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On ordinary differential inequalities

1. Introduction. In this paper we shall deal with the differential inequality

$$D^+y^{(n-1)} \geqslant f(x,y,y',\dots,y^{(n-1)}).$$

In the first section we shall give some lemmas on Dini derivatives which will be needed in the sequel.

In the second section we shall prove some theorems on the differential inequalities using the notion of first integral. Those theorems will be generalizations of the results from the paper [1].

2. The Dini derivative of composed function. Let the real functions $\varphi_1, \varphi_2, \dots, \varphi_n$ be defined in the interval $I = (a,b)$. Put

$$\Phi(x) := (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)) \quad \text{for } x \in I.$$

Let the function f be defined in an open set $D \subset R^n$, such that $\Phi(I) \subset D$.

LEMMA 1. Let the functions $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ be differentiable and let φ_n be right-hand continuous at the point $x_0 \in I$. Assume that f is differentiable at the point $y_0 = (\mathring{y}_1, \ldots, \mathring{y}_n) = (\varphi_1(x_0), \ldots, \varphi_n(x_0))$ and $\frac{\partial f}{\partial y_n}(y_0) \neq 0$.

If $\frac{\partial f}{\partial y_n}(y_0) > 0$, then

$$(1) \qquad D^+(f \circ \Phi)(x_0) = \sum_{i=1}^{n-1} \frac{\partial f}{\partial y_i}(y_0) \cdot \varphi_i'(x_0) + \frac{\partial f}{\partial y_n}(y_0) D^+ \varphi_n(x_0).$$

If $\frac{\partial f}{\partial y_n}(y_0) < 0$, then

$$(2) \qquad D^+(f \circ \Phi)(x_0) = \sum_{i=1}^{n-1} \frac{\partial f}{\partial y_i}(y_0) \cdot \varphi_i'(x_0) + \frac{\partial f}{\partial y_n}(y_0) D_+ \varphi_n(x_0).$$

P r o o f. It follows from the differentiablity of f that there exists a function r defined in an open neighbourhood U of the point $(0,\ldots,0) \in R^n$ such that $r(0,\ldots,0) = 0$, r is continuous in $(0,\ldots,0)$ and

$$f(y)-f(y_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(y_0)(y_i-\mathring{y}_i) + \sum_{i=1}^{n} |y_i-\mathring{y}_i|\, r(y-y_0)$$

for $y-y_0 \in U$, $y = (y_1,\ldots,y_n)$.

Hence, putting $y = \Phi(x_0+h)$ we have

$$(3) \qquad \frac{(f\circ\Phi)(x_0+h) - (f\circ\Phi)(x_0)}{h} =$$

$$= \sum_{i=1}^{n}\left[\frac{\partial f}{\partial y_i}(y_0) + \operatorname{sgn}(\varphi_i(x_0+h)-\mathring{y}_i)\cdot r(\Phi(x_0+h)-y_0)\right] \frac{\varphi_i(x_0+h)-\mathring{y}_i}{h}$$

for $h > 0$, $\Phi(x_0+h)-y_0 \in U$.

It is obvious that

(4) $$\lim_{h\to 0^+}\left[\frac{\partial f}{\partial y_i}(y_0)+\mathrm{sgn}(\varphi_i(x_0+h)-\mathring{y}_i)\cdot r(\Phi(x_0+h)-y_0)\right]\frac{\varphi_i(x_0+h)-\mathring{y}_i}{h}=$$

$$=\frac{\partial f}{\partial y_i}(y_0)\cdot\varphi_i'(x_0)$$

for $i = 1,\dots,n-1$.

If $\frac{\partial f}{\partial y_n}(y_0) > 0$, then

(5) $$\limsup_{h\to 0^+}\left[\frac{\partial f}{\partial y_n}(y_0)+\mathrm{sgn}(\varphi_n(x_0+h)-\mathring{y}_n)\cdot r(\Phi(x_0+h)-y_0)\right]\frac{\varphi_n(x_0+h)-\mathring{y}_n}{h}=$$

$$=\frac{\partial f}{\partial y_n}(y_0)\,D^+\varphi_n(x_0)\,.$$

If $\frac{\partial f}{\partial y_n}(y_0) < 0$, then

(6) $$\limsup_{h\to 0^+}\left[\frac{\partial f}{\partial y_n}(y_0)+\mathrm{sgn}(\varphi_n(x_0+h)-\mathring{y}_n)\cdot r(\Phi(x_0+h)-y_0)\right]\frac{\varphi_n(x_0+h)-\mathring{y}_n}{h}=$$

$$=\frac{\partial f}{\partial y_n}(y_0)\,D_+\varphi_n(x_0)\,.$$

Combining (3), (4) and (5), (6) we have (1) and (2).

A simple consequence of Lemma 1 is

COROLLARY 1. Let $g: I \to K$, $f: K \to R$ and let I, K be open intervals in R. Assume that g is right-hand continuous at $x_0 \in I$, f differentiable at $g(x_0)$ and $f'(g(x_0)) \neq 0$.

If $f'(g(x_0)) > 0$, then

(7) $$D^+(f\circ g)(x_0) = f'(g(x_0))\,D^+ g(x_0)\,.$$

If $f'(g(x_0)) < 0$, then

8 $$D^+(f\circ g)(x_0) = f'(g(x_0))\,D_+ g(x_0)\,.$$

R e m a r k 1. If $f'(g(x_0)) = 0$, then Corollary 1 is false, as we can see in the following example:

E x a m p l e 1. Let

$$g(x) = \begin{cases} \sqrt{|x|}\sin\frac{1}{x} & \text{for } x \in R \setminus A \\ 0 & \text{for } x \in A \end{cases},$$

where $A = \{x: \sin\frac{1}{x} > 0\} \cup \{0\}$,

and $f(y) = y^2$. Hence

$$(f \circ g)(x) = \begin{cases} |x|\sin^2\frac{1}{x} & \text{for } x \in R \setminus A \\ 0 & \text{for } x \in A \end{cases}.$$

The functions g, f satisfy the assumptions of Corollary 1 at the point $x_0 = 0$ except $f'(g(x_0)) \neq 0$. A simple calculation shows that $D^+g(0) = 0$, $D_+g(0) = -\infty$ and $D^+(f \circ g)(0) = 1$. Therefore, the functions g, f satisfy neither (7) nor (8) at the point $x_0 = 0$.

R e m a r k 2. Corollary 1 is not true without the assumption of right-hand continuity, as the following counterexample shows.

E x a m p l e 2. Let $B = \{x: x = \frac{1}{n}, n \in N\} \cup \{0\}$ and

$$g(x) = \begin{cases} x & \text{for } x \in B \\ -1 & \text{for } x \in R \setminus B \end{cases}, \qquad f(y) = 2y^2 + y.$$

Hence

$$(f \circ g)(x) = \begin{cases} 2x^2 + x & \text{for } x \in B \\ 1 & \text{for } x \in R \setminus B \end{cases}$$

The function g is right-hand discontinuous at the point $x_0 = 0$ and $D^+g(0) = 1$, but $f'(g(0)) = f'(0) = 1$ and $D^+(f \circ g)(0) = +\infty$. Therefore $D^+(f \circ g)(0) \neq f'(0)D^+g(0)$.

Now we shall prove the lemma about the Dini derivative of composed function in the case when the internal function is differentiable.

LEMMA 2. Let the functions g,f be defined as in Corollary 1 and let g be differentiable at the point $x_0 \in I$ and $g'(x_0) \neq 0$.

If $g'(x_0) > 0$ and g is continuous in a right-hand neighbourhood of the point x_0, then

$$D^+(f \circ g)(x_0) = D^+f(g(x_0)) \cdot g'(x_0).$$

If $g'(x_0) < 0$ and g is continuous in a left-hand neighbourhood of the point x_0, then

$$D^+(f \circ g)(x_0) = D_-f(g(x_0)) \cdot g'(x_0).$$

P r o o f. We shall prove the first part of the lemma. The second part one can prove analogically.

Let $g'(x_0) > 0$ and let g be continuous in a right-hand neighbourhood of the point x_0. Hence we get that there exists a positive number h_0 such that

(a) g continuous in $[x_0, x_0 + h_0]$;

(b) $g(x) > g(x_0)$ for $x \in (x_0, x_0+h_0]$.

From (b) we have

$$D^+(f \circ g)(x_0) = \limsup_{h \to 0^+} \left[\frac{f(g(x_0+h)) - f(g(x_0))}{g(x_0+h) - g(x_0)} \cdot \frac{g(x_0+h) - g(x_0)}{h} \right].$$

Hence, from the properties of upper limit and from the condition $g'(x_0) > 0$ we get the inequality

(9) $$D^+(f \circ g)(x_0) \leqslant D^+f(g(x_0)) \cdot g'(x_0).$$

On the other hand, it follows from the definition of upper limit that there exists a sequence $\{y_n\}$ such that $y_n \xrightarrow[n\to\infty]{} g(x_0)^+$ and

(10) $$D^+f(g(x_0)) = \lim_{n\to\infty} \frac{f(y_n) - f(g(x_0))}{y_n - g(x_0)}.$$

It follows from (a) and (b) that there exists a number l such that $g([x_0,x_0+h_0]) = [g(x_0),l]$. It is obvious that $l > g(x_0)$. Without loss of generality we may assume that $\{y_n\} \subset (g(x_0),l]$, hence there exists a sequence $\{h_n\}$ such that $h_n \xrightarrow[n\to\infty]{} 0^+$ and $g(x_0+h_n) = y_n$. Hence, from (10) and from the definition of upper limit we get

(11) $$g'(x_0) \cdot D^+f(g(x_0)) = \lim_{n\to\infty} \frac{f(g(x_0+h_n))-f(g(x_0))}{h_n} \leqslant D^+(f\circ g)(x_0).$$

From (9) and (11) we obtain

$$D^+(f \circ g)(x_0) = D^+f(g(x_0)) \cdot g'(x_0).$$

R e m a r k 3. If either $g'(x_0) = 0$ or g is discontinuous in every right-hand (left-hand) neighbourhood of x_0, then Lemma 2 is fails.

E x a m p l e 3. Let $g(x) = -x^2$ and

$$f(y) = \begin{cases} 0 & \text{for } y \geqslant 0 \\ \sqrt{-y} & \text{for } y < 0 \end{cases}.$$

Hence $(f \circ g)(x) = |x|$. The functions g,f satisfy the assumptions od Lemma 2 at the point $x_0 = 0$ except

$g'(x_0) \neq 0$. It is obvious that $D^+(f \circ g)(0) = 1$, $D^+f(0) = 0$, $D_-f(0) = -\infty$. Therefore, Lemma 2 is false.

E x a m p l e 4. Let

$$g_1(x) = \begin{cases} \frac{1}{k\pi} & \text{for } x \in \left(\frac{1}{(k+1)\pi}, \frac{1}{k\pi}\right], \ k \in N \\ 0 & \text{for } x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} g_1(x) & \text{for } x \in \left[0, \frac{1}{\pi}\right) \\ -g_1(-x) & \text{for } x \in \left(-\frac{1}{\pi}, 0\right) \end{cases}.$$

The function g is differentiable at the point $x_0 = 0$, $g'(0) = 1$, but g is discontinuous in every right-hand neighbourhood of x_0.

Let

$$f(y) = \begin{cases} y \sin \frac{1}{y} & \text{for } y \neq 0 \\ 0 & \text{for } y = 0 \end{cases}.$$

Hence $(f \circ g)(x) = 0$ for $x \in \left(-\frac{1}{\pi}, \frac{1}{\pi}\right)$. Therefore $D^+(f \circ g)(0) = 0$, but $D^+ f(0) = 1$. Thus $D^+(f \circ g)(0) \neq D^+f(g(0)) \cdot g'(0)$.

R e m a r k 4. Similar relations hold for the other Dini derivatives.

3. <u>Ordinary differential inequalities</u>. Let us consider the differential equation

(12) $\quad y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \quad$ for $x \in I$,

where I is an open interval.

We assume that the function f fulfills the following hypothesis H_1

(i) f is defined and continuous in a region $D \subset I \times R^n$;

(ii) the initial value problem is uniquely solvable in D on I;

(iii) there exists a first integral $R(x,y,y_1,\dots,y_{n-1})$ of equation (12) defined and differentiable in D and such that $\frac{\partial R}{\partial y_{n-1}} > 0$ in D.

A function $R: D \to R$ will be called a first integral of equation (12), if and only if for every solution φ of equation (12) $R(x,\varphi(x), \varphi'(x),\dots, \varphi^{(n-1)}(x)) = \text{const.}$ for $x \in I$.

D. Brydak has proved in his paper [1] that the function ψ, defined and n times differentiable in I, is a solution of the differential inequality

$$\psi^{(n)}(x) \geqslant f(x,\psi(x), \psi'(x),\dots,\psi^{(n-1)}(x))$$

if and only if the function

(13) $$\eta(x) = R(x,\psi(x), \psi'(x),\dots,\psi^{(n-1)}(x))$$

is an increasing function in I.

We are going to give a generalization of that theorem.

THEOREM 1. Let hypothesis H_1 be fulfilled. Let $\psi \in C^{(n-1)}(I)$ and let its graph lie in D.

The function ψ is a solution of the differential inequality

(14) $$D^+\psi^{(n-1)}(x) \geqslant f(x,\psi(x),\psi'(x),\dots,\psi^{(n-1)}(x)) \quad \text{for } x \in I,$$

if and only if the function $\eta(x)$, defined by formula (13), is an increasing function in I.

The function ψ is a solution of the inequality opposite to (14), if and only if, the function η is a decreasing function in I.

P r o o f. We are going to prove the theorem for inequality (14). The proof for the opposite inequality is similar.

Let $\psi \in C^{(n-1)}(I)$ and $x \in I$. In view of hypothesis H_1 (ii), there exists a unique solution φ of equation (12) satisfying the conditions

(15) $\varphi(x) = \psi(x), \quad \varphi^{(i)}(x) = \psi^{(i)}(x)$ for $i=1,2,\ldots,n-1$.

Denote

$$z_0 = (x, \psi(x), \psi'(x), \ldots, \psi^{(n-1)}(x)).$$

From H_1 (iii), (13) and from Lemma 1 we get

$$D^+\eta(x) = \frac{\partial R}{\partial x}(z_0) + \frac{\partial R}{\partial y}(z_0)\cdot\psi'(x) + \ldots + \frac{\partial R}{\partial y_{n-2}}(z_0)\psi^{(n-1)}(x) + \\ + \frac{\partial R}{\partial y_{n-1}}(z_0)\cdot D^+\psi^{(n-1)}(x)$$

and

$$\frac{\partial R}{\partial x}(z_0) + \frac{\partial R}{\partial y}(z_0)\cdot\varphi'(x) + \ldots + \frac{\partial R}{\partial y_{n-1}}(z_0)\cdot\varphi^{(n)}(x) = 0$$

because R is a first integral of equation (12).

Hence

$$D^+\eta(x) = \frac{\partial R}{\partial y_{n-1}}(z_0)\left[D^+\psi^{(n-1)}(x) - \varphi^{(n)}(x)\right].$$

If η is a solution of (14), then from (14), (15) and from definition of φ we get

$$D^+\eta(x) \geqslant \frac{\partial R}{\partial y_{n-1}}(z_o)\left[f(x,\psi(x),\ldots,\psi^{(n-1)}(x)) - f(x,\varphi(x),\ldots,\varphi^{(n-1)}(x))\right] = 0.$$

Thus η is an increasing function in I.

Conversely, if η is an increasing function in I, then

$$\frac{\partial R}{\partial y_{n-1}}(z_o)\left[D^+\psi^{(n-1)}(x) - \varphi^{(n)}(x)\right] = D^+\eta(x) \geqslant 0.$$

Hence, from (15) and from the definition of φ we have

$$D^+\psi^{(n-1)}(x) \geqslant \varphi^{(n)}(x) = f(x,\varphi(x),\ldots,\varphi^{(n-1)}(x)) =$$
$$= f(x,\psi(x),\ldots,\psi^{(n-1)}(x)).$$

This ends the proof of the theorem.

Now let us consider the second order differential inequality

(16) $$D^+\psi'(x) \geqslant f(x,\psi(x),\psi'(x)) \quad \text{for} \quad x \in I.$$

We assume the following hypothesis H_2

(i) f is defined and continuous in a region $D \subset I \times R^2$;

(ii) the initial value problem, as well as the boundary value problem for the differential equation

(17) $$y'' = f(x,y,y') \quad \text{for} \quad x \in I$$

is uniquely solvable in D on I;

(iii) there exists a first integral R of equation (17) defined and differentiable in D such that $\frac{\partial R}{\partial y_1} > 0$ in D.

Let F be the family of all solutions of equation (17).

The function ψ will be called a convex (concave) function with respect to the fanily F if

$$\psi(x) \leqslant \varphi(x) \qquad (\psi(x \geqslant \varphi(x))$$

for all $x_1, x_2, x \in I$ with $x_1 < x < x_2$, where φ is such a solution of the equation (17), that $\varphi(x_i) = \psi(x_i)$ $i = 1,2$.

In paper [1] there was proved the following

THEOREM 2. Let $\psi \in C^1(I)$ and hypothesis H_2 be fulfilled. Denote

(18) $$\eta(x) = R(x, \psi(x), \psi'(x)) \quad \text{for} \quad x \in I.$$

The function ψ is covex (concave) with respect to the family F if and only if the function η is increasing (decreasing) in I.

From Theorems 1 and 2 we get the following

THEOREM 3. Let the hypothesis H_2 be fulfilled and $\psi \in C^1(I)$, such that its graph lies in D.

The function ψ is a solution of inequality (16) if and only if it is convex with respect to the family F.

The function ψ is a solution of the inequality opposite to (16) if and only if it is concave with respect to the family F.

Let us consider the differential inequality

(19) $$D^+\psi(x) \geqslant f(x, \psi(x)) \quad \text{for} \quad x \in I.$$

We assume the following hypothesis H_3

(i) f is defined and continuous in a region $D \subset I \times R$;

(ii) the initial value problem for the differential equation

(20) $$y' = f(x,y) \quad \text{for} \quad x \in I$$

in uniquely solvable in D on I;

(iii) there exists a first integral R of equation (20), defined and strictly increasing with respect to second variable in D.

THEOREM 4. Let $\psi \in C(I)$ be such that its graph lies in D and let hypothesis H_3 be fulfilled.

The function ψ is a solution of inequality (19) if and only if the function

(21) $$\eta(x) = R(x, \psi(x))$$

is increasing in I.

P r o o f. Let $\psi \in C(I)$ be a solution of (19) and let $x_1, x_2 \in I$ be such that $x_1 < x_2$. From hypothesis H_3 there exists a unique solution of equation (20) satisfying the condition

(22) $$\psi(x_1) = \varphi(x_1).$$

Hence, by a basic theorem on ordinary differential inqualities [2; Theorem 9.5] $\psi(x) \geqslant \varphi(x)$ for $x \geqslant x_1$, consequently $\psi(x_2) \geqslant \varphi(x_2)$.

Whence, from (22), from the definition of first integral, hypothesis H_3 (iii) and (21) we have

$$\eta(x_1) = R(x_1, \psi(x_1)) = R(x_1, \varphi(x_1)) = R(x_2, \varphi(x_2)) \leqslant$$
$$\leqslant R(x_2, \psi(x_2)) = \eta(x_2).$$

Now let η be increasing in I and $x_0 \in I$. We have to show that $D^+\psi(x_0) \geqslant f(x_0, \psi(x_0))$. Let φ be a solution of (20) such that $\varphi(x_0) = \psi(x_0)$. Since η is an increasing function, thus

$$\eta(x) = R(x\psi(x)) \geqslant R(x_0,\psi(x_0)) = \eta(x_0) \quad \text{for} \quad x \geqslant x_0$$

by virtue of (21), and

$$R(x_0,\psi(x_0)) = R(x_0,\varphi(x_0)) = R(x,\varphi(x)) \quad \text{for} \quad x \in I,$$

because $\varphi(x_0) = \psi(x_0)$ and R is a first integral.

Therefore

$$R(x,\psi(x)) \geqslant R(x,\varphi(x)) \quad \text{for} \quad x \geqslant x_0.$$

It follows from hypothesis H_3 (iii) that

$$\psi(x) \geqslant \varphi(x) \quad \text{for} \quad x \geqslant x_0.$$

This implies

$$D^+\psi(x_0) \geqslant \varphi'(x_0),$$

and

$$\varphi'(x_0) = f(x_0,\varphi(x_0)) = f(x_0,\psi(x_0))$$

because φ is a solution of (20) and $\varphi(x_0) = \psi(x_0)$.

Thus

$$D^+\psi(x_0) \geqslant f(x_0,\psi(x_0)).$$

References

[1] Brydak D., Application of generalized convex functions to second order differential inequalities, General Inequalities 4, Birkhäser Verlag, Basel, Stuttgart and New York (to appear).

[2] Szarski J., Differential inequalities, PWN, Warszawa 1967.