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On a homogeneous functional inequality

In this paper we shall deal with the homogeneous functional inequality

(1) $$\psi[f(x)] \leqslant g(x)\psi(x)$$

related to the homogeneous functional equation

(2) $$\varphi[f(x)] = g(x)\varphi(x),$$

where f, g are given functions and ψ, φ are unknown functions.

D. Brydak has given in the paper [2] (cf. also [1]) some theorems about continuous solutions $\psi, \varphi: [0,\alpha) \to [0,\infty)$ of (1) and (2), respectively, vanishing only at the origin. In this paper we shall prove analogous theorems for continuous and non-negative solutions of (1) and (2) which can take the zero value not only at the origin.

In the sequel we shall assume the following hypothesis (H)

(i) $f: I \to I$ is a function strictly increasing and continuous in the interval $I = [0, \alpha)$. Moreover $0 < f(x) < x$ for $x \in I_0 = (0, \alpha)$.

(ii) The function $g: I \to \mathbb{R}$ is continuous in I and $g(x) > 0$ for $x \in I$.

(iii) There exists a non-void open subinterval J of I such that the sequence

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] \qquad \text{for } x \in I,\ n \in \mathbb{N}, \tag{3}$$

where f^i denotes the i-th iterate of f, converges to zero, uniformly in J.

1. If hypothesis (H) is fulfilled, then equation (2) has a continuous solution in I depending on an arbitrary function and every continuous solution φ of equation (2) in I satisfies the condition $\varphi(0) = 0$ (cf. [4], p.48, Theorem 2.2).

Let U denote the union of all open (relatively to I) subsets of I on which the sequence $\{G_n\}_{n \in \mathbb{N}}$ converges uniformly to zero.

The following lemma has been proved in [5]:

LEMMA 1. Under hypothesis (H) for every continuous solution $\varphi: I \to \mathbb{R}$ of (2) we have

$$\varphi(x) = 0 \quad \text{for } x \in I \setminus U.$$

For $a \in \mathbb{R}$, $A \subset I$ and an arbitrary $\varphi: I \to \mathbb{R}$, if $a = \lim_{x \to 0} \varphi|_A(x)$, then we shall write

$$a = (A)\lim_{x \to 0} \varphi(x).$$

Let us denote by Φ the family of all continuous, non-negative solutions φ of equation (2) in I such that the set $N(\varphi) = I \setminus \varphi^{-1}(\{0\})$ is dense in U.

We define the following relation $\sim$ in the family Φ :

$$\varphi_1 \sim \varphi_2 \Leftrightarrow \bigvee_{a \in \mathbb{R}} a = (N(\varphi_1) \cap N(\varphi_2)) \lim_{x \to 0} \frac{\varphi_1(x)}{\varphi_2(x)} .$$

The following lemma gives some properties of relation $\sim$:

LEMMA 2. The relation $\sim$ is an equivalence relation in the set Φ . If for $\varphi_1, \varphi_2 \in \Phi$, $\varphi_1 \sim \varphi_2$, then there exists $a \in \mathbb{R}$ such that $\varphi_1 = a\varphi_2$.

P r o o f. The proof of the first part of this lemma is very simple thus it will be omitted.

Let φ_1, φ_2 fulfil the assumptions of lemma 2. Let $x_0 \in N(\varphi_1) \cap N(\varphi_2)$. Then the sequence $x_n = f^n(x_0)$ converges to zero and $f^n(x_0) \in N(\varphi_1) \cap N(\varphi_2)$ for $n=1,2,\ldots$, because φ_1 and φ_2 are solutions of equation (2) in I. Hence

$$\lim_{n \to \infty} \frac{\varphi_1(x_n)}{\varphi_2(x_n)} = a.$$

But, in view of (2),

$$\frac{\varphi_1(x_n)}{\varphi_2(x_n)} = \frac{\varphi_1(x_0)G_n(x_0)}{\varphi_2(x_0)G_n(x_0)} = \frac{\varphi_1(x_0)}{\varphi_2(x_0)} ,$$

and $\varphi_1(x_0) = a\,\varphi_2(x_0)$. Since the set $N(\varphi_1) \cap N(\varphi_2)$ is dense in U, then $\varphi_1(x) = a\varphi_2(x)$ for $x \in U$. According to the Lemma 1, the proof of the lemma is complete.

Let $\bar{\varphi} \in \Phi$. We denote by $\Phi(\bar{\varphi})$ the family of all functions $\varphi \in \Phi$ such that $\varphi \sim \bar{\varphi}$.

It is obvious that:

THEOREM 1. $\Phi(\bar{\varphi})$ is a one-parameter family of functions (cf. [2] p.21).

If the set $N(\bar{\varphi})$ is not dense in U, then Theorem 1 fails to be true. We shall show it by the following

E x a m p l e. Consider the functional equation

$$\varphi\left(\frac{x}{2}\right) = \frac{1}{2}\varphi(x) \quad \text{for} \quad x \in I = [0,1).$$

Let φ_0 and $\bar{\varphi}_0$ be two functions defined in the interval $\left[\frac{1}{4},\frac{1}{2}\right]$ by

$$\varphi_0(x) = \begin{cases} -(x-\frac{1}{4})(x-\frac{3}{8}) & \text{for} \quad x \in \left[\frac{1}{4},\frac{3}{8}\right) \\ -(x-\frac{3}{8})(x-\frac{1}{2}) & \text{for} \quad x \in \left[\frac{3}{8},\frac{1}{2}\right] \end{cases}$$

and

$$\bar{\varphi}_0(x) = \begin{cases} 0 & \text{for} \quad x \in \left[\frac{1}{4},\frac{3}{8}\right) \\ \varphi_0(x) & \text{for} \quad x \in \left[\frac{3}{8},\frac{1}{2}\right] \end{cases}.$$

If we extend the functions φ_0 and $\bar{\varphi}_0$ to continuous solutions of equation (2) on I (cf. [4], p.48, Theorem 2.2), then we get the functions φ, $\bar{\varphi}$ such that

$$(N(\varphi) \cap N(\bar{\varphi})) \lim_{x \to 0} \frac{\varphi(x)}{\bar{\varphi}(x)} = 1.$$

We also have that $\varphi \neq a\bar{\varphi}$ for every $a \in \mathbb{R}$. In this example the set $N(\bar{\varphi})$ is not dense in $U = I$.

2. In this section of the paper we deal with inequality (1). We assume that hypothesis (H) is fulfilled. If ψ is a continuous non-negative solution of (1) in I, then there exists the limit

$$\lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for} \quad x \in I_0,$$

where $G_n(x)$ is defined by formula (3), and the function

$$\varphi_0(x) = \begin{cases} \lim\limits_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} & \text{for } x \in I_0 \\ 0 & \text{for } x = 0 \end{cases} \tag{4}$$

is a solution of equation (2) in I, upper semi-continuous in I and continuous at zero (cf. [3], p.10).

We are going to give the following sufficient conditions for the solution φ_0 to be continuous in the whole I.

THEOREM 2. Let hypothesis (H) be fulfilled. If ψ is a solution of inequality (1) in I such that there exists a continuous solution φ of equation (2) in I fulfilling the condition

$$\varphi^{-1}(\{0\}) = \bigcup_{i=0}^{\infty} f^{-i}(\psi^{-1}(\{0\})), \tag{5}$$

where f^{-i} denotes the i-th iterate of the functions f^{-1}, and there exists the limit

$$a = (N(\varphi)) \lim_{x\to 0} \frac{\psi(x)}{\varphi(x)}, \tag{6}$$

then φ_0, defined by (4), is continuous in I.

P r o o f. If $x \in N(\varphi)$, then the sequence $x_n = f^n(x)$ converges to zero and $x_n \in N(\varphi)$ for $n=1,2,\ldots$. It implies, by virtue of (6), that there exists a sequence ε_n such that $\varepsilon_n \to 0$ and

$$\psi[f^n(x)] = (a + \varepsilon_n)\,\varphi[f^n(x)].$$

Hence, in view of (2),

$$\psi[f^n(x)] = (a + \varepsilon_n)\varphi(x)G_n(x)$$

and, consequently,

$$\lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} = a\,\varphi(x) \quad \text{for} \quad x \in N(\varphi).$$

If $x \notin N(\varphi)$, then (5) and (2) imply that $\psi[f^n(x)] = 0$ for $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} = 0 = a\,\varphi(x).$$

Consequently $\varphi_0 = a\varphi$ in I and φ_0 is a continuous solution of equation (2) in I.

We conclude the paper with two simple remarks. Denote by Φ_ψ the family of the continuous solutions φ of equation (2) in I such that for the given solution ψ of inequality (1) in I conditions (5), (6) are fulfilled.

R e m a r k 1. If for a certain $\varphi \in \Phi_\psi$ the limit a defined by (6) is different from zero, then

$$(N(\varphi_0)) \lim_{x\to 0} \frac{\psi(x)}{\varphi_0(x)} = 1,$$

where φ_0 is defined by (4).

R e m a r k 2. Φ_ψ is a one-parameter family of functions.

References

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