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A note on smoothness of the Stäckel transformation

Introduction. In 1892 P. Stäckel [1] proved that the most general form of pointwise transformations of the class C^n that convert the set of all solutions y of each linear homogeneous differential equation of the n-th order ($n \geqslant 2$) into the set of all solutions z of an equation of the same type is given by

(1) $$z(t) = f(t)\cdot y(h(t)) ,$$

see also [2].

The aim of this note is to show that if we suppose that a linear transformation of the form (1), with continuous functions f and h, transforms the set of all solutions of a linear homogeneous differential equation of the n-th order onto the set of all solutions of an equation of the same type then the n-times differentiability of the functions f and h follows.

This is a generalization of the result for the second order equations derived in [3] to an arbitrary order.

THEOREM. Let n be an integer, $n \geqslant 2$, and $I \subset \mathbb{R}$, $J \subset \mathbb{R}$ be two open intervals.

Suppose

$$y_i : I \longrightarrow \mathbb{R}; \quad y_i \in C^n(I), \quad i=1,\dots,n$$

and

$$z_i : J \longrightarrow \mathbb{R}; \quad z_i \in C^n(J), \quad i=1,\dots,n$$

are two n-tuples of real functions, whose Wronskians $W[y]$ and $W[z]$ are different from zero on I and J, respectively.

Let

(2) $$z_i(t) = f(t)\cdot y_i(h(t)); \quad t \in J; \quad i=1,\dots,n;$$

be satisfied for continuous functions f and h defined on J such that $h(J) = I$.

Then

$$f \in C^n(J), \quad f(t) \neq 0 \quad \text{for all} \quad t \in J,$$

$$h \in C^n(J), \quad dh(t)/dt \neq 0 \quad \text{for all} \quad t \in J,$$

i.e., h is a C^n-diffeomorphism of J onto I.

P r o o f.

First we have

(3) $$z_1^2(t)+\dots+z_n^2(t) = f^2(t)\left[y_1^2(h(t))+\dots+y_n^2(h(t))\right],$$

where both expressions in square brackets are different from zero for each $t \in J$, otherwise $z_i(t_o) = 0$ or $y_i(h(t_o)) = 0$ at some $t_o \in J$ and for all $i=1,\dots,n$.

Then $W[z](t_o) = 0$ or $W[y](h(t_o)) = 0$ that contradicts to our assumption.

Thus the relation (3) gives $f(t) \neq 0$ for all $t \in J$. Due to continuity of f on J, f is always positive or always negative on J:

(4) $$f(t) = \varepsilon\Big(\sum_{i=1}^{n} z_i^2(t) \Big/ \sum_{i=1}^{n} y_i^2(h(t)\Big)^{1/2}, \quad \varepsilon = \pm 1, \; t \in J.$$

Consider an arbitrary $t_o \in J$. Since $W[y](h(t_o)) \neq 0$, there exists (not unique for $n > 2$) a function

$$\bar{y}(t) = \sum_{i=1}^{n} c_i y_i(t), \qquad c_i = \text{const.},$$

such that

$$\bar{y}(h(t_o)) = 0 \quad \text{and} \quad \bar{y}'(h(t_o)) = 1.$$

Furthermore, let $\bar{\bar{y}}$ be a function of the form

$$\bar{\bar{y}}(t) = \sum_{i=1}^{n} k_i y_i(t), \qquad k_i = \text{const.},$$

such that

$$\bar{\bar{y}}(h(t_o)) = 1 \quad \text{and} \quad \bar{\bar{y}}'(h(t_o)) = 0.$$

Due to continuity of $\bar{\bar{y}}$ and h, there exists a vicinity $V(t_o)$ of t_o, where $\bar{\bar{y}} \circ h$ is nonvanishing. Let $U(h(t_o))$ denote a vicinity of $h(t_o)$, where y is nonvanishing.

Evidently

$$\bar{z}(t) := f(t) \cdot \bar{y}(h(t)) = \sum_{i=1}^{n} c_i z_i(t),$$

$$\bar{\bar{z}}(t) := f(t)\,\bar{\bar{y}}(h(t)) = \sum_{i=1}^{n} k_i z_i(t)$$

are of the class $C^n(J)$, $\bar{z}(t_o) = 0$ and $\bar{\bar{z}}(t_o) = f(t_o) \neq 0$.

Consider the relation

(5) $$\bar{z}(t)/\bar{\bar{z}}(t) - \bar{y}(x)/\bar{\bar{y}}(x) = 0$$

for $(t,x) \in V(t_o) \times U(h(t_o))$. Now, (5) is satisfied for $(t_o, h(t_o))$, and

$$F(t,x) := \bar{z}(t)/\bar{\bar{z}}(t) - \bar{y}(x)/\bar{\bar{y}}(x) \in C^n(V(t_o) \times U(h(t_o))),$$

gives

$$F_x(h(t_o)) = -\bar{y}'(h(t_o)) \cdot \bar{\bar{y}}(h(t_o)) \cdot (y(h(t_o)))^{-2}$$
$$= -1 \neq 0.$$

Since

$$\bar{z}(t)/\bar{\bar{z}}(t) - \bar{y}(h(t))/\bar{\bar{y}}(h(t)) = 0 \quad \text{on} \quad V(t_o),$$

the Implicit Function Theorem yields $h \in C^n(V^*(t_o))$ for some vicinity $V^*(t_o) \subset V(t_o)$ of t_o. Because $t_o \in J$ was arbitrarily chosen,

$$h \in C^n(J).$$

Due to the relation (4), also

$$f \in C^n(J).$$

Finally, if $h'(t_o) = 0$ for some $t_o \in J$, then (2) implies

$$z_i'(t_o) = f'(t_o) y_i(h(t_o)) \quad \text{for} \quad i=1,\dots,n,$$

and hence $z_i'(t_o) = [f'(t_o)/f(t_o)] \cdot z_i(t_o)$ for all i. However, this gives $W[z](t_o) = 0$, contrary to our supposition. Hence

$$dh(t)/dt \neq 0 \quad \text{for all} \quad t \in J,$$

and due to continuity of dh/dt,

$dh(t)/dt$ is always positive, or always negative on J.

Because $h(J) = I$ was already supposed,

h is a C^n-diffeomorphism of J onzo I, Q.E.D.

Reférences

[1] Stäckel P., Uber Transformationen von Differential-gleichungen. J. Reine Angew. Math. (Crelle Journal) 111 (1893), 290-302.

[2] Wilczyński E.J., Projective Differential Geometry of Curves and Ruled Surfaces. Teubner, Leipzig 1906.

[3] Neuman F., Note on Kummer's transformation. Archivum Math. (Brno) 6 (1970), 185-188.