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On the stability of some class of functional equations

There are various definitions of the stability of functional equations. In papers [7] and [8] the authors consider the properties of these definitions and the problems of the stability of some classes of equations. In this paper we generalize those results to a possibly large class of equations.

I. Consider the equation:

$$(1)\quad \begin{aligned} &F(x,y,\varphi(x),\varphi(y),\varphi(k(x,y)),\varphi(l(x,y))) = \\ &= K(x,y,\varphi(x),\varphi(y),\varphi(k(x,y)),\varphi(l(x,y))) , \end{aligned}$$

where $\varphi : E_1 \to V_1$ is an unknown function, $k,l: E_1 \times E_1 \to E_1$, $F,K: E_1 \times E_1 \times V_1 \times V_1 \times V_1 \times V_1 \to V_1$ are given functions and E_1, V_1 are arbitrary sets.

The following definition of the stability of equation (1) is patterned after that given by D.H. Hyers (cf. [6]).

DEFINITION 1. Let ϱ_1 be a metric in V_1. Equation (1) is said to be stable iff for every positive ε there exists a positive δ such that for all functions $\phi: E_1 \to V_1$, if

$$\varrho_1(F(x,y,\phi(x),\phi(y),\phi k(x,y),\varphi l(x,y)), K(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y))) < \delta$$

for all $x,y \in E_1$,

then there exists a solution $\varphi: E_1 \to V_1$ of (1) such that

$$\varrho_1(\varphi(x),\phi(x)) < \varepsilon$$

for all $x \in E_1$.

In this definition nothing is assumed about the metric ϱ_1. In the following example we can see that this fact may bring about an unexpected situation:

E x a m p l e. Let there be a metric $\varrho(a,b) = |e^a - e^b|$ in the set R of real numbers. Consider the equations:

(2) $$f(x) + f(y) - f(xy) = 0,$$

(3) $$f(x) + f(y) = f(xy),$$

where $f: R \to R$. The function $f(x) \equiv 0$ is the unique solution of these equations.

Equation (3) is not stable in the sense of definition 1, because if we take an ε, $0 < \varepsilon < 1$ and a $\delta > 0$, and then choose an n such that $|\frac{1}{n} - 1| \geqslant \varepsilon$ and $|\frac{1}{n}(\frac{1}{n} - 1)| < \delta$, then the function

$$g(x) = \begin{cases} 0 & \text{for} \quad x \neq 0, \\ \ln \frac{1}{n} & \text{for} \quad x = 0, \end{cases}$$

satisfies condition $\varrho(g(x)+g(y),g(xy))<\delta$, but $\varrho(g(0),0)\geqslant\varepsilon$.

Equation (2), on the other hand, is stable in this sense, since putting $y = 0$ in the inequality $\varrho(g(x)+g(y)-g(xy),0)<\delta$, we have $\varrho(g(x),0)<\delta$, the stability follows with $\delta = \varepsilon$.

It follows from this example that the stability of equation (1) cannot be reduced to the stability of an equation of the form:

$$F(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y)) = a, \tag{4}$$

where a is an arbitrary element from V_1. Even if we assume that V_1 is a group with the unit e_1 and transform equation (1) into the equivalent equation:

$$\begin{aligned} &F(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y)) \cdot \\ &\cdot [K(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y))]^{-1} = e_1 \end{aligned} \tag{5}$$

of type (4), then it may happen that one of equations (1) and (5) is stable in the sense of definition 1 while the other is not.

It is interesting to find what conditions on the metric ϱ are sufficient for the situation from example 1 not to happen. The following lemma says about it:

LEMMA 1. If (V_1,ϱ_1) is a metric space, $(V_1,\cdot)$ is a group with the unit e_1, and the following condition is fulfilled:

$$(6) \qquad \bigwedge_{\varepsilon>0} \ \bigvee_{\delta>0} \ \bigwedge_{x,y\in V_1} \Big\{ \varrho_1(x,y)<\delta \Longrightarrow \bigwedge_{z\in V_1} \varrho_1(x\cdot z, y\cdot z)<\varepsilon \Big\},$$

then equation (1) is stable if and only if equation (5) is stable.

P r o o f. Let (1) be stable. Given an $\varepsilon>0$ we take δ_1 according to the stability of (1) and to δ_1 we choose δ according to condition (6). If

$$\varrho_1(F(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y))\cdot[K(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y))]^{-1},e_1)<\delta$$

for all $x,y\in E_1$, then in view of (6)

$$\varrho_1(F(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y)),K(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y)))<\delta_1$$

therefore there exists a solution φ of (1) such that $\varrho_1(\phi(x),\varphi(x))<\varepsilon$ for all $x\in E_1$. This proves the sufficiency because φ is also a solution of equation (5). The proof in the other direction is analogous.

II. The topological definition of stability proposed in [7] does not present those difficulties:

DEFINITION 2. Let V_1 be a topological space and $(V_1,\cdot)$ a group with the unit e_1. Equation (1) is said to be stable iff for every neighbourhood Δ_1 of e_1 there exists a neighbourhood Ω_1 of e_1 such that for all functions $\phi: E_1\to V_1$, if

$$F(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y))[K(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y))]^{-1}\in\Omega_1$$

for all $x,y\in E_1$,

then there exists a solution $\varphi: E_1 \to V_1$ of (1) such that

$$\Phi(x)[\varphi(x)]^{-1} \in \Delta_1 \quad \text{for all} \quad x \in E_1.$$

It is obvious that in general definitions 1 and 2 cannot be equivalent. In [7] it is shown that even if the topological space V_1 is metrizable by a metric ϱ_1 then definitions 1 and 2 are not necessarily equivalent. It is also proved there that in the case of Cauchy type equation $\varphi k(x,y) = K(\varphi(x), \varphi(y))$ these definitions are equivalent if the metric ϱ_1 satisfies condition (6). An analogous argument shows that this result is valid also for equations of type (1).

In connection with condition (6), we shall prove the following lemma. (The method of the proof follows the pattern in [1], pp.299-301).

LEMMA 2. If a topological group $(G,\cdot)$ is metrizable by the metric $\bar{\varrho}$, then there exists a metric ϱ equivalent to the metric $\bar{\varrho}$ and such that $\varrho(x,y) = \varrho(x\cdot z, y\cdot z)$ for all $x,y,z \in G$, and thus fulfilling (6) with $\delta = \varepsilon$.

P r o o f. If G is metrizable, then it has a countable basis of neighbourhoods of the unit e such that its intersection is the set $\{e\}$ (for example, we may take the family the balls $\{x \in G: \bar{\varrho}(x,e) < \frac{1}{n}\}$). Take a set $B \subset G$. Let $B^{-1} = \{x \in G: x^{-1} \in B\}$ and let $B^* = B \cap B^{-1}$. Note that if $x \in B^*$, then $x^{-1} \in B^*$, and that if $\beta = \{B_n: n \in N\}$ is a neighbourhood basis of the unit, then $\beta^* = \{B_n^*: B_n \in \beta\}$ is a neighbourhood basis of the unit, too. Thus we can assume that in G there exists a countable basis $\{B_n : n \in N\}$ of neighbourhoods of the unit such that for each set B from this basis, if $x \in B$ then $x^{-1} \in B$ and, moreover, putting $B_o = G$, we may assume that the condition

(7) $\quad B_n \cdot B_n \cdot B_n \subset B_{n-1} \quad$ for $\quad n = 1,2,3,\ldots$

is fulfilled. Hence we have also $B_n \subset B_{n-1}$. Now put

$$f(x) = \begin{cases} 2^{-n} & \text{for } \ x \in B_{n-1} \setminus B_n, \quad n = 1,2,3,\ldots, \\ 0 & \text{for } \ x = e, \end{cases}$$

and

$p(x) = \inf\{f(x_1) + \ldots + f(x_n) : x_1 \cdot \ldots \cdot x_n = x,\ n \in N\}, \quad x \in G.$

We shall prove that the function

$$\varrho(x,y) = p(xy^{-1})$$

is the metric which we are looking for.

First, we shall prove by induction the inequality:

$$f(x_1 \cdot \ldots \cdot x_n) \leqslant 2(f(x_1) + \ldots + f(x_n)) .$$

It is evident if $n = 1$. Assume it valid for an $n \geqslant 1$. Since for all $x \in G$ $f(x) \leqslant \frac{1}{2}$, we need consider only the case, where $s = f(x_1) + \ldots + f(x_{n+1}) < \frac{1}{4}$. If $f(x_1) \leqslant \frac{s}{2}$, then there exists a k, $1 \leqslant k \leqslant n$, such that

$$f(x_1) + \ldots + f(x_k) \leqslant \frac{s}{2},\ f(x_1) + \ldots + f(x_{k+1}) \geqslant \frac{s}{2},$$

$$f(x_{k+2}) + \ldots + f(x_{n+1}) \leqslant \frac{s}{2}.$$

Hence $f(x_1 \cdot \ldots \cdot x_k) \leqslant 2(f(x_1) + \ldots + f(x_k)) \leqslant 2 \cdot \frac{s}{2} = s$, $f(x_{k+1}) \leqslant s$, and $f(x_{k+2} \cdot \ldots \cdot x_{n+1}) \leqslant 2(f(x_{k+2}) + \ldots + f(x_{n+1})) \leqslant$ $\leqslant 2 \cdot \frac{s}{2} = s$. Take m such that $2^{-m} \leqslant s < 2^{-m+1}$. Then $f(x_1 \cdot \ldots \cdot x_k)$, $f(x_{k+1})$, $f(x_{k+2} \cdot \ldots \cdot x_{n+1}) < 2^{-m+1}$, whence by the definition of f we get $x_1\cdot\ldots x_k, x_{k+1}, x_{k+2}\cdot\ldots\cdot x_{n+1}\in B_{m-1}$. In view of (7) $x_1\cdot\ldots\cdot x_{n+1}\in B_{m-2}$ and hence $f(x_1\cdot\ldots\cdot x_{n+1})\leqslant 2^{-m+1}\leqslant 2s$. If, on the other hand, $f(x_1)>\frac{s}{2}$, then $f(x_2)+\ldots+f(x_{n+1})\leqslant\frac{s}{2}$ and $f(x_2\cdot\ldots\cdot x_{n+1})\leqslant 2(f(x_2)+\ldots+f(x_{n+1}))\leqslant 2\cdot\frac{s}{2}=s$. Since, of course, also $f(x_1)\leqslant s$, by a similar argument we get again $f(x_1\cdot\ldots\cdot x_{n+1})\leqslant 2s$.

In virtue of the inequality just proved we have

$$p(x)\leqslant f(x)\leqslant 2\,p(x).$$

From this inequality it results that $p(x)=0$ iff $f(x)=0$. Thus we have $\rho(x,y)=0\Longleftrightarrow p(x\cdot y^{-1})=0\Longleftrightarrow$ $\Longleftrightarrow f(x\cdot y^{-1})=0\Longleftrightarrow x\cdot y^{-1}=e\Longleftrightarrow x=y$. From the properties of the elements of the basis we get $f(x)=f(x^{-1})$, therefore $p(x)=p(x^{-1})$, and hence

$$\rho(x,y)=p(x\cdot y^{-1})=p((x\cdot y^{-1})^{-1})=p(y\cdot x^{-1})=\rho(y,x).$$

Since, moreover, $p(x\cdot y)\leqslant p(x)+p(y)$, we have

$$\rho(x,y)=p(x,y^{-1})=p((xz^{-1})(zy^{-1}))\leqslant p(xz^{-1})+$$
$$+p(zy^{-1})=\rho(x,z)+\rho(z,y).$$

In this way we have proved that the function ρ is really a metric. In virtue of the inequalities $p(x)\leqslant f(x)\leqslant 2p(x)$ we have moreover

$$B_n=\left\{x:\ f(x)\leqslant\frac{1}{2^{n+1}}\right\}\subset\left\{x:\ p(x)\leqslant\frac{1}{2^{n+1}}\right\}=\left\{x:\ \rho(e,x)\leqslant\frac{1}{2^{n+1}}\right\}$$
$$=\left\{x:\ 2\,p(x)\leqslant\frac{1}{2^n}\right\}\subset\left\{x:\ f(x)\leqslant\frac{1}{2^n}\right\}=B_{n-1},$$

which proves that this metric induces the original topology in G.

We have also

$$\rho(x\cdot z, y\cdot z) = p(x\cdot z\cdot(y\cdot z)^{-1}) = p(x\cdot y^{-1}) = \rho(x,y),$$

which completes the proof.

It arises from the lemma proved that if V_1 is a metrizable topological group it can be metrized in such a way that the stability of an equation of type (1) in the sense of definition 2 is equivalent to the stability of this equation in the sense of definition 1 with respect to this metric.

It should be observed that in general it is impossible to replace the equivalence of metrics ρ and $\bar{\rho}$ in lemma 2 by the uniformly equivalence of these metrics (see [5] p.321) i.e. by the following two conditions

$$\bigwedge_{\varepsilon>0} \bigvee_{\delta_1>0} \bigwedge_{x,y\in V_1} [\bar{\rho}(x,y) < \delta_1 \Rightarrow \rho(x,y) < \varepsilon],$$

$$\bigwedge_{\varepsilon>0} \bigvee_{\delta_2>0} \bigwedge_{x,y\in V_1} [\rho(x,y < \delta_2 \Rightarrow \bar{\rho}(x,y) < \varepsilon].$$

It is known (cf. [7]) that if the metric in definition 1 is changed into a uniformly equivalent one, then stabilities of functional equation (1) in the sense of this definition with respect to both these metrics are equivalent. (In [7] it is proved for equation of the type $\varphi(k(x,y)) = K(\varphi(x),\varphi(y))$, but the proof is exactly the same for equation of type (1)). If in G we would construct a metric ρ uniformly equivalent to $\bar{\rho}$ and fulfilling the condition $\rho(x,y) = \rho(x\cdot z, y\cdot z)$ for $x,y,z \in G$ then the stability of equation (1) in the sense of definition 1 with respect to the metric $\bar{\rho}$ would be equivalent to the stability of this equation in the sense of definition 1 with respect to the metric ρ , and hence also, to the stability in the sense of definition 2 (with respect to the topology generated by either of these metrics) . On the ground of lemma 1 the stability of equation (1) in the sense of definition 1 with respect to the metric ρ is equivalent to the stability of this equation in the sense of definition 2 with the topology induced by ρ . Thus the stability of equation (1) in the sense of definition 1 with respect to the metric $\bar{\rho}$ would be equivalent to the stability of this equation in the sense of definition 2 with respect to the topology induced by the metric $\bar{\rho}$. In general this cannot be achived because the metric $\bar{\rho}$ is arbitrary, and thus definitions 1 and 2 are not equivalent, as we know also from paper [7].

III. Consider now another equation of type (1):

$$G(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b)) = \\ = L(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b)), \tag{8}$$

where $\psi : E_2 \to V_2$ is an unknown function; $g,h: E_2 \times E_2 \to E_2$, $G,L: E_2 \times E_2 \times V_2 \times V_2 \times V_2 \times V_2 \to V_2$ are given functions, and E_2, V_2 are arbitrary sets.

Before we formulate the main theorems of this paper we prove the following:

LEMMA 3. Let $\alpha : E_1 \to E_2$ be a bijection, let $\beta_1 : V_1 \to V_2$ be a function, and let the following conditions be fulfilled:

(9) $$\alpha k(x,y) = g(\alpha(x),\alpha(y)),$$

(10) $$\alpha l(x,y) = h(\alpha(x),\alpha(y)),$$

(11) $$\beta_1 F(x,y,z,u,w,v) = G(\alpha(x),\alpha(y),\beta_1(z),\beta_1(u),\beta_1(w),\beta_1(v)),$$

(12) $$\beta_1 K(x,y,z,u,w,v) = L(\alpha(x),\alpha(y),\beta_1(z),\beta_1(u),\beta_1(w),\beta_1(v)),$$

for all $x,y \in E_1$ and $z,u,w,v \in V_1$.

If a function φ satisfies equation (1), then the function $\psi = \beta_1 \varphi \alpha^{-1}$ satisfies equation (8).

P r o o f. Put $\alpha(x) = a$ and $\alpha(y) = b$. We have

$$G(a,b,\beta_1\varphi\alpha^{-1}(a),\beta_1\varphi\alpha^{-1}(b),\beta_1\varphi\alpha^{-1}g(a,b),\beta_1\varphi\alpha^{-1}h(a,b)) =$$
$$\beta_1 F(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y)) =$$
$$\beta_1 K(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y)) =$$
$$L(a,b,\beta_1\varphi\alpha^{-1}(a),\beta_1\varphi\alpha^{-1}(b),\beta_1\varphi\alpha^{-1}g(a,b),\beta_1\varphi\alpha^{-1}h(a,b)).$$

This proves the lemma.

THEOREM 1. Let (V_1, ϱ_1) and (V_2, ϱ_2) be metric spaces. Let, moreover, $\alpha : E_1 \to E_2$ be a bijection and let $\beta_1 : V_1 \to V_2$, $\beta_2 : V_2 \to V_1$ be uniformly continuous functions. Assume that the conditions (9) - (12) and:

(13) $$\beta_2 G(a,b,c,d,e,f) = F(\alpha^{-1}(a),\alpha^{-1}(b),\beta_2(c),\beta_2(d),\beta_2(e),\beta_2(f)),$$

(14) $\beta_2 L(a,b,c,d,e,f) =$

$$= K(\alpha^{-1}(a),\alpha^{-1}(b),\beta_2(c),\beta_2(d),\beta_2(e),\beta_2(f)),$$

for all $a,b \in E_2$ and $c,d,e,f \in V_2$,

(15) $$\beta_1\beta_2 = id_{V_2},$$

are fulfilled.

If equation (1) is stable in the sense of definition 1, then equation (8) is stable in the sense of the same definition, too.

P r o o f. Fix $\varepsilon > 0$. Choose, step by step: ε_1 to ε by the uniform continuity of β_1, δ_1 to ε_1 by the stability of equation (1), δ to δ_1 by the uniform continuity of β_2.

Assume that

$$\varrho_2(G(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b)),$$
$$L(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b))) < \delta$$

for all $a,b \in E_2$.

By the uniform continuity of β_2 we have

$$\varrho_1(\beta_2 G(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b)),$$
$$\beta_2 L(a,b,\psi(a),\psi(b),\psi g(a,b),\psi h(a,b))) < \delta_1.$$

Putting $\alpha(x) = a$, $\alpha(y) = b$, applying (9), (10), (13), (14) and $\alpha^{-1}\alpha = id_{E_1}$ we get

$$\varrho_1(F(x,y,\beta_2\psi\alpha(x),\beta_2\psi\alpha(y),\beta_2\psi\alpha k(x,y),\beta_2\psi\alpha l(x,y)),$$
$$K(x,y,\beta_2\psi\alpha(x),\beta_2\psi\alpha(y),\beta_2\psi\alpha k(x,y),\beta_2\psi\alpha l(x,y))) < \delta_1.$$

Put now $\phi = \beta_2\psi\alpha$. In view of the stability of equation (1) there exists a function $\varphi : E_1 \to V_1$ satisfying equation (2) and such that

$$\varrho_1(\varphi(x),\phi(x)) < \varepsilon_1 \quad \text{for all} \quad x \in E_1.$$

By the uniform continuity of β_1 we have

$$\varrho_2(\beta_1\varphi(x),\beta_1\phi(x)) < \varepsilon$$

i.e.

$$\varrho_2(\beta_1\varphi\alpha^{-1}(a),\beta_1\phi\alpha^{-1}(a)) < \varepsilon$$

and finally in view of $\alpha\alpha^{-1} = id_{E_2}$

$$\varrho_2(\beta_1\varphi\alpha^{-1}(a),\psi(a)) < \varepsilon \ .$$

By lemma 3 the function $\beta_1\varphi\alpha^{-1}$ satisfies equation (8) which completes the proof.

R e m a r k. In the above proof the assumption that α is a bijection was essential only when transforming the first two variables of the functions G and L. Besides it is sufficient to know that $\alpha\alpha^{-1} = id_{E_2}$. Therefore in the case where the functions F and K do not depend on the first two variables (then the functions G and L satisfying (11) - (14) do not depend on the first two variables, either) it is sufficient to assume that there exist functions $\alpha_1 : E_1 \to E_2$ and $\alpha_2 : E_2 \to E_1$ such that

(16) $$\alpha_1\alpha_2 = id_{E_2},$$

conditions (9) - (14) are fulfilled with the function α_1 instead of α and function α_2 instead of α^{-1}, and

$$\alpha_2 g(a,b) = k(\alpha_2(a),\alpha_2(b)) ,$$

$$\alpha_2 h(a,b) = l(\alpha_2(a),\alpha_2(b)) .$$

In the case of the Cauchy equation the functions F and K do not depend on the first two variables, so in paper [8], where a theorem analogous to theorem 1 is proved for the Cauchy equation, only condition 16 is assumed, not the bijectivity of α .

For convenience, we shall use the following short notation:

$$F(x,y,\varphi(x),\dots) = F(x,y,\varphi(x),\varphi(y),\varphi k(x,y),\varphi l(x,y))$$

(and analogically for functions K, G, L).

An analogue of theorem 1 for the topological definition of the stability can be formulated as follows:

THEOREM 2. Let V_1 and V_2 be topological spaces and $(V_1,\cdot)$, $(V_2,\odot)$ groups with units e_1 and e_2, respectively. Moreover, let continuous homomorphisms β_1, β_2 and a bijection α satisfy conditions (9) - (15).

If equation (1) is stable in the sense of definition 2, then equation (8) is stable in the sense of the same definition, too.

P r o o f. Let Δ_2 be a neighbourhood of e_2. As in the preceding theorem, choose: a neighbourhood Δ_1 of e_1 such that $\beta_1(\Delta_1) \subset \Delta_2$ by the continuity of β_1, Ω_1 to Δ_1 by the stability of equation (1), Ω_2 such that $\beta_2(\Omega_2) \subset \Omega_1$ by the continuity of β_2 at e_2. Let

$$G(a,b,\psi(a),\dots) \odot [L(a,b,\psi(a),\dots)]^{-1} \in \Omega_2.$$

β_2 is a homomorphism, so

$$\beta_2 G(a,b,\psi(a),\dots) \cdot [\beta_2 L(a,b,\psi(a),\dots)]^{-1} \in \beta_2(\Omega_2) \in \Omega_1.$$

In virtue of (13) and (14), putting $a = \alpha(x)$, $b = \alpha(y)$, we have

$$F(x,y,\beta_2\psi\alpha(x),\ldots)\cdot[K(x,y,\beta_2\psi\alpha(x),\ldots)]^{-1}\in\Omega_1.$$

From the stability of (1), putting like in the proof of theorem 1 $\phi=\beta_2\psi\alpha$, we get

$$\phi(x)\cdot[\varphi(x)]^{-1}\in\Delta_1,$$

whence by the choice of Δ_1

$$\beta_1\phi(x)\odot[\beta_1\varphi(x)]^{-1}\in\beta_1(\Delta_1)\subset\Delta_2.$$

Since $x=\alpha^{-1}(a)$, we obtain

$$\beta_1\phi\alpha^{-1}(a)\odot[\beta_1\varphi\alpha^{-1}(a)]^{-1}\in\Delta_2$$

and this, together with lemma 3, completes the proof.

IV. Throughout the rest of this paper we assume that (V_1,ϱ_1) and (V_2,ϱ_2) are metric spaces.

The following definition of the stability is also considered (cf. [6], [8]):

DEFINITION 3. Equation (1) is said to be stable iff there exists a positive η such that for every positive ε and for all functions $\phi: E_1\to V_1$, if

$$\varrho_1(F(x,y,\phi(x),\ldots),K(x,y,\phi(x),\ldots))<\varepsilon \quad \text{for all } x,y\in E_1$$

then there exists a solution $\varphi: E_1\to V_1$ of (1) such that

$$\varrho_1(\varphi(x),\phi(x))<\eta\varepsilon \quad \text{for all } x\in E_1.$$

The following theorem refers with this definition:

THEOREM 3. Let $\alpha: E_1\to E_2$ be a bijection and let functions $\beta_1: V_1\to V_2$, $\beta_2: V_2\to V_1$ satisfy a Lipschitz condition. Let, moreover, conditions (8)–(15) be fulfilled.

If equation (1) is stable then equation (8) is stable, too.

P r o o f. Let equation (1) be stable. Assume that

$$\varrho_2(G(a,b,\psi(a),\ldots),L(a,b,\psi(a),\ldots)) < \varepsilon .$$

Since β_2 satisfies a Lipschitz condition there exists an η_2 such that

$$\varrho_1(\beta_2 G(a,b,\psi(a),\ldots),\beta_2 L(a,b,\psi(a),\ldots)) \leqslant$$
$$\leqslant \eta_2 \varrho_2(G(a,b,\psi(a),\ldots),L(a,b,\psi(a),\ldots)) <$$
$$< \eta_2 \varepsilon .$$

Applying conditions (9), (10), (13), (14) and putting $\phi = \beta_2 \psi \alpha$ and $\alpha(x) = a$, $\alpha(y) = b$ we get analogously as in theorem 1

$$\varrho_1(F(x,y,\phi(x),\ldots),K(x,y,\phi(x),\ldots) < \eta_2 \varepsilon .$$

It follows from the stability of (1) that there exists a φ such that

$$\varrho_1(\phi(x),\varphi(x)) < \eta\,\eta_2 \varepsilon \quad \text{for all} \quad x \in E_1 .$$

From Lipschitz condition (with a constant η_1) for β_1 we get

$$\varrho_2(\beta_1\phi(x),\beta_1\varphi(x)) \leqslant \eta_1 \varrho_1(\phi(x),\varphi(x)) < \eta\eta_1\eta_2\varepsilon$$

whence

$$\varrho_2(\beta_1\phi\alpha^{-1}(a),\beta_1\varphi\alpha^{-1}(a)) \quad \eta\eta_1\eta_2\varepsilon ,$$

which, in view of lemma 3, completes the proof.

V. The following definition of the stability is also considered (cf. [3]):

DEFINITION 4. Equation (1) is said to be stable if for all $\varphi : E_1 \to V_1$, if there exists a positive δ such that

$\rho_1(F(x,y,\varphi(x),\ldots),K(x,y,\varphi(x),\ldots)) < \delta$ for all $x,y \in E_1$

then φ is a solution of (1) or φ is bounded.

The corresponding theorem has now the form ($R^+ = \{x \in R: x \geqslant 0\}$):

THEOREM 4. Let $\alpha: E_1 \to E_2$ be a bijection, let $\beta_1: V_1 \to V_2$ be bounded on bounded sets, and let there exists an increasing function $h_2: R^+ \to R^+$ such that the following condition is fulfilled:

(17) $\rho_1(\beta_2(x),\beta_2(y)) \leqslant h_2(\rho_2(x,y))$ for all $x,y \in V_2$.

Let, moreover, conditions (9)-(15) be fulfilled. If equation (1) is stable then equation (8) is stable, too.

P r o o f. Let

$\rho_2(G(a,b,\psi(a),\ldots),L(a,b,\psi(a),\ldots)) < \delta$ for $a,b \in E_2$.

The function h_2 is increasing, whence

$$h_2\rho_2(G(a,b,\psi(a),\ldots),L(a,b,\psi(a),\ldots)) < h_2(\delta).$$

By condition (17) we get

$$\rho_1(\beta_2 G(a,b,\psi(a),\ldots),\beta_2 L(a,b,\psi(a),\ldots)) < h_2(\delta).$$

whence analogously as in theorems 1 and 3

$$\rho_1(F(x,y,\beta_2\psi\alpha(x),\ldots),K(x,y,\beta_2\psi\alpha(x),\ldots)) < h_2(\delta),$$

so $\beta_2\psi\alpha$ is a solution of (1) or $\beta_2\psi\alpha$ is bounded. In the former case, in view of lemma 3, $\beta_2\psi\alpha$ is a solution of equation (8). If the function $\beta_2\psi\alpha$ is bounded, then the function $\psi = \beta_1\beta_2\psi\alpha\alpha^{-1}$ is bounded, too, beceuse β_1 is bounded on bounded sets. This completes the proof.

R e m a r k s

1. The function $\beta_1 : V_1 \to V_2$ maps bounded sets onto bounded sets iff there exists an increasing function $h_1 : R^+ \to R^+$ and a point $a \in V_1$ such that the following condition is fulfilled:

(18) $\varrho_2(\beta_1(x), \beta_1(a)) \leqslant h_1(\varrho_1(x,a))$ for all $x \in V_1$.

In fact, let h_1 and a satisfy the above inequality and let $Z \subset V_1$ be a bounded set. Then there exists an $r > 0$ such that

$$\varrho_1(x,a) \leqslant r \quad \text{for all} \quad x \in Z.$$

Hence, in view of (18), we have for $x \in Z$

$$\varrho_2(\beta_1(x), \beta_1(a)) \leqslant h_1(\varrho_1(x,a)) \leqslant h_1(r),$$

and thus the set $\beta_1(Z)$ is bounded.

Conversely let $a \in V_1$ be an arbitrary point and let $K(r) = \{y \in V_1 : \varrho_1(y,a) \leqslant r\}$. Put $h_1(r) = \sup_{x \in K(r)} \varrho_2(\beta_1(x), \beta_1(a))$

In view of the property of β_1 the function h_1 is well defined. Further, fix an $x \in V_1$ and put $r = \varrho_1(x,a)$. Then $x \in K(r)$ and

$$\varrho_2(\beta_1(x), \beta_1(a)) \leqslant h_1(r) = h_1(\varrho_1(x,a)),$$

i.e. condition (18) is fulfilled. It is also evident that h_1 is increasing.

2. Note that the above condition regarding the function β_1 is equivalent to the following one: for every point $a \in V_1$ there exists an increasing function $h_1 : R^+ \to R^+$ such that the condition

$$\varrho_2(\beta_1(x), \beta_1(a)) \leqslant h_1(\varrho_1(x,a)) \quad \text{for all} \quad x \in V_1$$

is fulfilled.

3. Note that if a function β_2 satisfies the Lipschitz condition with a constant η_2, then it also fulfils the assumptions of theorem 4 with $h_2(t) = \eta_2 t$.

VI. In paper [4] the suthor consider a definition of the stability which is essentially more general than definition 1.

DEFINITION 5. Equation (1) is said to be iteratively stable relatively to function k iff for every positive ε there exists a positive δ such that for all functions $\phi: E_1 \to V_1$, if

$$\varrho_1(F_n(x,y,\phi), K_n(x,y,\phi)) < \delta \quad \text{for all } x,y \in E_1 \text{ and all } n \in N$$

then there exists a solution $\varphi : E_1 \to V_1$ of (1) such that

$$\varrho_1(\phi(x), \varphi(x)) < \varepsilon \qquad \text{for all} \quad x \in E_1,$$

where for arbitrary $\phi : E_1 \to V_1$ the sequence F_n (and analogously K_n) is defined as follows:

$$F_1(x,y,\phi) = F(x,y,\phi(x),\phi(y),\phi k(x,y),\phi l(x,y)),$$

$$F_{n+1}(x,y,\phi) =$$

$$= F(k_n(x,y),y,F_n(x,y,\phi),\phi(y),\phi k_{n+1}(x,y),\phi l(x,y)), n \in N,$$

and the sequence k_n is defined in the following way:

$$k_1(x,y) = k(x,y),$$

$$k_{n+1}(x,y) = k(k_n(x,y),y), \quad n \in N.$$

It is easy to prove inductively that φ satisfies equation (1) iff it satisfies the equation $F_n(x,y,\varphi) = K_n(x,y,\varphi)$ for all $n \in N$.

An iterative analogue of definition 3 reads as follows:

DEFINITION 6. Equation (1) is said to be iteratively stable relatively to function k iff there exists a positive η such that for every positive ε and all functions $\phi: E_1 \to V_1$, if

$\varrho_1(F_n(x,y,\phi), K_n(x,y,\phi)) < \varepsilon$ for all $x,y \in E_1$ and all $n \in N$, then there exists a solution $\varphi: E_1 \to V_1$ of (1) such that

$$\varrho_1(\phi(x), \varphi(x)) < \eta \cdot \varepsilon \qquad \text{for all } x \in E_1,$$

where F_n and K_n are defined as in definition 5.

Before formulating other theorems we prove the following:

LEMMA 4. Let a bijection α and function β_1, β_2 satisfy conditions (9), (10), (11), (13). Let, moreover, functions F_n, G_n, k_n, g_n be defined as in definition 5. The following equalities hold:

(19) $\alpha k_n(x,y) = g_n(\alpha(x), \alpha(y))$,

(20) $\beta_1 F_n(x,y,\phi) = G_n(\alpha(x), \alpha(y), \beta_1 \phi \alpha^{-1})$,

(21) $\beta_2 G_n(a,b,\psi) = F_n(\alpha^{-1}(a), \alpha^{-1}(b), \beta_2 \psi \alpha)$.

P r o o f. The proof will be by induction. For $n = 1$ relations (19)-(21) are evident. Assume that they hold for an $n \in N$. In view of (9) we have

$$\alpha k_{n+1}(x,y) = \alpha k(k_n(x,y), y) =$$
$$= g(\alpha k_n(x,y), \alpha(y)) =$$

$$= g(g_n(\alpha(x),\alpha(y)),\alpha(y)) =$$
$$= g_{n+1}(\alpha(x),\alpha(y)) .$$

Thus (19) is fulfilled.

Similarly by (11) we get

$$\beta_1 F_{n+1}(x,y,\phi) = \beta_1 F(k_n(x,y),y,F_n(x,y,\phi),\phi(y),\phi k_n(x,y),\phi l(x,y))$$
$$= G(\alpha k_n(x,y),\alpha(y),G_n(\alpha(x),\alpha(y),\beta_1\phi\alpha^{-1}),\beta_1\phi(y),\beta_1\phi k_n(x,y),\beta_1\phi l(x,y))$$
$$= G(g_n(\alpha(x),\alpha(y)),\alpha(y),G_n(\alpha(x),\alpha(y),\beta_1\phi\alpha^{-1}),$$
$$\beta_1\phi\alpha^{-1}\alpha(y),\beta_1\phi\alpha^{-1}g_n(\alpha(x),\alpha(y)),\beta_1\phi\alpha^{-1}h(\alpha(x),\alpha(y)))$$
$$= G_{n+1}(\alpha(x),\alpha(y),\beta_1\phi\alpha^{-1}).$$

Thus (20) is fulfilled, too.

The proof of (21) is analogous.

From lemma 4, taking in the proofs of theorems 1 and 3 functions F_n, G_n, K_n, L_n instead of functions F, G, K, L, respectively, we get two theorems:

THEOREM 5. Let α be a bijection, let β_1 and β_2 be uniformly continuous functions and let conditions (9) - (15) be fulfilled. If equation (1) is iteratively stable relatively to the function k in the sense of definition 5, then equation (8) is iteratively stable relatively to function g in the sense of the same definition, too.

THEOREM 6. Let α be a bijection, let β_1 and β_2 satisfy Lipschitz condition and let conditions (9) - (15) be fulfilled. If equation (1) is iteratively stable relatively to function k in the sense of definition 6, then equation (8) is iteratively stable relatively to function g in the sense of the same definition, too.

R e m a r k s

1. All the theorems of this paper can be proved analogously for the equation

$$F(x_1,\dots,x_n,\varphi(x_1),\ \ ,\varphi(x_n),\varphi f_1(x_1,\dots,x_n),\dots,\varphi f_k(x_1,\dots x_n)) =$$
$$= K(x_1,\dots,x_n,\varphi(x_1),\dots,\varphi(x_n),\varphi f_1(x_1,\dots,x_n),\dots,\varphi f_k(x_1,\dots,x_n)).$$

2. In the case where equations (1) and (8) have the form

$$(22)\qquad \varphi(k(x,y)) = K(\varphi(x),\varphi(y)),$$

$$(23)\qquad \psi(g(a,b)) = L(\psi(a),\psi(b)),$$

they are equations of homomorphisms φ and ψ of groupoid E_1 with the operation $k(x,y)$ into groupoid V_1 with the operation $K(u,v)$ and of groupoid E_2 with the operation $g(a,b)$ into groupoid V_2 with the operation $L(w,z)$, respectively. So the results of this paper extend those of papers [7] and [8].

Notice that if equation (1) is of form (22) and conditions (11), (12) and (15) are fulfilled then equation (8) must be of form (23).

3. Equation (1) contains as a particular case the equation

$$\varphi(k(x)) = K(x,\varphi(x))$$

whose stability was considered by D. Brydak (cf. [4]) and E. Turdza (cf. [9]). Equation (1) contains also the cosine equation whose stability was studied by J.A. Baker (cf.[2]).

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