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Inequalities like Opial's inequality

Introduction. I have not been able to see Opial's paper [1]. Simpler proofs of his inequality appeared soon afterwards, by various writers and under various hypotheses. What follows is a brief outline of the literature, showing what seem to be the main features of the subsequent developments, but omitting many facets.

According to Beesack [2], Opial's original inequality was (in a notation to suit the present paper):

If $F(x) > 0$ in (a,b), $F(a) = 0 = F(b)$ and $F \in C^1[a,b]$, then

$$(1) \qquad \int_a^b |F(x)F'(x)|\,dx \leq \frac{1}{4}(b-a) \int_a^b |F'(x)|^2 dx.$$

Beesack [2] also gave various extensions of this, in which the hypotheses $F(x) > 0$ and $F \in C^1[a,b]$ were reduced to the requirements that F be real-valued and absolutely continuous on [a,b]. In one of these extensions the right side of (1) was replaced by

$$C \int_a^b p(x)F'(x)^2 dx$$

where p is a positive continuous function.

Among others, Pedersen [3] gave a very simple proof of (1), with F complex-valued and absolutely continuous, and $F(a) = 0$. The hypothesis that $F(b) = 0$ was omitted and the constant $\frac{1}{4}(b - a)$ doubled in consequence; this particular feature had also crept into some of the earlier proofs.

Boyd and Wong [4] obtained an inequality of the form (2) (below) with $q = 1$, $r = p + 1$ and

$$Tf(x) = \int_a^x f(t)\, dt.$$

They gave a fuller reference list than the one at the end of the present paper.

Boyd [5] discussed best possible constants C and extremal functions f for inequalities of the form

$$\int_a^b |T f|^p |f|^q d\mu_1 \leqslant C \left(\int_a^b |f|^r d\mu_2 \right)^{(p+q)/r}, \tag{2}$$

where T is an integral transform; the measures μ_1 and μ_2 were in most cases the same.

Fitzgerald [6] extended (1) in a different way, replacing the right side by

$$C \int_a^b |F^{(m)}(x)|^2 \, dx,$$

where m is any positive integer, $F \in C^m[a,b]$ and certain derivatives of F vanish at a and b in addition. It was partly this work of Fitzgerald that moved me to make the investigations which led to this paper.

Hardy's Inequality, integral version [7: Theorem 327], states that if f is a complex-valued function in $L^r(0,\infty)$, $\|\cdot\|$ is the $L^r(0,\infty)$ norm and $r > 1$, then

$$(3) \qquad \left\| \frac{1}{x} \int_0^x f(t) \, dt \right\| \leqslant \frac{r}{r-1} \|f\|.$$

In this paper I obtain first a generalization of Hardy's Inequality, Theorem 1 (compare [7: Theorems 319 and 330]). Theorem 2 is a comparably general inequality which has something of the character of Boyd's result, (2) (above); it is proved from Theorem 1. Corollaries 1, 2, 3 are successive specializations of Theorem 2, the last of these resembles Opial's Inequality.

Theorem 2 is not, however, an exact generalization of Opial's Inequality. The specialization of it which is closest to that inequality is

$$(4) \qquad \int_a^b |F(x)F'(x)| \, dx \leqslant 2(b-a) \int_a^b |F'(x)|^2 \, dx$$

for complex-valued F locally absolutely continuous in $[a,b)$ with $F(a) = 0$; this is Corollary 3 with $m = 1$ and $n = 0$. The main differences between (1) and (4) are that $F(b) = 0$

is not required in (4) and the constant $2(b-a)$ in (4) is much larger than its counterpart in (1). The latter is partly due to the former, but the main reason for it is probably the wastage involved in proving the much more general result, Theorem 2. In particular, there is wastage at (10), simply to obtain a neat constant; and in specializing Theorem 2 there is further wastage at (13).

THEOREM 1. If $s \geqslant r \geqslant 1$, $0 \leqslant a < b \leqslant \infty$, is real, $\omega(x)$ is decreasing and positive in (a,b), $f(x)$ and $H(x,y)$ are measurable and non-negative on (a,b), $H(x,y)$ is homogeneous of degree -1,

$$(Hf)(x) = \int_a^x H(x,y)f(y)\,dy$$

and

$$\|f\|_r = \left(\int_a^b f(x)^r x^{\delta-1}\omega(x)\,dx\right)^{1/r},$$

then

$$\|Hf\|_r \leqslant C\|f\|_s,$$

where

$$C = \int_{a/b}^1 H(1,t)t^{-\delta/r}\left(\int_a^{bt} x^{\delta-1}\omega(x)\,dx\right)^{\frac{1}{r}-\frac{1}{s}} dt.$$

Here a/b is to mean 0 if $a = 0$ or $b = \infty$ or both; and bt is to mean ∞ if $b = \infty$.

P r o o f. (i) For $a < x < b$ the homogeneity of H gives

$$(Hf)(x) = \int_{a/x}^1 H(x,xt)f(xt)x\,dt =$$

$$= \int_{a/x}^1 H(1,t)f(xt)\,dt,$$

where $t = \frac{y}{x}$. As $a \leqslant y \leqslant x$, then $0 < t \leqslant 1$.

Using Minkowski's Inequality at (5), and the decreasing property of ω at (6),

$$\|Hf\|_r = \left(\int_a^b \left(\int_{a/x}^1 H(1,t) f(xt)\, dt \right)^r x^{\gamma-1} \omega(x)\, dx \right)^{1/r}$$

$$(5) \qquad \leqslant \int_{a/b}^1 \left(\int_{a/t}^b H(1,t)^r f(xt)^r\, x^{\gamma-1} \omega(x)\, dx \right)^{1/r} dt$$

$$(6) \qquad \leqslant \int_{a/b}^1 H(1,t)\, t^{-\gamma/r} \left(\int_{a/t}^b f(xt)^r (xt)^{\gamma-1} \omega(xt)\, t\, dx \right)^{1/r} dt$$

$$(7) \qquad = \int_{a/b}^1 H(1,t)\, t^{-\gamma/r} \left(\int_a^{bt} f(y)^r\, y^{\gamma-1} \omega(y)\, dy \right)^{1/r} dt$$

$$\leqslant \int_{a/b}^1 H(1,t)\, t^{-\gamma/r} dt \left(\int_a^b f(y)^r\, y^{\gamma-1} \omega(y)\, dy \right)^{1/r}.$$

If $s = r$ this is the required inequality.

(ii) Suppose that $s > r$. By Hölder's Inequality with indices s/r and $s/(s-r)$, the inner integral in (7) is

$$\int_a^{bt} [f(y)]^r \{y^{\gamma-1}\omega(y)\}^{\frac{r}{s}} \{y^{\gamma-1}\omega(y)\}^{1-\frac{r}{s}}\, dy$$

$$\leqslant \left(\int_a^{bt} [f(y)]^s y^{\gamma-1} \omega(y)\, dy \right)^{\frac{r}{s}} \left(\int_a y^{\gamma-1} \omega(y)\, dy \right)^{1-\frac{r}{s}}$$

$$(8) \qquad \leqslant \|f\|_s^r \left(\int_a^{bt} y^{\gamma-1} \omega(y)\, dy \right)^{1-\frac{r}{s}}.$$

The required inequality follows from (7) and (8).

R e m a r k s. The constant C in Theorem 1 is greatly simplified in the case $s = r$. Further it becomes independent of ω, as well as of f.

Hardy's Inequality (3) is the case of Theorem 1 in which $a = 0$, $b = \infty$, $\gamma = 1$, $s = r > 1$, $\omega(x) = 1$ and $H(x,y) = 1/x$.

THEOREM 2. If $p > 0$, $q > 0$, $p + q = r \geqslant 1$, $0 \leqslant a < b \leqslant \infty$, $\gamma < r$, $\omega(x)$ is decreasing and positive in (a,b), $f(x)$ is measurable and non-negative on (a,b), I^{α} is the Riemann-Liouville operator of fractional integration defined by

$$(I^{\alpha}f)(x) = \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\, dt \quad \text{for } \alpha > 0,$$

$$I^{0}f(x) = f(x),$$

and $I^{\beta}f$ is defined similarly for $\beta \geqslant 0$, then

$$\text{(9)} \qquad \int_a^b [(I^{\alpha}f)(x)]^p [(I^{\beta}f)(x)]^q x^{\gamma-\alpha p-\beta q-1} \omega(x)\, dx \leqslant$$

$$\leqslant C \int_a^b [f(x)]^r x^{\gamma-1} \omega(x)\, dx,$$

where

$$C = \left(\frac{\Gamma(1-\gamma/r)}{\Gamma(\alpha+1)-\gamma/r)}\right)^p \left(\frac{\Gamma(1-\gamma/r)}{\Gamma(\beta+1-\gamma/r)}\right)^q.$$

P r o o f. In Theorem 1 let

$$H(x,y) = \frac{(x-y)^{\alpha-1}}{x^{\alpha}\,\Gamma(\alpha)} \quad \text{for } x > y > 0 \text{ and } \alpha > 0.$$

Then $H f(x) = x^{-\alpha} I^{\alpha} f(x)$, and so

$$\| x^{-\alpha} I^{\alpha} f(x) \|_r \leqslant A \|f\|_r \quad \text{with} \quad A = \int_{a/b}^1 \frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)} t^{-\gamma/r}\, dt.$$

This inequality holds à forriori with A replaced by

$$\text{(10)} \qquad A_0 = \int_0^1 \frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)} t^{-\gamma/r}\, dt = \frac{\Gamma(1-\gamma/r)}{\Gamma(\alpha+1-\gamma/r)};$$

and the resulting inequality also holds for $\alpha = 0$ obviously.

Similarly, for $\beta \geq 0$,

$$\| x^{-\beta}(I^{\beta}f)(x)\|_r \leq B_o \|f\|_r \quad \text{with} \quad B_o = \frac{\Gamma(1-\gamma/r)}{\Gamma(\beta+1-\gamma/r)}.$$

Using Hölder's Inequality with indices r/p and r/q, the left side of (9) is

$$\int_a^b \{x^{-\alpha}(I^{\alpha}f)(x)\}^p \{x^{-\beta}(I^{\beta}f)(x)\}^q \, x^{\gamma-1}\omega(x) \, dx$$

$$\leq \left(\int_a^b \{x^{-\alpha}(I^{\alpha}f)(x)\}^r \, x^{\gamma-1}\omega(x) \, dx\right)^{\frac{p}{r}}$$

$$\cdot \left(\int_a^b \{x^{-\beta}(I^{\beta}f)(x)\}^r \, x^{\gamma-1}\omega(x) \, dx\right)^{\frac{q}{r}}$$

$$= \| x^{-\alpha}(I^{\alpha}f)(x)\|_r^p \, \| x^{-\beta}(I^{\beta}f)(x)\|_r^q$$

$$\leq A_o^p \|f\|_r^p \, B_o^q \|f\|_r^q = C\|f\|_r^r.$$

COROLLARY 1. If a, b, γ, p, q, r and ω are as in Theorem 2, m and n are integers, $m \geq n \geq 0$, F is complex-valued and has $(m-1)$th derivative locally absolutely continuous in $[a,b)$, and $F(a) = F'(a) = \ldots = F^{(m-1)}(a) = 0$, then

$$\int_a^b |F(x)|^p |F^{(m-n)}(x)|^q x^{\gamma-mp-nq-1}\omega(x) \, dx$$

(11)

$$\leq \frac{1}{\{(1-\gamma/r)_m\}^p \{(1-\gamma/r)_n\}^q} \int_a^b |F^{(m)}(x)|^r x^{\gamma-1}\omega(x) \, dx,$$

where $(k)_m = k(k+1)(k+2)\ldots(k+m-1)$ and $(k)_o = 1$.

P r o o f. In Theorem 2 let $f(x) = |F^{(m)}(x)|$, $\alpha = m$ and $\beta = n$. For $m \geqslant n > 0$, Taylor's Theorem with remainder gives, for $a < x < b$,

$$F^{(m-n)}(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} F^{(m)}(t)\,dt,$$

$$|F^{(m-n)}(x)| \leqslant \int_a^x \frac{(x-t)^{n-1}}{\Gamma(n)} f(t)\,dt = (I^n f)(x);$$

and the inequality between the extreme terms holds trivially if $n = 0$. The left side of (11) is thus less than or equal to

$$\int_a^b [(I^m f)(x)]^p [(I^n f)(x)]^q x^{\gamma-mp-nq-1} \omega(x)\,dx,$$

$$\leqslant \left(\frac{\Gamma(1-\gamma/r)}{\Gamma(m+1-\gamma/r)}\right)^p \left(\frac{\Gamma(1-\gamma/r)}{\Gamma(n+1-\gamma/r)}\right)^q \int_a^b f(x)^r x^{\gamma-1} \omega(x)\,dx$$

by Theorem 2, and this proves Corollary 1.

COROLLARY 2. If $p > 0$, $q > 0$, $p + q = r > 1$, $-\infty < a < b < \infty$, $\omega(x)$ is decreasing and positive in (a,b) and F, m and n are as in Corollary 1, then

$$\int_a^b |F(x)|^p |F^{(m-n)}(x)|^q \omega(x)\,dx$$

(12)

$$\leqslant \frac{(b-a)^{mp+nq}}{\{(1-1/r)_m\}^p \{(1-1/r)_n\}^q} \int_a^b |F^{(m)}(x)|^r \omega(x)\,dx.$$

P r o o f. (i) Suppose that $a = 0$. Taking $\gamma = 1$ in Corollary 1, the right side of (12) is greater than or equal to

$$b^{mp+nq} \int_0^b |F(x)|^p |F^{(m-n)}(x)|^q x^{-mp-nq} \omega(x)\,dx$$

$$= \int_0^b |F(x)|^p |F^{(m-n)}(x)|^q \left(\frac{b}{x}\right)^{mp+nq} \omega(x)\, dx$$

(13) $$\geqslant \int_0^b |F(x)|^p |F^{(m-n)}(x)|^q \omega(x)\, dx.$$

(ii) Suppose that $a \neq 0$. Then (12) follows from (i) by translation. For if $\tau(x) = \omega(x+a)$ and $E(x) = F(x+a)$, (i) gives

$$\int_0^{b-a} |E(s)|^p |E^{(m-n)}(s)|^q \tau(s)\, ds$$

$$\leqslant \frac{(b-a)^{mp+nq}}{\{(1-1/r)_m\}^p \{(1-1/r)_n\}^q} \int_0^{b-a} |E^{(m)}(s)|^r \tau(s)\, ds,$$

from which (12) follows by substituting $s = x - a$.

COROLLARY 3. If $-\infty < a < b < \infty$, m and n are integers, $m,n \geqslant 0$, F is complex-valued and has $(m-1)$th derivative locally absolutely continuous in $[a,b)$, and $F(a) = F'(a) = \dots = F^{(m-1)}(a) = 0$, then

$$\int_a^b |F(x) F^{(m-n)}(x)|\, dx \leqslant \frac{(b-a)^{m+n}}{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n} \int_a^b |F^{(m)}(x)|^2\, dx.$$

P r o o f. In Corollary 2 take $p = 1 = q$ and $\omega(x) = 1$.

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