (5) $r - w > 1$, we have

$$\lim_{n\to\infty} h(\varphi(x_{k_n})) = 0,$$

which contradicts (9).

Now we shall prove that there exists limit $\lim_{x\to 0^+} h(x) =: h(0)$ and (7) holds. Let x_{k_n} be a sequence chosen from arbitrary sequence $x_n \to 0^+$. It follows from (9) that there exists a subsequence $x_{k_{l_n}} := z_n$ chosen from the sequence x_{k_n}, such that

(10) $$\lim_{n\to\infty} h(z_n) =: s > 0,$$

and it is possible that the number s depends on the sequence x_n. From (6), (1), (4) and (2) we obtain

$$h(\varphi(x)) = \varphi^2(x)\,[\varphi(x)]^{-r} = G(x, \varphi(x))\,[h(x)]^{-r}\, x^{-r^2} =$$

$$= A(x, \varphi(x))\; x^{v}[h(x)]^{w}\, x^{rw}[h(x)]^{-r}\, x^{-r^2} =$$

$$= A(x, \varphi(x))\,[h(x)]^{w-r},$$

whence

(11) $$\lim_{n\to\infty} h(\varphi(z_n)) = \lim_{n\to\infty} A(z_n, \varphi(z_n))\,[h(z_n)]^{w-r} = d\, s^{w-r}.$$

Let us put $\delta := w - r$, $w_{n,k} := \varphi^k(z_n)$.

We shall prove, by induction, that

(12) $$\lim_{n\to\infty} h(w_{n,k}) = d^{\frac{1-\delta^k}{1-\delta}}\, s^{\delta^k}, \quad k = 1,2,\ldots .$$

It follows from (11) that (12) holds for $k = 1$. Suppose that (12) is fulfilled for a $k \geqslant 1$ and consider

$$h(w_{n,k+1}) = h(\varphi(w_{n,k})) .$$

As $\varphi^k(0) = 0$ for $k = 1,2,\ldots$, then $\lim_{n\to\infty} w_{n,k} = 0$ and, similarly as (11), we obtain from (6), (1) and (2)

$$\lim_{n\to\infty} h(w_{n,k+1}) = \lim_{n\to\infty} h(\varphi(w_{n,k})) =$$

$$= \lim_{n\to\infty} A(w_{n,k}, \varphi(w_{n,k}))\,[h(w_{n,k})]^{w-r} =$$

$$= d^{1+\delta\frac{1-\delta^k}{1-\delta}}\, s^{\delta^{k+1}} = d^{\frac{1-\delta^{k+1}}{1-\delta}}\, s^{\delta^{k+1}}$$

which ends the proof of (12).

Now let us put

(13) $$s = p\, d^{\frac{1}{1-\delta}}$$

where a number p, chosen to s, may depend on x_n.

Then, by virtue of (12), we have

$$(14) \qquad \lim_{n\to\infty} h(w_{n,k}) = d^{\frac{1}{1-\delta}} p^{\delta^k} = d^{\frac{1}{1+r-w}} p^{(w-r)^k}.$$

We shall prove that $p = 1$. If $p > 1$, then taking k odd, $k = 2m + 1$ and denoting

$$y_{n,k} := h(w_{n,k})$$

we have, by virtue of (14)

$$y_k := \lim_{n\to\infty} y_{n,k} = d^{\frac{1}{1+r-w}} p^{(-1)^k (r-w)^k},$$

hence

$$\lim_{m\to\infty} y_k = \lim_{m\to\infty} y_{2m+1} = 0.$$

Now Lemma 1 implies that there exist index sequences $i_n \to \infty$, $j_n \to \infty$, for which

$$\lim_{n\to\infty} y_{i_n, j_n} = \lim_{n\to\infty} h(\varphi^{i_n}(z_{j_n})) = 0,$$

a contradiction with respect to (9).

If $p \in (0,1)$, then taking k even, $k = 2m$ we obtain in a similar way

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} d^{\frac{1}{1+r-w}} p^{(r-w)^k} = 0$$

what also implies a contradiction with respect to (9).

In this way we have proved, that for an arbitrary sequence $x_n \to 0^+$ and an arbitrary subsequence $h(x_{k_n})$ of the sequence $h(x_n)$ it is possible to choose a subsequence $h(z_n) = h(x_{k_{l_n}})$ of the sequence $h(x_{k_n})$ which converges to the limit s (number s does not depend in fact on the sequence x_n). It implies that the limit $\lim_{n \to \infty} h(x_n)$ exists and it is equal to s, which is equivalent to the relation

$$\lim_{x \to 0^+} h(x) = s$$

which, by virtue of (13), implies (7) and the proof is ended.

References

[1] Kuczma M., Functional equations in a single variable, Polish Scientific Publishers, Warszawa 1968.

[2] Turdza E., Comparison theorems for a functional inequality, General Inequalities, Birkhauser Verlag Basel, 1978, vol 1, p.199-211.