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A modification of a construction of Z. Moszner

Let $(\mathcal{G}, \circ)$ denote an arbitrary group and let X be an arbitrary non-empty set. We are going to look for a function $f: X \times \mathcal{G} \to X$ fulfilling the condition

(1) $$f(x, \alpha\beta) = f(f(x,\alpha), \beta)$$

for $x \in X, \alpha, \beta \in \mathcal{G}$. Equation (1) is called the equation of translation. Equation (1) has been solved by Z.Moszner [1]. In the present paper we shall give a modification of that construction describing the set of all solutions of equation (1).

A. Construction of Z. Moszner. Every solution f of equation (1), and only solutions of (1), can be obtained in the following way:

1° Let us take an arbitrary function $g: X \to X$ such that $g \circ g = g$.

2° In the set $g(X)$ let us choose an arbitrary family of subsets $\{X_k\}_{k \in K}$ having the following properties:

a) $X_k \neq \emptyset$ for $k \in K$;

b) $X_{k_1} \cap X_{k_2} = \emptyset$ for $k_1 \neq k_2$;

c) $g(X) = \bigcup_{k \in K} X_k$

and

d) for every $k \in K$ there exists a subgroup $\mathcal{G}_k$ of the group $\mathcal{G}$ such that

(2) $$\operatorname{card} X_k = \operatorname{card}(\mathcal{G}/\mathcal{G}_k),$$

where

$$\mathcal{G}/\mathcal{G}_k = \{\mathcal{G}_k\alpha : \alpha \in \mathcal{G}\} .$$

3° Let us take an arbitrary one-to-one map g_k of the set X_k onto the set $\mathcal{G}/\mathcal{G}_k$ $(k \in K)$.

We define the function f by the following formula

(3) $$f(x,\alpha) := g_k^{-1}[g_k(g(x))\alpha]$$

for $x \in X$, $\alpha \in \mathcal{G}$, where $k \in K$ is such that $g(x) \in X_k$.

B. The modification of the construction of Z. Moszner

Let $\{\mathcal{G}_s\}_{s \in S}$ denote an arbitrary family of subgroups of $\mathcal{G}$. We do not assume that the map $S \ni s \longmapsto \mathcal{G}_s$ is one-to-one, thus the map $S \ni s \longmapsto \mathcal{G}/\mathcal{G}_s$ is also not necessarily one-to-one. We shall introduce the so called indexed quotient structure $(\mathcal{G}/\mathcal{G}_s, s) := \{(\mathcal{G}_s\alpha, s) : \alpha \in \mathcal{G}\}$ for $s \in S$. In this way we shall obtain a one-to-one map $S \ni s \longmapsto (\mathcal{G}/\mathcal{G}_s, s)$.

Further on we introduce a function λ_s ascribing to an arbitrary right-hand coset the same coset provided with the index s. Here is the definition of λ_s:

$$\mathcal{G}/\mathcal{G}_s \ni \mathcal{G}_s\alpha \mapsto \lambda_s(\mathcal{G}_s\alpha) := (\mathcal{G}_s\alpha, s) \in (\mathcal{G}/\mathcal{G}_s, s) \quad (\alpha \in \mathcal{G},\ s \in S).$$

The multiplication of indexed cosets by elements of the group $\mathcal{G}$ is defined in the natural way:

(4) $$(\mathcal{G}_s\alpha, s)\beta := \lambda_s(\mathcal{G}_s\alpha\beta)$$

for

$$\alpha, \beta \in \mathcal{G}, \quad s \in S.$$

THEOREM. The following construction is equivalent to the construction of Z.Moszner.

We choose

1) an arbitrary family $\{\mathcal{G}_s\}_{s \in S}$ of subgroups of $\mathcal{G}$ such that

$$\operatorname{card} \bigcup_{s \in S} (\mathcal{G}/\mathcal{G}_s, s) \leq \operatorname{card} X;$$

2) an arbitrary one-to-one map $\varphi: \bigcup_{s \in S} (\mathcal{G}/\mathcal{G}_s, s) \to X$

and

3) an arbitrary function $g: X \to X$ such that $g \circ g = g$

and

$$\varphi\Big(\bigcup_{s \in S} (\mathcal{G}/\mathcal{G}_s, s)\Big) = g(X).$$

We define the function f by the formula

(5) $$f(x, \alpha) = \varphi[\varphi^{-1}(g(x))\alpha]$$

for $x \in X$, $\alpha \in \mathcal{G}$.

P r o o f. 1. Let a solution f of equation (1) be obtained by the construction of Z.Moszner. Then there exists: a function $g: X \to X$, a decomposition $\{X_k\}_{k \in K}$ of the set $g(X)$, a family $\{\mathcal{G}_k\}_{k \in K}$ of subgroups of the group $\mathcal{G}$ and a family $\{g_k\}_{k\in K}$ of bijections fulfilling conditions 1^o, 2^o and 3^o of that construction, whereas the function f is defined by formula (3). We are going to prove that the function f can be obtained from the construction presented in the theorem that we are proving now.

To this end let us put $S = K$ and let us take the same function g and the same family of subgroups $\{\mathcal{G}_s\}_{s\in S}$ as in the construction of Z.Moszner. We define the function $\varphi : \bigcup_{s\in S} (\mathcal{G}/\mathcal{G}_s, s) \to X$ by the formula $\varphi = \bigcup_{s\in S} (g_s^{-1} \circ \lambda_s^{-1})$. Hence and from the definitions of the maps λ_s and g_s it follows that the map φ is an injection and $\varphi\left[\bigcup_{s\in S} (\mathcal{G} / \mathcal{G}_s, s)\right] = g(X) \subset X$. Therefore card $\bigcup_{s\in S} (\mathcal{G}/\mathcal{G}_s, s) \leqslant$ $\leqslant$ card X, thus conditions 1), 2) and 3) are fulfilled. To prove (5) let us choose an arbitrary $x \in X$ and an arbitrary $\alpha \in \mathcal{G}$. It follows from 1^o and 2^o that there exists exactly one $s \in S$ such that $g(x) \in X_s$. Hence and from the definition of the functions g_s we obtain the existence of $\beta \in \mathcal{G}$ such that $g_s(g(x)) = \mathcal{G}_s\beta$, thus $[g_s(g(x))]\alpha = \mathcal{G}_s\beta\alpha$, whence $g_s(f(x,\alpha)) = \mathcal{G}_s\beta\alpha$ by virtue of (3). It follows from the last equality, the definition of the functions λ_s and (4) that

$$(\lambda_s \circ g_s)(f(x,\alpha)) = \lambda_s(\mathcal{G}_s\beta\alpha) = [\lambda_s(\mathcal{G}_s\beta)]\alpha = [(\lambda_s \circ g_s)(g(x))]\alpha ,$$

therefore

$$f(x,\alpha) = (g_s^{-1} \circ \lambda_s^{-1}) \left([(\lambda_s \circ g_s)(g(x))\alpha]\right)$$

whence we obtain (5) by virtue of definition of φ.

2. Let us assume that f is defined by formula (5). We are going to prove that the function f can be obtained from the construction of Z.Moszner. To this end let us assume that $K = S$ and let us choose the same function g and the same family of subgroups $\{\mathcal{G}_k\}_{k\in K}$ as in the construction occurring in the theorem that we are proving now. Putting $X_k = \varphi((\mathcal{G}/\mathcal{G}_k,k))$ and $g_k = \lambda_k^{-1}\circ(\varphi\,|\,\mathcal{G}/\mathcal{G}_k,k))^{-1}$ for $k\in K$ we can easily check that conditions 1°, 2° and 3° of the construction of Z.Moszner and formula (3) are fulfilled.

References

[1] Moszner Z., Structure de l'automate plein, réduit et inversible, Aequationes Math., 9(1973), p.46-59.