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On n-homomorphisms in Brandt groupoids

Introduction. In the note [1] has been proved that if $(G,\cdot)$, $(K,\cdot)$ are groups, $RR \subset R$, $R \cup R^{-1} = G$ and $h: R \to K$ is a homomorphism then there exists exactly one homomorphism $\bar{h}: G \to K$ being an extension of h, and the form of $\bar{h}$ has been given. In the note [2] the authors generalized the results of the note [1] considering the Ehresmann groupoids instead of groups $(G,\cdot)$, $(K,\cdot)$. In this note we examine n-homomorphisms defined on R and their extensions considering the Brandt groupoid $(B,\cdot)$ instead of the group $(G,\cdot)$ and the semigroup $(M,\cdot)$ instead of the group $(K,\cdot)$.

Definitions and notations. In [3] W.Waliszewski gave the following definition of Ehresmann groupoid: The pair $(E,\cdot)$, where E is a nonempty set and $\cdot$ is a binary operation defined for some pairs $(x,y) \in E \times E$ will be called the Ehresmann groupoid if the following conditions are satisfied

a) If in equation $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ one of its sides or both of the products $y \cdot z$, $x \cdot y$ are defined, then both sides if this equation are defined and the equality holds,

b) For every element $x \in E$ there exists exactly one left unit f_x and exactly one right unit e_x such that $f_x \cdot x = x = x \cdot e_x$,

c) If the product $x \cdot y$ is defined, then $e_x = f_y$,

d) For every element $x \in E$ there exists exactly one element x^{-1} (inverse to x) such that $x \cdot x^{-1} = f_x$ and $x^{-1} \cdot x = e_x$.

An Ehresmann groupoid $(E, \cdot)$ is called a Brandt groupoid if the following condition is satisfied

e) For every elements $x,y \in E$ there exists an element $z \in E$ such that the products $x \cdot z$, $z \cdot y$ are defined. Every group is a Brandt groupoid, of course.

If $(E, \cdot)$ is an Ehresmann groupoid and A, B are subsets of E then we shall use the following notations

$AB := \{xy : x \in A,\ y \in B$ and the product xy is defined$\}$,

$A^{-1} := \{x : x \in E,\ x^{-1} \in A\}$,

$I_A := \{e \in E : ex = x$ or $xe = x$ for some $x \in A\}$,

$I := I_E$.

Below we give some properties of Ehresmann groupoid $(E,\cdot)$ which will be used in the sequel (they can be easily verified by the reader).

a) $(x^{-1})^{-1} = x$ for $x \in E$,

b) $f_{x_1} = f_{x_1 \cdot \ldots \cdot x_n}$ for $x_1, \ldots, x_n \in E$, $x_1 \cdot \ldots \cdot x_n$ defined,

c) $f_x = e_{x^{-1}}$, $e_x = f_{x^{-1}}$ for $x \in E$,

d) $(A^{-1})^{-1} = A$ for $A \subset E$,

e) if the product $x \cdot y$ is defined then $e_x = f_y$ for $x, y \in E$,

f) $e_x = f_x = x = x^{-1}$ for $x \in I$,

g) $(A \cap B)\cdot(C \cap D) \subset AC \cap AD \cap BC \cap BD$ for $A,B,C,D \subset E$,

h) $(A \cup B)\cdot(C \cup D) = AC \cup AD \cup BC \cup BD$ for $A,B,C,D \subset E$,

i) $(AB)^{-1} = B^{-1} \cdot A^{-1}$ for $A, B \subset E$,

j) $(A \cap B)^{-1} = A^{-1} \cap B^{-1}$, $(A \cup B)^{-1} = A^{-1} \cup B^{-1}$, $(A \setminus B)^{-1} = A^{-1} \setminus B^{-1}$ for $A, B \subset E$.

Moreover it is well known that every Ehresmann groupoid is a sum of disjoint Brandt groupoids. By this property the results of this paper can be used for Ehresmann groupoids.

D_f will denote the domain of the function f and $f^{-1}(x)$ the element inverse to $f(x)$.

If $(K,\cdot)$, $(L,\cdot)$ are multiplicative systems then a function g the domain of which is contained in K and the range of which is contained in L will be called an n-homomorphism iff the equality $g(x_1 \cdot \ldots \cdot x_n) = g(x_1) \cdot \ldots \cdot g(x_n)$ holds whenever $x_1, \ldots, x_n \in K$ and the product $x_1 \cdot \ldots \cdot x_n$ is defined.

In the case $n = 2$ we say homomorphism instead of 2-homomorphism.

The function $g: x \to -x^2$, $x \in \mathbb{R} \setminus \{0\}$, is a simple example of 3-homomorphism of $(\mathbb{R} \setminus \{0\}, \cdot)$ into $(\mathbb{R} \setminus \{0\}, \cdot)$.

A function g will be called an extension of the function f if $f \subset g$. In this case we say also that g is an extension of f onto the set D_g.

In this note R will denote a subset of a Brandt groupoid B such that the following condition is satisfied

$$RR \subset R, \qquad R \cup R^{-1} = B. \tag{1}$$

It is easy to verify that the following condition is fulfilled

$$I = I_R, \qquad I \subset R \cap R^{-1}, \qquad R^{-1} \cdot R^{-1} \subset R^{-1}. \tag{2}$$

$(M, \cdot)$ will denote a semigroup such that $M = H \cup \{0\}$, where $(H, \cdot)$ is a group and $x \cdot 0 = 0 \cdot x = 0$ for $x \in M$.

Finally, for a function g the range of which is contained in the set M we put

$$N_g := \{x : x \in D_g, \ g(x) \neq 0\}.$$

Homomorphisms. In this section we consider homomorphisms of $(R, \cdot)$ into $(M, \cdot)$ and their extensions.

LEMMA 1. If $(E, \cdot)$ is an Ehresmann groupoid being a subgroupoid of $(B, \cdot)$ and $N := R \cap E$, then the following conditions are fulfilled

a) $NN \subset N, \quad N^{-1} \cdot N^{-1} \subset N^{-1},$ (3)

b) $N \cup N^{-1} = E,$ (4)

c) $R \setminus N = R \setminus E,$ (5)

d) $R \setminus N \cup (R \setminus N)^{-1} \cup E = B,$ (6)

e) $R \setminus N \cdot N \subset R \setminus N,$ (7)

f) $N \cdot R \setminus N \subset R \setminus N.$ (8)

Moreover, if the following condition

$$R \setminus N \cdot R \setminus N \subset R \setminus N \tag{9}$$

holds, then the following conditions are fulfilled

g) $R \setminus E \cdot E \subset R \setminus E,$ (10)

h) $E \cdot R \setminus E \subset R \setminus E,$ (11)

i) $(R \cup E) \cdot (R \cup E) \subset R \cup E.$ (12)

P r o o f. We have

$N \cdot N = (R \cap E) \cdot (R \cap E) \subset RR \cap EE \subset R \cap E = N,$

$N^{-1} \cdot N^{-1} = (N \cdot N)^{-1} \subset N^{-1},$

$N \cup N^{-1} = (R \cap E) \cup (R \cap E)^{-1} = (R \cap E) \cup (R^{-1} \cap E^{-1}) =$

$= (R \cap E) \cup (R^{-1} \cap E) = (R \cup R^{-1}) \cap E = B \cap E = E,$

$R \setminus N = R \setminus (R \cap E) = R \setminus R \cup R \setminus E = R \setminus E,$

$R \setminus N \cup (R \setminus N)^{-1} \cup E = R \setminus E \cup (R \setminus E)^{-1} \cup E =$

$= R \setminus E \cup R^{-1} \setminus E \cup E = R \cup R^{-1} \cup E = B \cup E = B,$

thus conditions (3) - (6) hold. Suppose now that $R \setminus N \cdot N \not\subset R \setminus N$. Since $R \setminus N \cdot N \subset RR \subset R$, there exist elements $x \in R \setminus N$, $y \in N$ such that their product xy is defined and $xy \in N$. Then $xy, y \in E$ and, since E is an Ehresmann groupoid, also $x \in E$. Hence $x \in R \cap E = N$, which is a contradiction. Thus condition (7) is satisfied.

Analogously one can prove that (8) holds.

Now let us assume that condition (9) is fulfilled. Suppose that (10) is not satisfied. Then there exist elements $x \in R \setminus E$, $y \in E$ such that the product xy is defined and $xy \notin R \setminus E$. By (5) and (6) there are possible two cases

a) $xy \in (R \setminus E)^{-1}$ or b/ $xy \in E$.

In the case a) $y^{-1} \cdot x^{-1} \in R \setminus E$, whence, by (5), (9) and because $x \in R \setminus E$, we get $y^{-1} \in R \setminus E$, which is a contradiction, too. Thus condition (10) is fulfilled. Analogously one can prove that (11) holds. Finally we have

$$(R \cup E) \cdot (R \cup E) = RR \cup RE \cup ER \cup EE \subset R \cup RE \cup ER \cup E \subset$$
$$\subset R \cup (R \setminus E \cup E) \cdot E \cup E \cdot (R \setminus E \cup E) \cup E =$$
$$= R \cup R \setminus E \cdot E \cup E \cup E \cdot R \setminus E \subset R \cup R \setminus E \cup E = R \cup E,$$

which completes the proof.

LEMMA 2. Let $h: R \to M$ be a homomorphism and $E_h := N_h \cup N_h^{-1}$. Then the following conditions are fulfilled

a) E_h is an Ehresmann groupoid unless $E_h = \emptyset$,

b) $N_h = R \cap E_h$,

c) $R \setminus N_h \cdot R \setminus N_h \subset R \setminus N_h$,

d) the function $\bar{h}$ defined as follows

$$\bar{h}(x) := \begin{cases} 0 & \text{for } x \in R \setminus E_h, \\ h(x) & \text{for } x \in N_h, \\ h^{-1}(x^{-1}) & \text{for } x \in N_h^{-1} \setminus N_h \end{cases}$$

is the only extension of the homomorphism h onto the set $R \cup E_h$.

P r o o f. We shall write N, E instead of N_h, E_h. Let E be nonempty, let x,y belong to E and let the product xy be defined. The following cases are possible

$$a') \ x,y \in N, \quad b') \ x \in N^{-1}, \ y \in N^{-1}, \quad c') \ x \in N, \ y \in N^{-1}, \quad d') \ x \in N^{-1}, \ y \in N.$$

In the case a') we have $x \in N$, $y \in N$, $N \subset R$, $xy \in R$, $h(x) \neq 0 \neq h(y)$. Since h is a homomorphism, we have $h(xy) \neq 0$, which implies $xy \in N$. Thus $N \cdot N \subset N$, whence $N^{-1} \cdot N^{-1} \subset N^{-1}$. Let now $x \in N$, $y \in N^{-1}$. Then the following two cases are possible

$$c_1') \ xy \in R, \qquad c_2') \ xy \in R^{-1}.$$

In the case c_1'), since h is a homomorphism defined on R, we have $h(xy) \cdot h(y^{-1}) = h(x) \neq 0$. Thus $xy \in N$. In the case c_2') $y^{-1} \cdot x^{-1} \in R$ and we have $h(y^{-1} \cdot x^{-1}) \cdot h(x) = h(y^{-1}) \neq 0$, which gives $y^{-1} \cdot x^{-1} \in N$. Hence $xy \in N^{-1}$. Thus $N \cdot N^{-1} \subset E$. Analogously one can show, that $N^{-1} \cdot N \subset E$. From the above considerations it follows that $E \cdot E \subset E$.

Let now $x \in E$. Then $x \in N$ or $x \in N^{-1}$ and $h(x) \neq 0$ or $h(x^{-1}) \neq 0$. Since $e_x, f_x \in R$ and h is a homomorphism defined on R we obtain $h(f_x) \cdot h(x) \cdot h(e_x) = h(x) \neq 0$ or $0 \neq h(x^{-1}) = h(e_x) \cdot h(x^{-1}) \cdot h(f_x)$, whence $e_x, f_x \in E$. By the definition of the set E we obtain $E = E^{-1}$. Thus we have shown that $(E, \cdot)$ is an Ehresmann groupoid . Now we shall show that $N = R \cap E$. N is contained in $R \cap E$, of course. Let us suppose $R \cap E \not\subset N$. Then there exists an element $x \in R \cap E$ such that $x \notin N$. Thus $x \in R \cap N^{-1}$, whence $x^{-1} \in N$,

$f_x \in N$, $h(x^{-1}) \neq 0$. Since h is a homomorphism we have $h(x) \cdot h(x^{-1}) = h(f_x) \neq 0$, which is a contradiction. Thus the equality b) holds. Now we shall prove that condition c) is fulfilled. Of course, $R \setminus N \cdot R \setminus N \subset R$. Let us suppose that there exist elements $x, y \in R \setminus N$ such that the product xy is defined and $xy \in N$. Then $h(xy) \neq 0$, whence, because h is a homomorphism, $h(x) \neq 0$. This contradicts the fact that $x \in R \setminus N$ and therefore condition c) is satisfied.

The function $\bar{h}$ is well defined on the set $R \cup E$, because $(R \setminus E) \cup N \cup (N^{-1} \setminus N) = R \cup E$ and the sets $R \setminus E$, N, $N^{-1} \setminus N$ are disjoint. Moreover, by the definition of $\bar{h}$ we conclude that $h \subset \bar{h}$. From the equality

$$\begin{aligned}(R \cup E) \cdot (R \cup E) &= (R \setminus N \cup N \cup N^{-1}) \cdot (R \setminus N \cup N \cup N^{-1}) = \\ &= (R \setminus N) \cdot (R \setminus N) \cup (R \setminus N) \cdot N \cup (R \setminus N) \cdot (N^{-1} \setminus N) \cup \\ &\quad N \cdot (R \setminus N) \cup N \cdot N \cup N \cdot (N^{-1} \setminus N) \cup (N^{-1} \setminus N) \cdot (R \setminus N) \cup \\ &\quad \cup (N^{-1} \setminus N) \cdot N \cup (N^{-1} \setminus N) \cdot (N^{-1} \setminus N)\end{aligned}$$

and from lemma 1 we conclude that for $x, y \in R \cup E$ such that the product xy is defined the following cases are possible

1) $x \in R \setminus N$, $y \in R \setminus N$, $xy \in R \setminus N$,

2) $x \in R \setminus N$, $y \in N$, $xy \in R \setminus N$,

3) $x \in R \setminus N$, $y \in N^{-1} \setminus N$, $xy \in R \setminus N$,

4) $x \in N$, $y \in R \setminus N$, $xy \in R \setminus N$,

5) $x \in N$, $y \in N$, $xy \in N$,

6) $x \in N$, $y \in N^{-1} \setminus N$, $xy \in N$,

7) $x \in N$, $y \in N^{-1} \setminus N$, $xy \in N^{-1} \setminus N$,

8) $x \in N^{-1} \setminus N$, $y \in R \setminus N$, $xy \in R \setminus N$,

9) $x \in N^{-1} \setminus N$, $y \in N$, $xy \in N$,

10) $x \in N^{-1} \setminus N$, $y \in N$, $xy \in N^{-1} \setminus N$,

11) $x \in N^{-1} \setminus N$, $y \in N^{-1} \setminus N$, $xy \in N^{-1} \setminus N$.

The equality $\bar{h}(x) \cdot \bar{h}(y) = \bar{h}(xy)$ holds in the cases 1), 2), 4), 5) because of the inclusion $h \subset \bar{h}$. We shall show that this equality holds in cases 7) and 9). In other cases proofs are analogical. In case 7) we have $x \in N$, $y^{-1} \in N$, $y^{-1} \cdot x^{-1} \in N$. Hence from the definition of $\bar{h}$ and because h is a homomorphism we get

$$h(y^{-1} \cdot x^{-1}) \cdot h(x) = h(y^{-1}),$$

$$h(y^{-1} \cdot x^{-1}) = h(y^{-1}) \cdot h^{-1}(x),$$

$$h^{-1}((xy)^{-1}) = h(x) \cdot h^{-1}(y^{-1}),$$

$$\bar{h}(xy) = \bar{h}(x) \cdot \bar{h}(y).$$

In case 9) we get $x^{-1}, y, xy \in N$. Analogously as above we have

$$h(x^{-1}) \cdot h(xy) = h(y),$$

$$h(xy) = h^{-1}(x^{-1}) \cdot h(y),$$

$$\bar{h}(xy) = \bar{h}(x) \cdot \bar{h}(y).$$

Thus we showed that $\bar{h}$ is a homomorphism.

Finally let $\bar{h}_1$, $\bar{h}_2$ be extensions of h onto the set $R \cup E$. For $x \in N^{-1} \setminus N$ we have $x^{-1} \in N$ and $\bar{h}_1(x^{-1}) = h(x^{-1}) = \bar{h}_2(x^{-1})$. Hence, because E is an Ehresmann groupoid, $x, x^{-1}, f_x \in E$ and $\bar{h}_1$, $\bar{h}_2$ are homomorphisms, we get

$$\bar{h}_1(x) = \bar{h}_1^{-1}(x^{-1}) = \bar{h}_2^{-1}(x^{-1}) = \bar{h}_2(x),$$

which together with $h \subset \bar{h}_1$, $h \subset \bar{h}_2$ means that $\bar{h}$ defined in lemma 2 is the only extension of h onto the set $R \cup E$ and completes the proof.

THEOREM 1. Let $h: R \to M$ be a function and $E_h := N_h \cup N_h^{-1}$. Then h is a homomorphism if and only if the following conditions are fulfilled

a) E_h is an Ehresmann groupoid unless $E_h = \emptyset$,

b) $R \cap E_h = N_h$,

c) $R \setminus E_h \cdot R \setminus E_h \subset R \setminus E_h$,

d) there exists a homomorphism $\bar{h}: E_h \to H$ such that $\bar{h}(x) = h(x)$ for $x \in N_h$.

P r o o f. In virtue of lemma 2 one can conclude that conditions a) - d) are fulfilled when h is a homomorphism. Let now conditions a) - d) be satisfied, $x, y \in R$ and let the product xy be defined. Then $xy \in R$. By a), c) we can use lemma 1. From the equality $R = R \setminus N \cup N$ and conditions b), d), (3), (5), (7), (8) it follows that there are possible the following cases

1) $x \in R \setminus N$, $y \in R \setminus N$, $xy \in R \setminus N$,

2) $x \in R \setminus N$, $y \in N$, $xy \in R \setminus N$,

3) $x \in N$, $y \in R \setminus N$, $xy \in R \setminus N$,

4) $x \in N$, $y \in N$, $xy \in N$,

where $N := N_h$. It is easy to verify that in each of the above cases the equality $h(x) \cdot h(y) = h(xy)$ holds, which completes the proof.

LEMMA 3. Let $h: R \to M$ be a homomorphism and let $\bar{R}$ be a subset of B such that $R \subset \bar{R}$ and $\bar{R} \cdot \bar{R} \subset \bar{R}$. Let $\bar{h}: \bar{R} \to M$ be a homomorphism being an extension of h. Then $E_h = E_{\bar{h}}$, where $E_h := N_h \cup N_h^{-1}$, $E_{\bar{h}} := N_{\bar{h}} \cup N_{\bar{h}}^{-1}$.

P r o o f. Since $h \subset \bar{h}$, we have $N_h \subset N_{\bar{h}}$ and $E_h \subset E_{\bar{h}}$. Let $x \in E_{\bar{h}}$. First let us consider the case, where $x \in R$. Then $x \in R \cap E_{\bar{h}} \subset \bar{R} \cap E_{\bar{h}}$ and, by lemma 2, $x \in N_{\bar{h}}$ and $\bar{h}(x) \neq 0$. Since $h \subset \bar{h}$ and $x \in R$, the equality $h(x) = \bar{h}(x)$ holds. Thus $x \in R$ and $h(x) \neq 0$, which means that $x \in N_h \subset E_h$. In the case $x^{-1} \in R$ we get $x^{-1} \in \bar{R} \cap E_{\bar{h}} = N_{\bar{h}}$ and $\bar{h}(x^{-1}) \neq 0$. Hence and by the equality $h(x^{-1}) = \bar{h}(x^{-1})$ we have $h(x^{-1}) \neq 0$. Therefore $x^{-1} \in N_h$, whence $x \in N_h^{-1} \subset$ $\subset E_h$, which completes the proof.

THEOREM 2. Let $h: R \to M$ be a homomorphism. Then there exists a homomorphism $\bar{h}: B \to M$ being an extension of h if and only if $E_h = B$ or $N_h = \emptyset$, where $E_h := N_h \cup N_h^{-1}$.

P r o o f. It is easy to see that conditions $h = 0$ and $N_h = \emptyset$ are equivalent. Now let $h \neq 0$. Let $\bar{h}: B \to M$ be a homomorphism which is an extension of h, and suppose that $B \not\subset E_h = E_{\bar{h}}$ (cf. Lemma 3). Then there exist elements $a, b \in B$ such that $\bar{h}(a) \neq 0$ and $\bar{h}(b) = 0$. B is a Brandt groupoid, therefore there exists an element $c \in B$ such that the products bc and ca are defined. By the equality $c^{-1} \cdot b^{-1} \cdot bca = a$ and because $\bar{h}$ is a homomorphism defined on B we obtain the contradiction: $0 \neq \bar{h}(a) = \bar{h}(c^{-1}b^{-1}) \cdot$ $\cdot \bar{h}(b) \cdot \bar{h}(ca) = 0$, thus the equality $B = E_h$ holds. If the equality $E_h = B$ holds then $N_h = R \cap E_h = R \cap B = R$. Thus $h(x) \neq 0$ for $x \in R$. From lemma 2 it follows that there exists an extension of h onto the set $\bar{R} \cup E_h = R \cup B = B$, which completes the proof.

n-homomorphisms. In this section we assume that n is an integer and $n > 2$.

THEOREM 3. A function $g: R \to M$ is an n-homomorphism if and only if there exist functions $k: I \to H$ and $h: R \to M$ such that the following consitions are fulfilled

a) $[k(e)]^n = k(e)$ for $e \in I$, (13)

b) $k(f_x) \cdot h(x) = h(x) \cdot k(e_x)$ for $x \in R$, (14)

c) h is a homomorphism, (15)

d) $g(x) = k(f_x) \cdot h(x)$ for $x \in R$. (16)

Moreover, if for the triplets of functions (g, k_1, h_1), (g, k_2, h_2), where $g: R \to M$, $k_1, k_2: I \to H$, $h_1, h_2: R \to M$, conditions (13) - (16) hold, then $h_1 = h_2$.

P r o o f. Let $g: R \to M$ be an n-homomorphism. Let us put

$$k(e) := \begin{cases} g(e) & \text{for } e \in I \text{ and } g(e) \neq 0, \\ 1 & \text{for } e \in I \text{ and } g(e) = 0, \end{cases} \tag{17}$$

$$h(x) := k^{-1}(f_x) \cdot g(x) \quad \text{for } x \in R, \tag{18}$$

where 1 is the unit of the group H.
It is easy to see that $k(I) \subset H$ and $k^n(e) = k(e)$ for $e \in I$, whence

$$k^{n-2}(e) = k^{-1}(e) \quad \text{for } e \in I. \tag{19}$$

By the definition of the function h we get

$$h(x) = 0 \quad \text{iff} \quad g(x) = 0 \quad \text{for} \quad x \in R, \tag{20}$$

From the equalities

$$g^{n-1}(f_x)\cdot g(x) = g(f_x^{n-1}\cdot x) = g(x) = g(x\cdot e_x^{n-1}) = \\ = g(x)\cdot g^{n-1}(e_x) \quad \text{for} \quad x \in R \tag{21}$$

we conclude that the following condition

$$g(x) \neq 0 \quad \text{implies} \quad g(f_x) \neq 0 \quad \text{and} \quad y(e_x) \neq 0 \tag{22}$$

holds for $x \in R$.

Let $x \in R$. In the case $h(x) = 0$ equality (14) holds, of course. If $h(x) \neq 0$, then in virtue of (20), (18), (22), (17) we obtain

$$k(f_x) = g(f_x) \quad \text{and} \quad k(e_x) = g(e_x), \tag{23}$$

whence

$$k(f_x)\cdot h(x) = g(x) = g^{n-2}(f_x)\cdot g(x)\cdot g(e_x) = \\ = k^{-1}(f_x)\cdot g(x)\cdot k(e_x) = h(x)\cdot k(e_x).$$

Thus condition (14) is satisfied.

Let now $x,y \in R$ and let the product xy be defined. Then $xy \in R$, $f_x = f_{xy}$, $e_x = f_y$. If $g(x) = 0$ or $g(y) = 0$ then $0 = g(x)\cdot g^{n-2}(e_x)\cdot g(y) = g(xy)$, whence, by (20), $h(xy) = 0$. Thus the equality $h(x)\cdot h(y) = h(xy)$ in this case holds. If $g(x) \neq 0$ and $g(y) \neq 0$ then $g(f_x) \neq 0$, $g(e_x) \neq 0$ and we have $h(x)\cdot h(y) = k^{-1}(f_x)\cdot g(x)\cdot k^{-1}(f_y)\cdot g(y) = k^{-1}(f_x)\cdot g(x)\cdot g^{n-2}(f_y)\cdot g(y) = k^{-1}(f_x)\cdot g(xy) = h(xy)$

Thus condition (15) is satisfied. The equality (16) follows from (18).

Let now for functions $g: R \to M$, $k: I \to H$, $h: R \to M$ conditions (13) - (16) be fulfilled and let $x_1,\dots,x_n \in R$ be such that the product $x_1\cdot\ldots\cdot x_n$ is defined. Then $f_{x_1} = f_{x_1\cdot\ldots\cdot x_n}$, $e_{x_i} = f_{x_{i+1}}$ for $i = 1,\dots,n-1$. We get by (13)-(16)

$$
\begin{aligned}
g(x_1)\cdot\ldots\cdot g(x_n) &= k(f_{x_1})\cdot h(x_1)\cdot\ldots\cdot k(f_{x_n})\cdot h(x_n) = \\
&= k(f_{x_1})\cdot k^{n-1}(f_{x_1})\cdot h(x_1)\cdot\ldots\cdot h(x_n) = \\
&= k^{n}(f_{x_1})\cdot h(x_1\cdot\ldots\cdot x_n) = \\
&= k(f_{x_1})\cdot h(x_1\cdot\ldots\cdot x_n) = g(x_1\cdot\ldots\cdot x_n),
\end{aligned}
$$

which means that g is an n-homomorphism.

If for the triplets of functions (g,k_1,h_1), (g,k_2,h_2) conditions (13)-(16) are fulfilled, then

$$g(x) = k_1(f_x)\cdot h_1(x) = k_2(f_x)\cdot h_2(x) \quad \text{for } x\in R. \qquad (24)$$

If $x \in R$ and $g(x) = 0$ then $h_1(x) = 0 = h_2(x)$.

If $x \in R$ and $g(x) \neq 0$ then $h_1(f_x) = 1 = h_2(f_x)$, where 1 is the unit of the group H, and by (24) we get $g(f_x) = k_1(f_x) = k_2(f_x)$. Hence, by (24) we have $h_1(x) = h_2(x)$, which completes the proof.

THEOREM 4. Let functions $g: R \to M$, $k: I \to H$, $h: R \to M$ be such that conditions (13)-(15) are satisfied and let

$$g(x) = k(f_x)\cdot h(x) \quad \text{for } x \in R. \qquad (25)$$

Moreover let $\bar{R} \subset B$ be such that $R \subset \bar{R}$ and $\bar{R}\cdot\bar{R} \subset \bar{R}$. Then the following conditions

a) there exists a homomorphism $\bar{h}: \bar{R} \to M$ being an extension of h,

b) there exists an n-homomorphism $\overline{g}: \overline{R} \to M$ being an extension of g

are equivalent.

Moreover, if an extension of the homomorphism g onto the set $\overline{R}$ exists, then it is unique.

P r o o f. Let $\overline{g}: \overline{R} \to M$ be an n-homomorphism which is an extension of the n-homomorphism $g: R \to M$. Then there exists a function $\overline{k}: I \to H$ and a homomorphism $\overline{h}: \overline{R} \to M$ such that

$$\overline{g}(x) = \overline{k}(f_x) \cdot \overline{h}(x) \quad \text{for} \quad x \in \overline{R}. \tag{26}$$

We shall show that $\overline{h}$ is an extension of h. Let $x \in R$. If $g(x) = 0$ then $\overline{g}(x) = 0$, and by (25) and (26) $h(x) = 0 = \overline{h}(x)$. Thus in this case the equality $h(x) = \overline{h}(x)$ holds. If $g(x) \neq 0$ then $g(f_x) \neq 0$, $\overline{g}(f_x) \neq 0$ and $h(f_x) = 1 = \overline{h}(f_x)$.

Hence and from (25) and (26) be obtain $k(f_x) = g(f_x) = \overline{g}(f_x) = \overline{k}(f_x)$, whence by (25) and (26) we have $\overline{h}(x) = \overline{k}^{-1}(f_x) \cdot \overline{g}(x) = k^{-1}(f_x) \cdot g(x) = h(x)$. Thus condition b) implies condition a). To prove that condition a) implies condition b) it is enough to show (in virtue of theorem 3) the condition

$$k(f_x) \cdot \overline{h}(x) = \overline{h}(x) \cdot k(e_x) \quad \text{for} \quad x \in \overline{R} \setminus R \tag{27}$$

holds. Let $x \in \overline{R} \setminus R$. If $\overline{h}(x) = 0$, then equality (27) holds. Now let $\overline{h}(x) \neq 0$. Then we have $x^{-1} \in R \subset \overline{R}$ and $f_x \in \overline{R}$. Hence and by the fact, that $\overline{h}$ is a homomorphism we conclude that $h(x^{-1}) = \overline{h}^{-1}(x)$. For functions k, h condition (14) is fulfilled, whence

$$k(f_{x^{-1}}) \cdot h(x^{-1}) = h(x^{-1}) \cdot k(e_{x^{-1}}),$$
$$k(e_x) \cdot \bar{h}^{-1}(x) = \bar{h}^{-1}(x) \cdot k(f_x),$$
$$\bar{h}(x) \cdot k(e_x) = k(f_x) \cdot \bar{h}(x).$$

Thus condition (27) is fulfilled. By theorem 3 we conclude that if $\bar{g}: \bar{R} \to M$ is an n-homomorphism which is an extension of g, then it is the only such extension. If the Brandt groupoid $(B, \cdot)$ is a group, then $f_x = e_x = 1$ for $x \in B$, where 1 is the unit of the group $(B, \cdot)$, and therefore from theorems 1, 2 and lemma 2 we obtain the following

COROLLARY. If the Brandt groupoid $(B, \cdot)$ is a group with the unit 1, then the function $g: R \to M$ is an n-homomorphism if and only if there exist an element $a \in H$ and a homomorphism $h: R \to M$ such that the following conditions are fulfilled

a) $a^n = a$,

b) $a \cdot h(x) = h(x) \cdot a$ for $x \in R$,

c) $g(x) = a \cdot h(x)$ for $x \in R$.

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