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On the solution of the equation $[n(x+y)-n(x-y)]^2 = 16n(x)n(y)$

§1. In this paper X denotes an abelian group and K denotes a commutative field subjected to the following conditions

a) if $2x = 0$ then $x = 0$ for $x \in K$, (1)

b) if $5x = 0$ then $x = 0$ for $x \in K$, (2)

c) to each $x \in K$ rhere exists $y \in K$ such that $x = y^2$. (3)

We shall consider the functional equation

$$[n(x+y) - n(x-y)]^2 = 16\, n(x)n(y) \quad \text{for } x,y \in X, \tag{4}$$

where n is a function mapping X into K. In the case when the values of function n are real numbers equation (4) was examined in [2] and [3].

LEMMA 1. Let $n: X \to K$ be a function satisfying (4) and let $M: X \times X \to K$ be a function defined as follows

$$M(x,y) := n(x+y) - n(x-y) \quad \text{for} \quad x,y \in X. \tag{5}$$

Then the following conditions are satisfied

a) $n(0) = 0,$ (6)

b) $n(x) = n(-x)$ for $x \in X,$ (7)

c) $n(2x) = 4n(x)$ for $x \in X,$ (8)

d) $[n(x) - n(y)]^2 = n(x+y)n(x-y)$ for $x,y \in X,$ (9)

e) $n(x+y) + n(x-y) = 2n(x) + 2n(y)$ for $x,y \in X,$ (10)

f) M is a function symmetric and additive with respect to each variable. (11)

P r o o f. If in (4) we set 0 instead of x, y then by (1) we obtain (6). If in (4) we set 0 instead of x then by (6) we obtain (7). Now let us put in (4) x inxtead of x and y. Using (6) we get

$$[n(2x)]^2 = 16[n(x)]^2 \quad \text{for} \quad x \in X, \tag{12}$$

whence

$$n(2x) = 4n(x) \quad \text{or} \quad n(2x) = -4n(x) \quad \text{for} \quad x \in X. \tag{13}$$

If $n(x) = 0$, then by (12) $n(2x) = 0$ and conclude that in this case (8) holds. Let us suppose now that for some $x \in X$ the following condition is fulfilled

$$n(x) \neq 0 \quad \text{and} \quad n(2x) = -4n(x). \tag{14}$$

By (14) and (13) we get

$$n(4x) = 16n(x) \quad \text{or} \quad n(4x) = -16n(x). \tag{15}$$

If in (4) we set 2x instead of x and x instead of y then, using (14) we obtain

$$[n(3x) - n(x)]^2 = -64[n(x)]^2. \tag{16}$$

If in (4) we replace x by 3x and y by x then, using (14),

(15) and (1) we have

$$[n(4x) + 4n(x)]^2 = 16n(3x)n(x),$$

$$16n(3x)n(x) = [20n(x)]^2 \quad \text{or} \quad 16n(3x)n(x) = [-12n(x)]^2,$$

whence

$$n(3x) = 25n(x) \quad \text{or} \quad n(3x) = 9n(x).$$

Hence and from (16) we get

$$2^7 \cdot 5[n(x)]^2 = 0 \quad \text{or} \quad 2^7[n(x)]^2 = 0,$$

which by (1) and (2) gives $n(x) = 0$ what contradicts to (14). From the above considerations and (13) we conclude that condition (8) is satisfied.

Putting in (4) $x+y$ instead of x and $x-y$ instead of y and using (8) and (1) we obtain

$$[n(2x) - n(2y)]^2 = 16\, n(x+y)n(x-y),$$

$$16[n(x) - n(y)]^2 = 16\, n(x+y)n(x-y),$$

$$[n(x) - n(y)]^2 = n(x+y)n(x-y),$$

thus condition (9) is fulfilled.

Now we shall prove condition (10). In virtue of (4) and (9) we have

$$[n(x+y) + n(x-y)]^2 = [n(x+y) - n(x-y)]^2 + 4n(x+y)n(x-y) =$$

$$= 16n(x)n(y) + 4[n(x) - n(y)]^2 =$$

$$= [2n(x) + 2n(y)]^2 ,$$

whence

$$n(x+y) + n(x-y) = 2n(x) + 2n(y)$$

or

$$n(x+y) + n(x-y) = -2n(x) - 2n(y) \quad \text{for} \quad x,y \in X. \qquad (17)$$

Let us suppose that there exist x,y $X\in$ such that

$$n(x+y) + n(x-y) = -2n(x) - 2n(y). \tag{18}$$

Let $a,b\in K$ be such that

$$a^2 = n(x), \qquad b^2 = n(y). \tag{19}$$

Using (4), (18) and (19) we get

$$[n(x+y) - n(x-y)]^2 = 16\ a^2b^2 \tag{20}$$

and

$$n(x+y) + n(x-y) = -2a^2 - 2b^2, \tag{21}$$

whence

$$n(x+y) = -(a-b)^2 \quad \text{and} \quad n(x-y) = -(a+b)^2 \tag{22}$$

or

$$n(x+y) = -(a+b)^2 \quad \text{and} \quad n(x-y) = -(a-b)^2. \tag{23}$$

Assume (22) (in the case (23) the proof is analogous and therefore is omitted).. If $b = 0$ then using (19), (8) we have

$$4a^4 = [-a^2-a^2]^2 = [n(x+y) - n(x)]^2 = n(2x+y)n(y) = 0.$$

Hence, in virtue of (1) we get $a = 0$ and (10) holds.

If $a = 0$ then using (22), (19), (8) we obtain

$$4b^4 = (-b^2-b^2)^2 = [n(x+y - n(y)]^2 = n(x+2y)n(x) = 0,$$

whence now $b = 0$ and (10) again holds.

Now let us consider the case $a \cdot b \neq 0$. By (19), (4), (8), (7) we get

$$16\ n(x+2y)\cdot a^2 = 16\ n(x+2y)n(x) = [n(2x+2y) - n(2y)]^2 =$$
$$= 16[n(x+y) - n(y)]^2 = 16[(a-b)^2 + b^2]^2,$$
$$16\ n(x-2y)\cdot a^2 = 16\ n(x-2y)n(x) = [n(2x-2y)^2 - n(2y)]^2 =$$
$$= 16[n(x-y) - n(y)]^2 = 16[(a+b)^2 + b^2]^2.$$

Hence in virtue of (19), (8), (1), (4) we have

$$16^2\cdot 64\ a^6\cdot b^2 = 16^2\cdot 64\ a^4\ n(x)n(y) = 16^2\cdot a^4\cdot 16n(x)n(2y) =$$
$$= 16^2\cdot a^4[n(x+2y) - n(x-2y)]^2 = [16\ a^2n(x+2y) - 16\ a^2\cdot$$
$$\cdot\ n(x-2y)]^2 = [16\langle(a-b)^2 + b^2\rangle^2 - 16\langle(a+b)^2 + b^2\rangle^2]^2 =$$
$$= 16^2[(-2b)(2a)(2a^2+4b^2)]^2 = 16^2\cdot 16\cdot 4\ a^2b^2(a^2+2b^2)^2,$$

whence

$$a^4 = (a^2 + 2b^2)^2, \quad \text{i.e.} \quad a^2 + b^2 = 0.$$

Hence and from (22) we get

$$n(x+y) + n(x-y) = -(a-b)^2 - (a+b)^2 = -2(a^2 + b^2) = 0 =$$
$$2(a^2 + b^2) = 2a^2 + 2b^2 = 2n(x) + 2n(y),$$

which means that in this case condition (10) is also fulfilled. The proof of (10) is completed.

Condition (11) was proved by S. Kurepa in [1] in the case when K is the set of real numbers. In our case the proof is analogical and therefore we omit it.

THEOREM 1. A function $n: X \to K$ satisfies (4) iff there exist and additive function $g: X \to K$ and a constant $a \in K$ such that

$$16\ n(x) = a[g(x)]^2 \quad \text{for} \quad x \in X. \tag{24}$$

P r o o f. Using (1) it is easy to verify that (24) implies (4). If $n = 0$ then (24) holds, of course. If there exists $y \in X$ such that $n(y) \neq 0$ then we put $a = [n(y)]^{-1}$,

$$g(x) = M(x,y) \quad \text{for} \quad x \in X,$$

where M is a function defined in (5). Using (4) and (11) we get

$$16\, n(x)n(y) = [n(x+y) - n(x-y)]^2 = [M(x,y)]^2 = [g(x)]^2,$$

whence

$$16\, n(x) = a[g(x)]^2,$$

which completes the proof.

§2. Now we shall consider the case where X is the additive group of the set of complex numbers and K is the set of complex numbers. It is well known, that the continuous additive functions $f: C \to C$ are of the form

$$f(z) = a \cdot \mathrm{re}\, z + b \cdot \mathrm{im}\, z \quad \text{for} \quad z \in C, \tag{25}$$

where $a, b \in C$ are arbitrary constants.

The following lemma is well known (see for instance [3], p.217).

LEMMA 2. An additive function $f: C \to C$ is continuous iff there exist $p, q \in C$ such that

$$f(z) = pz + q\bar{z} \quad \text{for} \quad z \in C. \tag{26}$$

In virtue of theorem 1 and lemma 2 we obtain the following theorem.

THEOREM 2. A function $n: C \to C$ is a continuous solution of equation (4) iff it is of the form

$$n(z) = (az + b\bar{z})^2 \quad \text{for} \quad z \in C,$$

where a, b are arbitrary complex constants.

References

[1] Kurepa S., The Cauchy functional equation and scalar product in vector space, Glasnik Mat-Fiz-Astr., Zagreb 1964, 23-36.

[2] Kurepa S., On a nonlinear functional equation, Glasnik Mat-Fiz, Zagreb 1965, 243-250.

[3] Aczél J., Lectures on functional equations and their applications, New York and London, 1966.

[4] Grząślewicz A., On the solution of the system of functional equations related to quadratic functionals, Glasnik Mat-Fiz, Zagreb, 1979, 77-82.