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Decompositions of bioperands

In this paper we shall describe some decompositions of bioperands. First of all we introduce some appropriate definitions. The description of the notations as a left [right] operand and a bioperand is based on the paper [1], vol. 2.

DEFINITION 1. If M is a non-empty set and S is a semigroup, we say that M is a left operand over the semigroup S if there is a mapping (left outer product) $(s,x) \mapsto sx$ from $S \times M$ into M such that

$$s_1(s_2x) = (s_1s_2)x$$

for all $s_1, s_2 \in S$ and $x \in M$.

Similarly, a non-empty set M is a right operand over a semigroup T if there is a mapping (right outer product) $(x,t) \mapsto xt$ from $M \times T$ into M such that

$$(x\ t_1)\ t_2 = x(t_1 t_2)$$

for all $t_1, t_2 \in T$ and $x \in M$.

If S and T are semigroups we say that a non-empty set M is a bioperand over the semigroups S and T if it is a left operand over the semigroup S, a right operand over the semigroup T and

$$(s\ x)t = s(x\ t)$$

for all $s \in S$, $t \in T$, $x \in M$.

If S and T are monoids we put $1\ x = x\ 1 = x$ for arbitrary $x \in M$ in all above cases.

Further on we shall write a left operand $_SM$, a right operand M_T, and a bioperand $_SM_T$.

The set M is said to be a fibre of the bioperand $_SM_T$ [left operand $_SM$, right operand M_T].

Next, we introduce definitions which are modelled on the suitable definitions of the semigroup theory and the algebraic object theory ([1], [3]).

DEFINITION 2. Let S_1 and T_1 be subsemigroups of semigroups S and T, respectively.

A non-empty subset N of the fibre M of the bioperand $_SM_T$ is said to be a left [right] invariant subset in the bioperand $_SM_T$ relative to the subsemigroup S_1 [T_1], if $S_1N \subset N$ [$N\ T_1 \subset N$].

A non-empty subset N of the fibre M is said to be an invariant subset in the bioperand $_SM_T$ relative to the subsemigroups S_1 and T_1, if $S_1N \subset N$ and $N\ T_1 \subset N$.

If $S_1 = S$ and $T_1 = T$ we simply say that N is an [left, right] invariant subset in the bioperand ${}_S M_T$.

DEFINITION 3. Let S_1 and T_1 be subsemigroups of semigroups S and T, respectively.

If a non-empty subset N of the fibre M of the bioperand ${}_S M_T$ with the left and right outer products restricted to the subset N and the subsemigroups S_1 and T_1 (respectively) is a bioperand ${}_{S_1} N_{T_1}$ over the semigroups S_1 and T_1, then the bioperand ${}_{S_1} N_{T_1}$ is said to be a FS-subbioperand of the bioperand ${}_S M_T$.

If $S_1 = S$ and $T_1 = T$ then a FS-subbioperand is said to be a subbioperand.

Analogously we can define a FS-suboperand and a suboperand.

An idea of the symbols FS- (abbreviations for: fibre, structure) has been inspired by the paper [3], p.12.

It is easy to see a simple connection between notions of definitions 2 and 3.

DEFINITION 4. A family $(M_i : i \in I)$ of subsets of a set M satisfying the following conditions:

(i) $\phi \notin (M_i : i \in I)$,

(ii) $\bigcup (M_i : i \in I) = M$,

(iii) $\bigwedge_{i,j \in I} [i \neq j \Rightarrow M_i \cap M_j = \phi]$,

is called a decomposition of the set M.

The decomposition received by means of the equivalence relation $\rho \subset M \times M$ will be called a ρ-decomposition.

The decomposition of the fibre M of the bioperand ${}_SM_T$ is said to be a decomposition of the bioperand ${}_SM_T$.

DEFINITION 5. If every subset M_i (for $i \in I$) in the decomposition $(M_i : i \in I)$ of the bioperand ${}_SM_T$ is invariant [relative to the subsemigroups $S_i \subset S$ and $T_i \subset T$ for $i \in I$], then the family $(M_i : i \in I)$ is called a decomposition of the bioperand ${}_SM_T$ into subbioperands [FS-subbioperands].

In the similar way we define a decomposition of the bioperand into suboperands [FS-suboperands].

DEFINITION 6. If there exists a decomposition of the bioperand ${}_SM_T$ into at least two different (with respect to the fibre) subbioperands [FS-subbioperands], then we say that the bioperand ${}_SM_T$ is decomposable into subbioperands [FS-subbioperands], otherwise, it is called indecomposable.

Analogously we define a bioperand decomposable [indecomposable] into suboperands [FS-suboperands].

Let S be a semigroup. Extend the binary operation on S to the set $S \cup \{1\}$, where 1 is an element not contained in S, putting $1\,1 = 1$ and $1\,s = s\,1 = s$ for every $s \in S$. Clearly, $S \cup \{1\}$ is a monoid with respect to this extended operation and 1 is its identity element. If S is a semigroup, then S^1 denotes the monoid obtained by the adjunction of an identity element to S if S does not contain an identity element, and $S^1 = S$ if S is a monoid.

The equivalence relation on M generated by any relation $\rho \subset M \times M$ will be denoted by ρ^e ([2], p.19). The symbol $\rho(x)$ denotes that equivalence class of the equivalence relation ρ which contains x. An equivalence class of the equivalence relation ρ will be also called ρ-class.

In the bioperand ${}_SM_T$ we define relations $\mathcal{L}^*$ and $\mathcal{R}^*$ by the rules:

$$x \mathcal{L}^* y \Leftrightarrow \bigvee_{s \in S^1} (s x = y),$$

$$x \mathcal{R}^* y \Leftrightarrow \bigvee_{t \in T^1} (x t = y),$$

for all $x, y \in M$.
The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are reflexive and transitive.

By means of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ we define in the bioperand ${}_SM_T$ a relation

$$\mathcal{D}^* = (\mathcal{L}^* \cup \mathcal{R}^*)^e.$$

Every $\mathcal{D}^*$-class is an invariant subset in the bioperand ${}_SM_T$. The decomposition of the fibre M by means of the relation $\mathcal{D}^*$ is a decomposition of the bioperand ${}_SM_T$ into the subbioperands.

THEOREM 1. The $\mathcal{D}^*$-decomposition of any bioperand ${}_SM_T$ is the only decomposition of this bioperand into indecomposable subbioperands.

P r o o f. We have already noticed above that in every bioperand ${}_SM_T$ the relation $\mathcal{D}^*$ determines the decomposition of this bioperand into subbioperands. We shall preve that it is the only decomposition of the bioperand ${}_S M_T$ into indecomposable subbioperands.

Let $(N_i : i \in I)$ be any decomposition of the bioperand ${}_S M_T$ into subbioperands. Let N_i for $i \in I$ be any class in this decomposition. We shall show that if $x \in N_i$ then $\mathcal{D}^*(x) \subset N_i$. Suppose that there exists an element $y \in M \setminus N_i$ such that $(x,y) \in \mathcal{D}^*$. Then by the Proposition 4.26 of [2], p.21, there exists a sequence $z_1, z_2, \ldots, z_n \in M$ such that $z_1 = x$, $z_n = y$, $(z_k, z_{k+1}) \in (\mathcal{L}^* \cup \mathcal{R}^*) \cup (\mathcal{L}^* \cup \mathcal{R}^*)^{-1}$ for $k = 1,2,\ldots,n-1$. Since $z_1 = x \in N_i$ and $z_n = y \in M \setminus N_i$ there exists $k \in \{1,2,\ldots,n-1\}$ such that $z_k \in N_i$ and $z_{k+1} \in M \setminus N_i$. If $(z_k, z_{k+1}) \in \mathcal{L}^* \cup \mathcal{R}^*$, then either $s z_k = z_{k+1}$ or $z_k t = z_{k+1}$ for some elements $s \in S^1$ and $t \in T^1$. As the subset N_i is invariant so $z_{k+1} \in N_i$. This contradicts the assumption that $z_{k+1} \in M \setminus N_i$. We receive the analogous contradiction considering the case where $(z_k, z_{k+1}) \in (\mathcal{L}^* \cup \mathcal{R}^*)^{-1}$. Hence an arbitrary subset N_i of the decomposition $(N_i : i \in I)$ is either a $\mathcal{D}^*$-class or a union of $\mathcal{D}^*$-classes.

This completes the proof.

We define relations $\mathcal{L}^{**}$ and $\mathcal{R}^{**}$ in the bioperand ${}_S M_T$ by the rules

$$\mathcal{L}^{**} = (\mathcal{L}^*)^e \quad \text{and} \quad \mathcal{R}^{**} = (\mathcal{R}^*)^e.$$

Every $\mathcal{L}^{**}$-class [$\mathcal{R}^{**}$-class] is a left [right] invariant subset in the bioperand ${}_S M_T$.

The decomposition of the set M by means of the relation $\mathcal{L}^{**}$ [$\mathcal{R}^{**}$] is a decomposition of the bioperand ${}_S M_T$ into the left [right] suboperands.

Let us observe that $\mathcal{L}^{**} \subset \mathcal{D}^{*}$ and $\mathcal{R}^{**} \subset \mathcal{D}^{*}$.

Every subbioperand of the $\mathcal{D}^{*}$-decomposition of the bioperand ${}_S M_T$ is a union of some left [right] suboperands of the bioperand ${}_S M_T$.

THEOREM 2. The $\mathcal{L}^{**}$- decomposition [$\mathcal{R}^{**}$- decomposition] of any bioperand ${}_S M_T$ is the only decomposition of this bioperand into indecomposable left [right] suboperands.

The proof of this theorem is similar to the proof of Theorem 1.

In a bioperand ${}_S M_T$ we define relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ by the rules:

$$x \mathcal{L} y \Longleftrightarrow S^1 x = S^1 y,$$
$$x \mathcal{R} y \Longleftrightarrow x T^1 = y T^1,$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R},$$

for all $x, y \in M$.

The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ will be called Green's relations. These relations are a generalization of the well-known Green's relations in the semigroup theory ([2], p.38). The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ are equivalences.

LEMMA. For arbitrary elements x and y in the fibre M of the bioperand ${}_S M_T$ the following conditions are satisfied:

(i) $x \mathcal{L} y \Leftrightarrow \bigvee_{s_1, s_2 \in S^1} (s_1 y = x \wedge s_2 x = y)$,

(ii) $x \mathcal{R} y \Leftrightarrow \bigvee_{t_1, t_2 \in T^1} (y t_1 = x \wedge x t_2 = y)$.

These results are direct consequences of the definitions of the relations $\mathcal{L}$ and $\mathcal{R}$.

Let s be any fixed element in the semigroup S. We define a mapping $l_s : M \to M$ as follows

$$l_s(x) = s\, x$$

for all $x \in M$.

The mapping l_s is said to be a left translation in the bioperand ${}_S M_T$.

For any fixed $t \in T$ we define a mapping $r_t : M \to M$ as follows

$$r_t(x) = x\, t, \qquad x \in M.$$

We call it a right antitranslation in the bioperand ${}_S M_T$.

In the sequel of our consideration the following two theorems will be useful.

THEOREM 3. Let ${}_S M_T$ be any arbitrary bioperand over semigroups S and T. Let $x, y \in M$ be elements such that $x \mathcal{R} y$, i.e. $x\, t_1 = y$ and $y\, t_2 = x$ for some elements $t_1, t_2 \in T^1$.
Then the mappings $r_{t_1} | \mathcal{L}(x)$ and $r_{t_2} | \mathcal{L}(y)$ are mutually inverse bijections preserving $\mathcal{R}$-classes, i.e. the arguments and their corresponding values belong to the same $\mathcal{R}$-class.

THEOREM 4. Let ${}_SM_T$ be an arbitrary bioperand over semigroups S and T. Let $x, y \in M$ be elements such that $x \mathcal{L} y$, i.e. $s_1 x = y$ and $s_2 y = x$ for some elements $s_1, s_2 \in S^1$. Then the mappings $l_{s_1} | \mathcal{R}(x)$ and $l_{s_2} | \mathcal{R}(y)$ are mutually inverse bijections preserving $\mathcal{L}$-classes, i.e. the arguments and their corresponding values belong to the same $\mathcal{L}$-class.

The proofs of the Theorems 3 and 4 are quite similar to the proofs of Green's Lemmas ([2], p.42-43).

THEOREM 5. Green's relation $\mathcal{R}$ determines a decomposition of the bioperand ${}_SM_T$ into left FS-suboperands.

P r o o f. Let R be any fixed $\mathcal{R}$-class in the bioperand ${}_SM_T$. Let $x, y \in R$ and $x \mathcal{L} y$. Therefore $s x = y$ for an element $s \in S^1$. It follows immediately from Theorem 4 that $l_s | R: R \to R$ is a bijection which will be denoted by l_s for short. The set of all left translations l_s determined in this way we denote by $T_l(R)$. We shall prove that $T_l(R)$ is a group. Let $l_{s_1}, l_{s_2} \in T_l(R)$. For any fixed element $z \in R$ we denote $w = (l_{s_1} l_{s_2})(z)$. Since l_{s_1} and l_{s_2} are bijections preserving $\mathcal{L}$-classes so $w \mathcal{L} z$. Moreover, $w = (l_{s_1} l_{s_2})(z) = l_{s_1 s_2}(z)$ so $(s_1 s_2) z = w$. Hence $l_{s_1 s_2} \in T_l(R)$, i.e. $l_{s_1} l_{s_2} \in T_l(R)$. It follows from Theorem 4 that for every left translation $l_s \in T_l(R)$ there exists a left tranlsation $l_{s'} \in T_l(R)$ such that l_s and $l_{s'}$ are mutually inverse bijections. Therefore $T_l(R)$ is a group.

We define a subset S_R of the semigroup S^1 as follows

$$S_R = \{s \in S^1 : l_s \in T_1(R)\}.$$

It is clear that S_R is the subsemigroup of the semigroup S^1. It follows immediatelly from these considerations that the $\mathcal{R}$-class R is the left FS-suboperand ${}_{S_R}R$ over the semigroup S_R of the bioperand ${}_SM_T$.
Therefore we can consider every $\mathcal{R}$-class in the bioperand ${}_SM_T$ as a left FS-suboperand of the bioperand ${}_SM_T$, which ends the proof.

Using Theorem 3, in a quite similar way, we can prove the following

THEOREM 6. Green's relation $\mathcal{L}$ determines a decomposition of the bioperand ${}_SM_T$ into right FS-suboperands.

In a similar manner as in the semigroup theory ([1], vol.1) we can construct Schützenberger's groups over $\mathcal{H}$-classes in a bioperand.
Let H be an arbitrary $\mathcal{H}$- class in the bioperand ${}_SM_T$ and let $x_o \in H$ be an arbitrary fixed element. Let y be an arbitrary element in $\mathcal{H}$-class H. Since $x_o \mathcal{H} y$ so $x_o \mathcal{L} y$ and $x_o \mathcal{R} y$. Then there exist elements $s_1, s_2 \in S^1$ and $t_1, t_2 \in T^1$ such that $s_1 s_o = y$, $s_2 y = x_o$, $x_o t_1 = y$, $y\, t_2 = x_o$. Hence $l_{s_1}(x_o) = y$, $l_{s_2}(y) = x_o$, $r_{t_1}(x_o) = y$, $r_{t_2}(y) = x_o$. By Theorems 3 and 4 the left translations $l_{s_1}|H$, $l_{s_2}|H$ and the right antitranslations $r_{t_1}|H$, $r_{t_2}|H$ are bijections of the subset H onto itself such that

$$l_{s_1} l_{s_2} = l_{s_2} l_{s_1} = id(H) \quad \text{and} \quad r_{t_1} r_{t_2} = r_{t_2} r_{t_1} = id(H).$$

Let us denote by $T_l(H)_{x_o}$ and $T_r(H)_{x_o}$ the sets of all left translations and right antitranslations, respectively, defined in the above way on the $\mathcal{H}$-class H for the fixed element x_o and for an arbitrary element $y \in H$.

Let z_o be any fixed element of $\mathcal{H}$-class H. It is easy to see that $T_l(H)_{x_o} = T_l(H)_{z_o}$ and $T_r(H)_{x_o} = T_r(H)_{z_o}$. So we shall write $T_l(H)$ and $T_r(H)$ instead of $T_l(H)_{x_o}$ and $T_r(H)_{x_o}$.

THEOREM 7. Let H be any $\mathcal{H}$-class in the bioperand ${}_SM_T$. The sets of all left translations $T_l(H)$ and all right antitranslations $T_r(H)$ are groups.

P r o o f. We shall prove that the set of all left translations $T_l(H)$ is a group. Let $l_{s_1}, l_{s_2} \in T_l(H)$ and x be any fixed element of $\mathcal{H}$-class H. Then $(l_{s_1} l_{s_2})(x) = l_{s_1 s_2}(x) \in H$, i.e. $l_{s_1} l_{s_2} \in T_l(H)$. We have seen that every left translation from the set $T_l(H)$ is a bijection on the subset H which has an inverse being a left translation from the set $T_l(H)$. Therefore the set $T_l(H)$ is a group. Analogously we can show that the set $T_r(H)$ is a group.

The groups $T_l(H)$ and $T_r(H)$ will be called Schützenberger's groups over $\mathcal{H}$-class H in the bioperand ${}_SM_T$.

THEOREM 8. The Green's relation $\mathcal{H}$ determines the decomposition of the bioperand ${}_SM_T$ into FS-subbioperands.

P r o o f. Let H be an arbitrary fixed $\mathcal{H}$-class in the bioperand ${}_SM_T$. We define the subsets $S_H \subset S^1$ and $T_H \subset T^1$ as follows:

$$S_H = \{s \in S^1 :\ l_s \in T_l(H)\},$$
$$T_H = \{t \in T^1 :\ r_t \in T_r(H)\}.$$

The sets S_H and T_H are subsemigroups of the semugroups S^1 and T^1, respectively. Therefore every $\mathcal{H}$-class in the bioperand ${}_S M_T$ can be considered as a FS-subbioperand ${}_{S_H} H_{T_H}$ over the semugroups S_H and T_H of the bioperand ${}_S M_T$.

This completes the proof.

References

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