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On the equivalence of definitions of a semidirect product of loops

This paper is a kind of sequel to the paper [2].

In the paper we shall consider the equivalence of three definitions of a semidirect product of loops.

Definitions of loop, subloop, normal subloop will be used according to [1].

In the sequel we shall identify isomorphic algebraic structures. According to [2] we may formulate the following two definitions.

DEFINITION 1. A loop T is called a semidirect product of a loop L on a loop K if and only if the following conditions are fulfilled:

1^o there exist subloops L^* and K^* of the loop T isomorphic to the loops L i K, respectively,

2^o K^* is a normal subloop of the loop T,

3° $L^{*} K^{*} = T$,

4° $L^{*} \cap K^{*} = \{1\}$.

DEFINITION 2. Let L and K be loops and let $\varphi : L \times K \times L \times K \rightarrow K$ be any mapping fulfilling the conditions:

(a_1) $\varphi(1,1,l,k) = \varphi(l,1,1,k) = \varphi(l,k,1,1) = k$,

(a_2) $\varphi(1,k_1,1,k_2) = k_1 k_2$,

(a_3) $\varphi(l_1,1,l_2,1) = 1$,

(a_4) the mapping $\varphi(l_1,k_1,l_2,\cdot) : K \rightarrow K$ is a bijection,

(a_5) the mapping $\varphi(l_1,\cdot,l_2,k_2) : K \rightarrow K$ is a bijection,

for any $l,l_1,l_2 \in L$ and $k,k_1,k_2 \in K$.

Consider an algebraic structure $[L \times K, \circ]_{\varphi}$ with a binary operation $\circ$ defined by

$$\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle = \langle l_1 l_2, \varphi(l_1,k_1,l_2,k_2)\rangle$$

for any $\langle l_1,k_1\rangle, \langle l_2,k_2\rangle \in L \times K$.

The algebraic structure $[L \times K, \circ]_{\varphi}$ is a loop with the identity element $\langle 1,1\rangle$.

The loop $[L \times K, \circ]_{\varphi}$ is said to be a semidirect product of the loop L on the loop K.

The third definition will be closely modelled on the definitions of a semidirect product of groups ([3]) and semigroups ([4]).

First, we shall define a structure $(L \times K, \circ)_{\alpha}$ and we shall give the necessary and sufficient conditions under which this structure is a loop.

Let L and K be any loops, let F(K) be a set of all mappings of K into K. Let $\alpha : L \to F(K)$ be any mapping. If $l \in L$ and $k \in K$ then k^l denotes the image of k under the mapping $\alpha(l)$.

Consider an algebraic structure $(L \times K, \circ)_\alpha$ where a binary operation $\circ$ is defined by

$$\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle = \langle l_1 l_2, k_1^{l_2} k_2\rangle$$

for any $\langle l_1,k_1\rangle, \langle l_2,k_2\rangle \in L \times K$.

THEOREM 1. The algebraic structure $(L \times K, \circ)_\alpha$ is a loop if and only if the following conditions are fulfilled:

(i) for any fixed $l \in L$ the mapping $k \mapsto k^l$ is a bijection from K onto K,

(ii) $\bigvee_{k_0 \in K} \bigwedge_{l \in L} \bigwedge_{k \in K} (k_0^l = 1 \wedge k^1 k_0 = k)$.

P r o o f. 1. Let $(L \times K, \circ)_\alpha$ be a loop. For any pairs $\langle l,k\rangle, \langle l_1,k_1\rangle \in L \times K$ there exists a unique solution of the equation

$$\langle x,y\rangle \circ \langle l,k\rangle = \langle l_1,k_1\rangle.$$

According to the definition of the operation $\circ$ this equation may be rewritten as $\langle xl, y^l k\rangle = \langle l_1,k_1\rangle$. So the equation $y^l k = k_1$ has a unique solution and for any fixed $l \in L$ the mapping $y \to y^l$ is a bijection from K onto K. The condition (i) is fulfilled.

Since $(L \times K, \circ)_\alpha$ is a loop thus there exists a pair $\langle l_0,k_0\rangle \in L \times K$ such that $\langle l_0,k_0\rangle \circ \langle l,k\rangle = \langle l,k\rangle \circ \langle l_0,k_0\rangle = \langle l,k\rangle$, for an arbitrary $\langle l,k\rangle \in L\times K$.

This means that

$$\langle l_o l, k_o^{l} k\rangle = \langle l,k\rangle \quad \text{and} \quad \langle l l_o, k^{l_o} k_o\rangle = \langle l,k\rangle.$$

Hence $l_o = 1$, $k_o^{l} = 1$, $k^{1}k_o = k$, and thus the condition (ii) is fulfilled.

2. It is a routine matter to verify that if the conditions (i) and (ii) hold, then $(L\times K, \circ)_\alpha$ is a loop with the identity element $\langle 1,k_o\rangle$.

We give an example of a structure $(L\times K,\circ)_\alpha$ in which the identity element has the form $\langle 1,k_o\rangle$, where $k_o\in K$ and $k_o \neq 1$.

E x a m p l e 1.

Consider the group $L_2 = \{1,2\}$ given by the table:

	1	2
1	1	2
2	2	1

Define the mapping $h\colon L_2 \to L_2$ as follows: $h(1) = 2$, $h(2) = 1$. Let $\alpha : L_2 \to F(L_2)$ and $\alpha(1) = \alpha(2) = h$. The binary operation in the structure $(L_2\times L_2, \circ)_\alpha$ is given by the table:

$\circ$	$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$
$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 1,1\rangle$	$\langle 2,2\rangle$	$\langle 2,1\rangle$
$\langle 1,2\rangle$	$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$
$\langle 2,1\rangle$	$\langle 2,2\rangle$	$\langle 2,1\rangle$	$\langle 1,2\rangle$	$\langle 1,1\rangle$
$\langle 2,2\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$	$\langle 1,1\rangle$	$\langle 1,2\rangle$

It is clear that $(L_2 \times L_2, \circ)_\alpha$ is a group with the identity element $\langle 1,2\rangle$.

DEFINITION 3. The loop $(L \times K, \circ)_\alpha$ is said to be a semidirect product of the loop L on the loop K.

In the sequal the symbol $(L \times K, \circ)_\alpha$ will denote a semidirect product in the sense of definition 3 of the loop L on the loop K.

The question arises whether these three definitions are equivalent.

It has been proved in [2] that definitions 1 and 2 are equivalent. We shall prove that definition 3 implies definitions 1 and 2, but not conversely.

THEOREM 2. Every loop $(L \times K, \circ)_\alpha$ is a semidirect product, in the sense of definition 1, of the loop L on the loop K.

P r o o f. Let $\langle 1,k_o\rangle$, $k_o \in K$ be the identity element of the loop $(L \times K, \circ)_\alpha$. It is easy to verify that $L^* = \{\langle l,k_o\rangle : l \in L\}$ and $K^* = \{\langle 1,k\rangle : k \in K\}$ are subloops of the loop $(L \times K, \circ)_\alpha$ isomorphic to the loops L and K, respectively.

Thus condition 1^o holds.

To prove condition 2^o we must show the following equalities:

(a) $\langle l,k\rangle \circ K^* = K^* \circ \langle l,k\rangle$,

(b) $\langle l_1,k_1\rangle \circ [\langle l_2,k_2\rangle \circ K^*] = [\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle] \circ K^*$,

(c) $[K^* \circ \langle l_1,k_1\rangle] \circ \langle l_2,k_2\rangle = K^* \circ [\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle]$, for any pairs $\langle l,k\rangle, \langle l_1,k_1\rangle, \langle l_2,k_2\rangle \in L\times K$.

(a) Let $\langle l_1,k_1\rangle \in \langle l,k\rangle \circ K^*$ and so $\langle l_1,k_1\rangle = \langle l,k\rangle \circ \langle 1,k_2\rangle = \langle l,k^1k_2\rangle$ where $k_2 \in K$. There exists an element $k_3 \in K$ such that $k^1k_2 = k_3^l k$ and so we have $\langle l_1,k_1\rangle = \langle l,k_3^l k\rangle = \langle 1,k_3\rangle \circ \langle l,k\rangle \in K^* \circ \langle l,k\rangle$. Hence $\langle l,k\rangle \circ K^* \subset K^* \circ \langle l,k\rangle$.

We can equally well obtain the converse inclusion.

(b) Let $\langle l,k\rangle \in \langle l_1,k_1\rangle \circ [\langle l_2,k_2\rangle \circ K^*]$. Then $\langle l,k\rangle = \langle l_1,k_1\rangle \circ [\langle l_2,k_2\rangle \circ \langle 1,k_3\rangle] = \langle l_1l_2, k_1^{l_2}(k_2^1k_3)\rangle$, $k_3 \in K$. There exists $k_4 \in K$ such that $k^{l_2}(k_2^1k_3) = (k_1^{l_2}k_2)^1k_4$. Hence $\langle l,k\rangle = \langle l_1l_2, (k_1^{l_2}k_2)^1k_4\rangle = [\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle] \circ \langle 1,k_4\rangle \in [\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle] \circ K^*$.

A similar argument establishes the converse inclusion.

(c) A closely analogous argument leads to the equality (c).

Let us observe that $\langle l,k\rangle = \langle l,k_0\rangle \circ \langle 1,k\rangle$ for any pair $\langle l,k\rangle \in L\times K$. Hence condition 3^0 of definition 1 holds.

Evidently, condition 4^0 is also fulfilled.

This completes the proof.

Let the loop T be a semidirect product in the sense of definition 1 of the loop L on the loop K. The loops L^* and K^* have the same sense as in definition 1. Consider a semidirect product $(L\times K,\circ)_\alpha$ with the identity element $\langle 1,k_0\rangle$. Denote by $L_1^* = \{\langle l,k_0\rangle : l \in L\}$ and $K_1^* = \{\langle 1,k\rangle : k \in K\}$ the subloops of the loop $(L\times K,\circ)_\alpha$.

In the sequel we shall use the following

PROPOSITION. Let the loop T be a semidirect product in the sense of definition 1 of the loop L on the loop K. Let $\lambda : T \to (L \times K, \circ)_{\alpha}$ be an isomorphism such that $\lambda(L^{*}) = L_1^{*}$ and $\lambda(K^{*}) = K_1^{*}$.

Then

$$l_1(l_2k) = (l_1l_2)k$$

for any $l_1, l_2 \in L$ and $k \in K$.

P r o o f. Let $l_1 = \lambda^{-1}(\langle l_1', k_o\rangle)$, $l_2 = \lambda^{-1}(\langle l_2', k_o\rangle)$, $k = \lambda^{-1}(\langle 1, k'\rangle)$, where $\langle l_1', k_o\rangle, \langle l_2', k_o\rangle \in L_1^{*}$ and $\langle 1, k'\rangle \in K_1^{*}$.

Then, $l_1(l_2k) = \lambda^{-1}(\langle l_1', k_o\rangle)[\lambda^{-1}(\langle l_2', k_o\rangle)\lambda^{-1}(\langle 1, k'\rangle)] =$
$= \lambda^{-1}(\langle l_1', k_o\rangle \lambda^{-1}(\langle l_2', k'\rangle) = \lambda^{-1}(\langle l_1' l_2', k'\rangle) =$
$= \lambda^{-1}(\langle l_1' \, l_2', k_o\rangle)\lambda^{-1}(\langle 1, k'\rangle) = [\lambda^{-1}\langle(l_1', k_o\rangle\rangle \lambda^{-1}(\langle l_2', k_o\rangle)]$
$\lambda^{-1}(\langle 1, k'\rangle) = (l_1l_2)k.$

We shall give an example of a loop $[L \times K, \circ]_{\varphi}$ for which there is no a mapping $\alpha : L \to F(K)$ such that the loops $(L \times K, \circ)_{\alpha}$ and $[L \times K, \circ]_{\varphi}$ are isomorphic.

E x a m p l e 2.

Consider the loop $K = \{1,2,3,4,5\}$ given by the table.

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

Define a mapping $\varphi : K\times K\times K\times K \to K$ as follows

$$\varphi(l_1,k_1,l_2,k_2) = \begin{cases} (k_1\ 2)(2\ k_2) & \text{for } l_1\neq 1 \text{ and } l_2\neq 1, \\ k_1k_2 & \text{for } l_1=1 \text{ or } l_2=1. \end{cases}$$

The mapping φ satisfies the conditions (a_1) - (a_5) of definition 2.

By means of the mapping φ we construct the semidirect product $[K\times K,\circ]_\varphi$ of the loop K on the loop K.

Suppose that there exists a mapping $\alpha : K \to F(K)$ such that the loops $[K\times K,\circ]_\varphi$ and $(K\times K,\circ)_\alpha$ are isomorphic, i.e. there exists an isomorphism $\lambda: [K\times K,\circ]_\varphi \to (K\times K,\circ)_\alpha$.

The subloops $L^* = \{\langle l,1\rangle : l\in K\}$ and $K^* = \{\langle 1,k\rangle : k\in K\}$ of the loop $[K\times K,\circ]_\varphi$ satisfy conditions 1^0 - 4^0 of definition 1. As the loop K is isomorphic to the loops L^* and K^* so the loop $[K\times K,\circ]_\varphi$ is a semidirect product in the sense of definition 1 of the loop K on the loop K. Let $\langle 1,k_0\rangle$ be the identity element of the loop $(K\times K,\circ)_\alpha$.
The sets $L_1^* = \{\langle l,k_0\rangle : l\in K\}$ and $K_1^* = \{\langle 1,k\rangle : k\in K\}$ are subloops of the loop $(K\times K,\circ)_\alpha$.

The following cases can take place:

(a) $\lambda(L^*) = L_1^*$ and $\lambda(K^*) = K_1^*$,

(b) $\lambda(L^*) \neq L_1^*$ or $\lambda(K^*) \neq K_1^*$.

We shall show that the case (b) does not hold.
As L^* is not a normal subloop of the loop $[K\times K,\circ]_\varphi$ so $\lambda(L^*) \neq K_1^*$.

It is sufficient to prove that the loop $[K \times K, \circ]_\varphi$ has no five-element subloop different from L^* and K^*.

Let P be a subloop of the loop $[K \times K, \circ]_\varphi$. Let $\pi_1(P) = \{ l \in K : \bigvee_{k \in K} \langle l,k \rangle \in P \}$ and $\pi_2(P) = \{ k \in K : \bigvee_{l \in K} \langle l,k \rangle \in P \}$.

The set $\pi_1(P)$ is a subloop of the loop K. The loop K has the following subloops: $\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}$, $\{1,2,3,4,5\}$. All subsets of the loop K containing at least two elements different from 1 generate the loop K. Let us define on the set $K = \{1,2,3,4,5\}$ a binary operation as follows

$$k_1 \cdot k_2 = (k_1 2)(2 k_2)$$

for any $k_1, k_2 \in K$.

On the right side of the equality is the binary operation in the loop K.

The binary operation $\cdot$ has the following table:

$\cdot$	1	2	3	4	5
1	1	2	4	5	3
2	2	1	5	3	4
3	5	4	3	2	1
4	3	5	1	4	2
5	4	3	2	1	5

The set K with the binary operation $\cdot$ is a quasigroup which will be denote by K_2.

The quasigroup K_2 has the following subquasigroups: $\{1\}$, $\{3\}, \{4\}, \{5\}, \{1,2\}, \{1,2,3,4,5\}$. Each subset of K_2 different from $\{1,2\}$ with two or more elements generate the quasigroup K_2.

Consider the following cases:

1) If $\pi_1(P) = \{1\}$ and $\pi_2(P) = \{1,2,3,4,5\}$ then $P = K^*$.

2) If $\pi_1(P)$ is a two-element subloop then the subset $\pi_2(P)$ have to have at least three elements. Because each three-element subset of the set K generates the loop K and the quasigroup K_2, so there does not exist a five-element subloop P satisfying above assumptions.

3) If $\pi_1(P) = \{1,2,3,4,5\}$ then putting $\pi_2(P) = \{1\}$ we get the subloop L^*. The subloop L^* is a unique five-element subloop P for which $\pi_2(P)$ is a one-element subset. It is easy to see that putting $\pi_2(P) = \{1,2\}$ we do not receive a five-element subloop P. It is a routine matter to check that taking for $\pi_2(P)$ other subset containing at least two elements we do not obtain a five-element subloop P because these subsets generate the loop K or the quasigroup K_2.

Therefore, the subloops L^* and K^* are the unique five-element subloops of the loop $[K \times K, \circ]_\varphi$.

Then only case (a) is possible.

Let $\langle 2,1\rangle, \langle 3,1\rangle \in L^*$ and $\langle 1,4\rangle \in K^*$. Then $[\langle 2,1\rangle \circ \langle 3,1\rangle] \circ \langle 1,4\rangle = \langle 5,1\rangle \circ \langle 1,4\rangle = \langle 5,4\rangle$ and $\langle 2,1\rangle \circ$ $\circ [\langle 3,1\rangle \circ \langle 1,4\rangle] = \langle 2,1\rangle \circ \langle 3,4\rangle = \langle 5,5\rangle$ which contradicts the Proposition.

There exists no a mapping $\alpha : K \to F(K)$ for which the loops $[K \times K, \circ]_\varphi$ and $(K \times K, \circ)_\alpha$ are isomorphic.

It follows from the above example that definition 3 is not equivalent to definitions 1 and 2.

In the sequel of the paper we assume that loops L and and K are groups.

In order to get definitions of a semidirect product of a group L on a group K corresponding to definitions 1, 2, and 3 for loops, some natural additional requirement are imposed. For example, we shall use notions a group, a normal subgroup, instead of a loop, a normal subloop, respectively.

In definition 3 we shall assume that the mapping α is a homomorphism and the set F(K) is a group of all automorphisms Aut(K) of the group K.

These assumptions are equivalent to the following conditions:

(1) for any fixed element $l \in L$ the mapping $k \to k^l$ is a bijection on K,

(2) $(k_1k_2)^l = k_1^lk_2^l$ and $(k^{l_1})^{l_2} = k^{l_1l_2}$ for all $l, l_1, l_2 \in L$ and $k, k_1, k_2 \in K$.

In monography [3] there is proved the equivalence of definitions 1 and 3 of a semidirect product of groups.

Let $[L \times K, \circ]_\varphi$ be an algebraic structure constructed according to definition 2, where L and K are groups.

Now, we shall prove that every group $[L \times K, \circ]_\varphi$ has the form $(L \times K, \circ)_\alpha$ and conversely, every group $(L \times K, \circ)_\alpha$ can be considered as a group in the form $[L \times K, \circ]_\varphi$.

THEOREM 3. Let L and K be arbitrary groups. Then $[L \times K, \circ]_\varphi$ is a group if and only if $\varphi(l_1, k_1, l_2, k_2) = k_1^{l_2} k_2$ for all $l_1, l_2 \in L$, $k_1, k_2 \in K$ and conditions (1) and (2) are fulfilled.

P r o o f. (i) Suppose that $[L \times K, \circ]_\varphi$ is a group. For arbitrary pairs $\langle l_1, k_1\rangle, \langle l_2, k_2\rangle \in L \times K$ we have $\langle l_1, k_1\rangle \circ \langle l_2, k_2\rangle = \langle l_1 l_2, \varphi(l_1, k_1, l_2, k_2)\rangle$. On the other hand, $\langle l_1, k_1\rangle \circ \langle l_2, k_2\rangle = (\langle l_1, 1\rangle \circ \langle 1, k_1\rangle) \circ (\langle l_2, 1\rangle \circ \langle 1, k_2\rangle) =$

$$= (\langle l_1, 1\rangle \circ \langle l_2, 1\rangle) \circ [(\langle l_2^{-1}, 1\rangle \circ \langle 1, k_1\rangle \circ \langle l_2, 1\rangle) \circ \langle 1, k_2\rangle].$$

Notice that

(3) $\langle l_2^{-1}, 1\rangle \circ \langle 1, k_1\rangle \circ \langle l_2, 1\rangle = \langle 1, \varphi(l_2^{-1}, k_1, l_2, 1)\rangle$.

Then $\langle l_1, k_1\rangle \circ \langle l_2, k_2\rangle = \langle l_1 l_2, 1\rangle \circ [\langle 1, \varphi(l_2^{-1}, k_1, l_2, 1)\rangle \circ$

$$\circ \langle 1, k_2\rangle] = \langle l_1 l_2, 1\rangle \circ \langle 1, \varphi(l_2^{-1}, k_1, l_2, 1) k_2\rangle =$$

$$= \langle l_1 l_2, \varphi(l_2^{-1}, k_1, l_2, 1) k_2\rangle.$$

Hence

$$\langle l_1 l_2, \varphi(l_1, k_1, l_2, k_2)\rangle = \langle l_1 l_2, \varphi(l_2^{-1}, k_1, l_2, 1) k_2\rangle$$

and so $\varphi(l_1, k_1, l_2, k_2) = \varphi(l_2^{-1}, k_1, l_2, 1) k_2$.

Putting $k_1^{l_2} = \varphi(l_2^{-1}, k_1, l_2, 1)$ we have $\varphi(l_1, k_1, l_2, k_2) = k_1^{l_2} k_2$.

It follows from the condition (a_5) that for any fixed element l_2 the mapping $k_1^{l_2}$ is a bijection on K.

We shall show that $(k_1k_2)^{l} = k_1^{l}k_2^{l}$ and $(k^{l_1})^{l_2} =$ $= k^{l_1l_2}$ for all $l, l_1, l_2 \in L$, $k, k_1, k_2 \in K$.

Applying (3) we get

$$\langle 1,\varphi(l^{-1},k_1k_2,l,1)\rangle = \langle l^{-1},1\rangle\circ\langle 1,k_1k_2\rangle\circ\langle l,1\rangle =$$
$$= \langle l^{-1},1\rangle\circ\langle 1,k_1\rangle\circ\langle 1,k_2\rangle\circ\langle l,1\rangle = (\langle l^{-1},1\rangle\circ\langle 1,k_1\rangle\circ$$
$$\circ\langle l,1\rangle)\circ(\langle l^{-1},1\rangle\circ\langle 1,k_2\rangle\circ\langle l,1\rangle) = \langle 1,\varphi(l^{-1},k_1,l,1)\rangle\circ$$
$$\circ\langle 1,\varphi(l^{-1},k_2,l,1)\rangle = \langle 1,\varphi(l^{-1},k_1,l,1)\varphi(l^{-1},k_2,l,1)\rangle.$$

Then $\varphi(l^{-1},k_1k_2,l,1) = \varphi(l^{-1},k_1,l,1)\varphi(l^{-1},k_2,l^{-1},1)$,

i.e. $(k_1k_2)^{l} = k_1^{l}\, k_2^{l}$.

Furthermore, $\langle 1,\varphi((l_1l_2)^{-1},k,l_1l_2,1)\rangle = \langle(l_1l_2)^{-1},1\rangle\circ$

$$\circ\langle 1,k\rangle\circ\langle l_1l_2,1\rangle = \langle l_2^{-1},1\rangle\circ(\langle l_1^{-1},1\rangle\circ\langle 1,k\rangle\circ\langle l_1,1\rangle)\circ$$
$$\circ\langle l_2,1\rangle = \langle l_2^{-1},1\rangle\circ\langle 1,\varphi(l_1^{-1},k,l_1,1)\rangle\circ\langle l_2,1\rangle =$$
$$= \langle 1,\varphi(l_2^{-1},\varphi(l_1^{-1},k,l_1,1),l_2,1)\rangle.$$

Hence, $\varphi((l_1l_2)^{-1},k,l_1l_2,1) = \varphi(l_2^{-1},\varphi(l_1^{-1},k,l_1,1),l_2,1)$,

i.e.

$$k^{l_1l_2} = (k^{l_1})^{l_2}.$$

(ii) Let $\varphi(l_1,k_1,l_2,k_2) = k_1^{l_2}k_2$ for all $l_1,l_2 \in L$, $k_1,k_2 \in K$ and let be conditions (1) and (2) fulfilled.

We shall prove that the mapping φ satisfies the conditions (a_1) - (a_5) of definition 2.

It is a routine matter to verify that the conditions (a_1) - (a_3) hold.

Notice that the mapping $g(x) = \varphi(l_1,k_1,l_2,x) = k_1^{l_2}x$ is a bijection of the group K onto itself. The fuction $h(x) =$ $= \varphi(l_1,x,l_2,k_2) = x^{l_2}k_2$ is also a bijection of the group K onto itself.

Therefore conditions (a_4) and (a_5) are fulfilled.

This completes the proof.

In view of Theorem 3 the definitions of a semidirect product of groups corresponding to definitions 2 and 3 for loops are equivalent.

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