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A remark on the stability of the Cauchy equation

Following the famous problem of S. Ulam regarding the stability of the Cauchy equation (cf. [4]) many authors were and are interested in deducing suitable properties of a given function f from properties of its Cauchy difference

$$f(x+y) - f(x) - f(y)$$

(as a function of two variables). While solving a functional equation, R. Ger [2] came to the following algebraic case of such a stability problem. What can be said about the function $f: \mathbb{R} \to \mathbb{R}$ for which its Cauchy difference takes values in a subgroup Z of the additive group of all real numbers $\mathbb{R}$. He conjectured that f must be of the form $g+h$, where g is an additive function and h takes values in the subgroup Z only. It is so in the case where Z is a linear space over the rationals. In general it is not true. Indeed, K. Nikodem observed that an example of G. Godini ([3, Example 2]) shows it is not true in the case where Z is the group of all integers. In this example the function considered is not measurable. It turns out that in any such type counter-example a suitable function cannot be measurable as we have the following theorem.

THEOREM. If $f: \mathbb{R} \to \mathbb{R}$ then the Cauchy difference of the function f is Lebesgue measurable and takes integer values only iff there exists an additive function $g: \mathbb{R} \to \mathbb{R}$ such that f-g is Lebesgue measurable and takes integer values only.

P r o o f. Suppose that the Cauchy difference of the function f is Lebesgue measurable and takes integer values only. Taking into account the measurability of this Cauchy difference and making use of a theorem of M. Laczkovich ([5, Theorem 5]) we can represent the function f as a sum of an additive function $a: \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $m: \mathbb{R} \to \mathbb{R}$:

$$f = a + m.$$

Of course Cauchy differences of the functions f and m are equal.

Now we are going to show that if $U \subset \mathbb{R}$ is an open set then so is also the set

$$\{x \in \mathbb{R}: (m(x)+\mathbb{Z}) \cap U \neq \emptyset\}.$$

We will present two proofs of this fact. The second proof, supposed to be more direct, will be given at the end of the paper.

Let us define the function $M: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ by putting

$$M(x) = m(x) + \mathbb{Z}.$$

In other words

$$M = \pi \circ m,$$

where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical map:

$$\pi(x) = x + \mathbb{Z}.$$

Then

$$M^{-1}(\mathcal{U}) = m^{-1}(\pi^{-1}(\mathcal{U})) \qquad \forall \mathcal{U} \subset \mathbb{R}/\mathbb{Z}.$$

Hence and from the continuity of the function π and measurability of the function m it follows that the function M is measurable (i.e., the counter-image under the function M of every open set in the space $\mathbb{R}/\mathbb{Z}$ is Lebesgue measurable). Moreover, since the Cauchy difference of the function m takes integer values only, M is an additive function. Applying a theorem of P. Fischer and Z. Słodkowski ([1, Theorem 1]; in fact this theorem is stated for additive functions mapping an Abelian Polish group into an Abelian Polish group but the proof given there works for additive functions mapping an Abelian Polish group into a Lindelöf topological group) we infer that M is a continuous function.

Now let us consider M as a multi-valued function which transforms a real number x into the set $m(x)+\mathbb{Z}$. Since

$$\{x\in\mathbb{R}: M(x)\cap U \neq \emptyset\} = \{x\in\mathbb{R}: (m(x)+\mathbb{Z})\cap U\neq\emptyset\} =$$

$$= m^{-1}(U+\mathbb{Z}) = m^{-1}(\pi^{-1}(\pi(U))) \ \forall\, U\subset\mathbb{R}$$

we see that for every open set $U\subset\mathbb{R}$ the set

$$\{x\in\mathbb{R}: M(x)\cap U\neq\emptyset\}$$

is also an open set. In other words, the function M considered as a multi-valued function is lower semicontinuous. Consequently, as it results from a theorem of G. Godini ([3, Theorem 2]), there exists an additive and continuous function $\mu:\mathbb{R}\to\mathbb{R}$ such that

$$M(x) = \mu(x)+\mathbb{Z} \quad \forall\, x\in\mathbb{R}.$$

Putting

$$g = a + \mu$$

we see that g is an additive function and

$$f(x) - g(x) = m(x)-\mu(x)\in\mathbb{Z} \quad \forall\, x\in\mathbb{R}.$$

It is clear that if there exists an additive function $g:\mathbb{R}\to\mathbb{R}$ such that f-g is Lebesgue measurable and takes integer values only then the Cauchy difference of the function f is Lebesgue measurable and takes integer values only.

The theorem is proved.

Now we will present the above promised second proof of this crucial fact that the set

$$\{x\in\mathbb{R}: (m(x)+\mathbb{Z})\cap U\neq\emptyset\}$$

is open provided so is the set $U\subset\mathbb{R}$. At first let us assume that $U\subset\mathbb{R}$ is an open neighbourhood of zero and choose an open neighbourhood W of zero such that

$$W - W \subset U.$$

Since $\bigcup_{k=1}^{\infty} m^{-1}(kW)=\mathbb{R}$ and the function m is Lebesgue measurable, there exists a positive integer k such that the set $m^{-1}(kW)$ has a positive Lebesgue measure. Hence and from the fact that

$$m^{-1}(kW)\subset km^{-1}(W+\tfrac{1}{k}\mathbb{Z}) = \bigcup_{j\in Z} km^{-1}(W+\tfrac{j}{k})$$

we infer that there exists a (rational) number r such that the set $m^{-1}(W+r)$ has a positive Lebesgue measure. Applying now Steinhaus theorem ([7, Théorème VIII]; cf. also [6, Theorem 4.8]) we get that

$$0\in \mathrm{Int}(m^{-1}(W+r) - m^{-1}(W+r)).$$

But

$$m^{-1}(W+r) - m^{-1}(W+r) \subset m^{-1}((W-W)+\mathbb{Z}) \subset m^{-1}(U+\mathbb{Z})=$$
$$=\{x\in \mathbb{R}: (m(x)+\mathbb{Z})\cap U\neq\emptyset\}$$

and so

$$0\in \mathrm{Int}\{x\in \mathbb{R}: (m(x)+\mathbb{Z})\cap U\neq\emptyset\}$$

(provided U is an open neighbourhood of zero).

Suppose now that $U\subset\mathbb{R}$ is an open set, let x_0 be a real number such that $(m(x_0)+\mathbb{Z})\cap U\neq\emptyset$ and fix a $u\in(m(x_0)+\mathbb{Z})\cap U$. It follows from the previous part of this proof that

$$0\in \mathrm{Int}\{x\in \mathbb{R}: (m(x)+\mathbb{Z})\cap (U-u)\neq\emptyset\}.$$

Moreover,

$$m(x+x_0)+\mathbb{Z} = m(x)+m(x_0)+\mathbb{Z} = (m(x)+u)+(m(x_0)-u+\mathbb{Z})=$$
$$= m(x)+u+\mathbb{Z} \qquad \forall\ x\in\mathbb{R}$$

which shows that

$$(m(x+x_0)+\mathbb{Z})\cap U = [(m(x)+\mathbb{Z})\cap(U-u)]+u \quad \forall x\in\mathbb{R}.$$

Consequently

$$x_0\in x_0+\mathrm{Int}\{x\in\mathbb{R}:\ (m(x)+\mathbb{Z})\cap(U-u)\neq\emptyset\} =$$

$$= x_0+\mathrm{Int}\{x\in\mathbb{R}:\ (m(x+x_0)+\mathbb{Z})\cap U\neq\emptyset\} =$$

$$= \mathrm{Int}(x_0+\{x\in\mathbb{R}:\ (m(x+x_0)+\mathbb{Z})\cap U\neq\emptyset\}) =$$

$$= \mathrm{Int}\{x\in\mathbb{R}:\ (m(x)+\mathbb{Z})\cap U\neq\emptyset\}$$

which we wanted to show.

References

[1] Fischer P. and Slodkowski Z., Christensen zero sets and measurable convex functions, Proc. Amer. Math. Soc. 79 (1980), 449-453.

[2] Ger R., On a functional equation related to the theory of jets, in preparation.

[3] Godini G., Set-valued Cauchy functional equation, Rev. Roumaine Math. Pures Appl. 20 (1975), 1113-1121.

[4] Hyers D.H., On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941), 222-224.

[5] Laczkovich M., Functions with measurable differences, Acta Math. Acad. Sci. Hungar. 35 (1980), 217-235.

[6] Oxtoby J.C., Measure and Category, Springer-Verlag, New York - Heidelberg - Berlin, 1971.

[7] Steinhaus H., Sur les distances des points dans les ensembles de mesure positive, Fund. Math. 1 (1920), 93-104.