JÓZEF TABOR

Rational iteration groups

Abstract. Let f be a continuous strictly increasing function mapping a closed interval $I \subset R$ onto itself. In the paper a construction of a rational iteration group f^t, $t \in Q$ of f is presented. A necessary and sufficient condition for this group to be continuous is also given. It is shown, by applying these results, that every continuous rational iteration group can be extended to a continuous real iteration group. Finally Zdun's problem is investigated: Can every real iteration group f^t, $t \in \mathbb{R}$ be written in the form $f^t = \tilde{f}^{\varphi(t)}$, where $\tilde{f}^t$, $t \in \mathbb{R}$ is a continuous real iteration group and φ is an additive function. The answer is "no". A necessary and sufficient condition for this to be possible is also given.

1. Let I be a closed interval in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $f\colon I \to I$ a strictly increasing continuous function such that $f(I) = I$ (if $-\infty \in I$ or $\infty \in I$ we assume that $f(-\infty) = \lim_{x\to\infty} f(x) = -\infty$ or $f(\infty) = \lim_{x\to\infty} f(x) = \infty$ respectively). By $F[f]$ we denote the set of fixed points of f. The set $F[f]$ is closed (cf. [4]) and hence $I \setminus F[f]$ consists of at most denumerably many disjoint open intervals. For any fixed $x \in I \setminus F[f]$ we denote by $(a(x), b(x))$ a maximal open interval such that $x \in (a(x), b(x)) \subset I \setminus F[f]$. In other words $(a(x), b(x))$ denotes an interval such that $x \in (a(x), b(x))$, $f(a(x)) = a(x)$, $f(b(x)) = b(x)$ and $f(z) \neq z$ for $z \in (a(x), b(x))$. If $x \in F[f]$ then we put $a(x) = b(x) = x$.

We introduce the following definitions (cf [4] and [5]).

DEFINITION 1. The iterates f^n, $n \in \mathbb{Z}$ of f are defined as follows

$$f^0 = id_I, \qquad f^{n+1} = f \circ f^n \qquad \text{for} \quad n = 0, 1, 2, \ldots,$$
$$f^{n-1} = f^{-1} \circ f^n \qquad \text{for} \quad n = 0, -1, -2, \ldots,$$

where f^{-1} denotes the function inverse to f. Throughout this paper upper indices will denote iterates.

DEFINITION 2. A family of continuous functions $\{f^t\colon I \to I,\ t \in \mathbb{R}\}$ ($\{f^t\colon I \to I,\ t \in Q\}$) is called a real (rational) iteration group of f whenever $f^t \circ f^s = f^{t+s}$ for $t, s \in \mathbb{R}$ ($t, s \in Q$) and $f^1 = f$.

DEFINITION 3. A real (rational) iteration group $\{f^t,\ t \in \mathbb{R}\}$ ($\{f^t,\ t \in Q\}$) is said to be continuous if for every $x \in I$ the mapping $\mathbb{R} \ni t \to f^t(x) \in I$ $(Q \ni t \to f^t(x) \in I)$ is continuous.

A real (rational) iteration group will be written shortly as r.i.g. (q.i.g.) and a continuous real (rational) iteration group as c.r.i.g. (c.q.i.g.) respectively.

2. In this section we are going to investigate a rational iteration group. At first we shall describe the construction of a rational iteration group of f.

THEOREM 1. Every q.i.g. of f can be obtained by putting

(i) $$f^1 = f,$$

(ii) $$f^{\frac{1}{(n+1)!}} = \psi_n \quad \text{for} \quad n \in N,$$

where ψ_n is an arbitrary continuous and strictly increasing solution of the equation

$$(\psi_n)^{n+1} = f^{\frac{1}{n!}} \text{ 1/}.$$

(iii) $$f^{\frac{m}{n}} = \left(f^{\frac{1}{n!}}\right)^{m(n-1)!} \quad \text{for} \quad n \in N,\ m \in \mathbb{Z}.$$

P r o o f. Every q.i.g. of f satisfies obviously the conditions (i) - (iii).

We are going to prove now that the construction (i) - (iii) defines a q.i.g. of f. Since $f(I) = I$ we have by (ii) $f^{\frac{1}{n!}}(I) = I$ for $n \in N$ (cf. [4] p.297), which implies that the function $\left(f^{\frac{1}{n!}}\right)^{m(n-1)!}$, $n \in N$, $m \in \mathbb{Z}$ maps I onto itself.

1/ The general continuous strictly monotonic solution of the equation $\varphi^n = g$ is given in [4].

To show that the function f^r, $r \in Q$ is "well defined" we must prove that f^r is independent of the representation of r in the form $r = \frac{m}{n}$, i.e. we must prove the equality

$$(1) \qquad f^{\frac{m}{n}} = f^{\frac{mp}{np}} \quad \text{for} \quad n,p \in N,\ m \in Z.$$

Making use of the properties of iterates for integer exponents we obtain from (iii)

$$f^{\frac{mp}{np}} = \left[f^{\frac{1}{(np)!}}\right]^{(np-1)!mp} =$$

$$= \left\{\left[f^{\frac{1}{np\,!}}\right]^{\frac{(np)!}{n!}}\right\}^{(n-1)!m} =$$

$$= \left\{\left[f^{\frac{1}{(np)!}}\right]^{np(np-1)\dots(n+1)}\right\}^{(n-1)!m} .$$

But by (ii)

$$\left[f^{\frac{1}{np\,!}}\right]^{np} = f^{\frac{1}{(np-1)!}},$$

$$\left[f^{\frac{1}{(np-1)!}}\right]^{np-1} = f^{\frac{1}{(np-2)!}}, \quad \dots ,$$

$$\left[f^{\frac{1}{(n+1)!}}\right]^{n+1} = f^{\frac{1}{n!}}.$$

Applying this equalities we get

$$\left\{\left[f^{\frac{1}{(np)!}}\right]^{np(np-1)\dots(n+1)}\right\}^{(n-1)!m} =$$

$$= \left\{\left(\left[f^{\frac{1}{(np)!}}\right]^{np}\right)^{(np-1)\dots(n+1)}\right\}^{(n-1)!m} =$$

$$= \left\{\left[f^{\frac{1}{(np-1)!}}\right]^{(np-1)\dots(n+1)}\right\}^{(n-1)!m} =$$

$$= \dots = \left\{f^{\frac{1}{n!}}\right\}^{(n-1)!m} = f^{\frac{m}{n}}.$$

Thus (1) is valid.

It is necessary to prove yet that $f^s \circ f^t = f^{s+t}$ for $s,t \in Q$. We may assume that $s = \frac{m_1}{n}$, $t = \frac{m_2}{n}$, $n \in N$, $m_1, m_2 \in Z$. Then we get from (iii)

$$f^s \circ f^t = f^{\frac{m_1}{n}} \circ f^{\frac{m_2}{n}} = \left(f^{\frac{1}{n!}}\right)^{m_1 (n-1)!} \circ \left(f^{\frac{1}{n!}}\right)^{m_2 (n-1)!} =$$

$$= \left(f^{\frac{1}{n!}}\right)^{(m_1+m_2)(n-1)!} = f^{\frac{m_1+m_2}{n}} = f^{s+t}.$$

R e m a r k. A continuous and strictly monotonic solution of the equation $\varphi^n = f$ in a non-trivial case where $F[f] \neq I$ depends on an arbitrary function (cf.[3], Theorem 15.7). Hence, it follows from Theorem 1 that a non-trivial case a q.i.g. of f depends on denumerably many arbitrary functions.

THEOREM 2. If $\{f^t, t \in Q\}$ is a q.i.g. of f then the function $Q \ni t \longrightarrow f^t(x)$ is constant for $x \in F[f]$, strictly decreasing if $f(x) < x$ and strictly increasing if $f(x) > x$.

P r o o f. As it is known (cf. [4], p.299), if a function $g\colon I \longrightarrow I$ is continuous strictly increasing and such that $g(I) = I$, then for every strictly monotonic and continuous solution φ of the equation $\varphi^n = g$ we have

$$\varphi(x) = x \quad \text{for} \quad x \in F[g]$$

and if $g(x) < x$ $(g(x) > x)$ then $\varphi(z) < z$ ($\varphi(z) > z$ respectively) for $z \in (a(x), b(x))$.

Consider now the q.i.g. $\{f^t, t \in Q\}$ of f. If $x \in F[f]$ then making use of (ii), in view of the above remark, we get $f^{\frac{1}{n!}}(x) = x$ for $n \in N$ and consequently by (iii) $f^{t}(x) = x$ for $t \in Q$.

Consider now an $x_0 \in I \setminus F[f]$ and suppose that $f(x_0) < x_0$ (in the other case considerations run similarly). Then making use of (ii) and the remark at the beginning of the proof we obtain by induction

$$f^{\frac{1}{n!}}(x) < x \quad \text{for} \quad x \in (a(x_0), b(x_0)), \quad n \in N,$$

whence, by applying (iii) we get

$$f^{t}(x) < x \quad \text{for} \quad x \in (a(x_0), b(x_0)), \quad t \in Q, \ t > 0.$$

Let $t_1 < t_2$, $t_1, t_2 \in Q$. We have in view of the last inequality,

$$f^{t_2 - t_1}(x_0) < x_0$$

and consequently, as f^{t} is strictly increasing

$$f^{t_2}(x_0) = f^{t_1}(f^{t_2 - t_1}(x_0)) < f^{t_1}(x_0),$$

which means that the function $Q \ni t \to f^{t}(x_0)$ is strictly increasing. □

3. We shall consider a c.q.i.g. of f.

THEOREM 3. Assume that f^{t}, $t \in Q$ is a q.i.g. of f and

(2) $$\lim_{n \to \infty} f^{\frac{1}{n!}}(x) = x \quad \text{for all} \quad x \in I.$$

Then f^{t}, $t \in Q$ is a c.q.i.g.

P r o o f. We obtain immediately from (2)

(3) $$\lim_{n \to \infty} f^{-\frac{1}{n!}}(x) = x \quad \text{for all} \quad x \in I.$$

Making use of Theorem 2, (2) and (3) we get

$$\lim_{t\to 0} f^t(x) = x \quad \text{for} \quad x \in I$$

and consequently,

$$\lim_{t\to t_0} f^t(x) = \lim_{t\to t_0} f^{t-t_0}(f^{t_0}(x)) = f^{t_0}(x) \quad \text{for} \quad x \in I,$$

which means that the function $Q \ni t \to f^t(x)$ is continuous.

The following corollary follows directly from Theorem 2 and Theorem 3.

COROLLARY 1. Let f^t, $t \in Q$ be a q.i.g. of f. If there exist $\lim_{t\to 0} f^t(x)$ for all $x \in I$ then f^t, $t \in Q$ is a c.q.i.g.

P r o o f. Consider any $x \in I$ and suppose that the function $Q \ni t \to f^t(x)$ is decreasing (the proof in the other case is similar).

Then we have

$$\lim_{t\to 0^+} f^t(x) \geqslant f^0(x) = x \geqslant \lim_{t\to 0^-} f^t(x),$$

whence, by applying the assumption we get

$$\lim_{t\to 0^+} f^t(x) = x \quad \text{for all} \quad x \in I,$$

which, in virtue of Theorem 3, proves the assertion. ▭

THEOREM 4. If $F[f] \neq I$, then there exists a discontinuous q.i.g. of f.

P r o o f. Consider any $x_0 \in I \setminus F[f]$. Without loss of generality we may assume that $f(x_0) < x_0$. Then $f(x) < x$ for all $x \in (a(x_0), b(x_0))$. It follows from Lemma 15.6 of [4] (p.297) that for arbitrary $x_1 \in (a(x_0), b(x_0))$

with $f(x_0) < x_1 < x_0$ every continuous strictly increasing function $\varphi: [x_1, x_0] \to [a(x_0), b(x_0)]$ such that $x_0 > \varphi(x_0) > x_1$, $\varphi(x_1) = f(x_0)$ and $\varphi^n(x_0) = f(x_0)$ can be extended to a solution of the equation $\varphi^n = f$.

We are going to construct a q.i.g. of f. Put

$$\varepsilon_i = \frac{x_0 - f(x_0)}{2} \cdot \frac{1}{2^i} \quad \text{for} \quad n=1,2,\ldots$$

and choose $f^{\frac{1}{n!}}$, $n=2,3,\ldots$ in such a manner that

$$f^{\frac{1}{(n+1)!}}(x_0) = f^{\frac{1}{n!}}(x_0) + \varepsilon_{n+1} \quad \text{for} \quad n=1,2,\ldots$$

Then we have for $n \geqslant 2$

$$f^{\frac{1}{n!}}(x_0) = f(x_0) + \varepsilon_2 + \ldots + \varepsilon_n < f(x_0) + \sum_{i=1}^{\infty} \varepsilon_i =$$

$$= f(x_0) + \frac{x_0 - f(x_0)}{2} = \frac{x_0 + f(x_0)}{2} < x_0.$$

By applying Theorem 1 (more precisely by (iii)) we obtain q.i.g. of f. Since

$$f^{\frac{1}{n!}}(x_0) < \frac{x_0 + f(x_0)}{2} < x_0,$$

we have $\lim\limits_{n\to\infty} f^{\frac{1}{n!}}(x_0) \neq x_0$, which proves that this group is discontinuous. $\square$

We are going to give another condition equivalent to the continuity of a q.i.g. But first we shall prove a very useful lemma.

LEMMA 1. Let f^t, $t \in Q$ be a q.i.g. of f. If the set $\{f^t(x_0),\ t \in Q\}$ is dense in $[a(x_0), b(x_0)]$, then the oscillation of the function $Q \ni t \to f^t(x_0)$ vanishes at every $t_0 \in \mathbb{R}$.

P r o o f. For $x_0 \in F[f]$ the statement follows directly from Theorem 2. Assume that $x_0 \in F[f]$. Then, in virtue of Theorem 2 the function $Q \ni t \to f^t(x_0)$ is strictly monotonic. We may assume that it is decreasing. Hence for every $t_0 \in \mathbb{R}$ there exist $\lim_{t \to t_0^+, t \in Q} f^t(x_0)$, $\lim_{t \to t_0^-, t \in Q} f^t(x_0)$ and obviously

$$\lim_{t \to t_0^+} f^t(x_0) \leqslant \lim_{t \to t_0^-} f^t(x_0) .$$

Suppose (contrary to our assertion) that the oscillation of the function $Q \ni t \to f^t(x_0)$ at the point $t_0 \in \mathbb{R}$ is greater then zero. Then there must be

$$\lim_{t \to t_0^+} f^t(x_0) \quad \lim_{t \to t_0^-} f^t(x_0)$$

and

$$f^{\tau}(x_0) \leqslant \lim_{t \to t_0^+} f^t(x_0) \quad \text{for } \tau > t_0,\ \tau \in Q,$$

$$f^{\tau}(x_0) \quad \lim_{t \to t_0^-} f^{r}(x_0) \quad \text{for } \tau < t_0,\ \tau \in Q.$$

But $f^{\tau}(x_0) \in (a(x_0), b(x_0))$ for every $\tau \in Q$.
Hence the last three inequalities prove that the set $\{f^t(x_0),\ t \in Q\}$ is not dense in $[a(x_0), b(x_0)]$, which contradicts our assumption.

THEOREM 5. A q.i.g. of f is continuous iff for every $x \in I$ the set $\{f^t(x),\ t \in Q\}$ is dense in $[a(x), b(x)]$.

P r o o f. Consider an $x_o \in I$ and assume that the set $\{f^t(x_o), t \in Q\}$ is dense in $[a(x_o), b(x_o)]$. By Lemma 1 the oscillation of the function $Q \ni t \to f^t(x_o)$ vanishes at every $t \in \mathbb{R}$, which proves that it is continuous (cf.[2] p.52).

Assume now that f^t, $t \in Q$ is a c.q.i.g. of f. It means by definition that for every $x \in I$ the function $Q \ni t \to f^t(x)$ is continuous. Consider an $x_o \in I \setminus F[f]$. We may assume that $f(x_o) < x_o$. Suppose, to argue by contradiction, that the set $\{f^t(x_o), t \in Q\}$ is not dense in $[a(x_o), b(x_o)]$. It means that there exist $c, d \in [a(x_o), b(x_o)]$, $a(x_o) \leq c < d \leq b(x_o)$ such that $f^t(x_o) \leq c$ or $f^t(x_o) \geq d$ for every $t \in Q$. The function $Q \ni t \to f^t(x_o)$ is strictly decreasing. Moreover, $\lim_{n \to \infty} f^n(x_o) = a(x_o)$ and $\lim_{n \to \infty} f^{-n}(x_o) = b(x_o)$ (cf. [4], p.21). Hence $a(x_o) < c$, $d < b(x_o)$ and so there exist $t_1, t_2 \in Q$ such that $f^{t_1}(x_o) \leq c$ and $f^{t_2}(x_o) \geq d$. Put

$$t_o = \sup\{t \in Q\colon f^t(x_o) <\}c ,$$

where the supremum is taken in $\mathbb{R}$.

Making use of the monotonicity of the function $Q \ni t \to f^t(x_o)$ we get

(4) $f^t(x_o) < c$ for $t > t_o$, $f^t(x_o) > d$ for $t < t_o$.

Consider two rational sequences $\{t_n\} \to t_o$, $t_n < t_o$ and $\{\tau_n\} \to t_o$, $\tau_n > t_o$. Then we have by (4)

$$f^{t_n}(x_o) > d, \quad f^{\tau_n}(x_o) < c$$

and consequently, as every function f^t, $t \in Q$ is strictly increasing.

$$(5) \qquad f^{\tau_n - t_n}(d) = f^{\tau_n}(f^{-t_n}(d)) < f^{\tau_n}(x_0) \leq c.$$

But $\tau_n - t_n \to 0$ and so there must be

$$\lim_{n\to\infty} f^{\tau_n - t_n}(d) = d,$$

which contradicts (5). $\square$

4. In this section we shall prove that a c.q.i.g. of f defines uniquely a c.r.i.g. of f.

THEOREM 6. Every c.q.i.g. of f can be extended to a c.r.i.g. of f. This extension is unique.

P r o o f. Let f^t, $t \in Q$ be c.q.i.g. of f and let $x \in I$. Then, in view of Theorem 5, the set $\{f^t(x): t \in Q\}$ is dense in $[a(x), b(x)]$ and so by Lemma 1 the oscillation of the function $Q \ni t \longrightarrow f^t(x)$ vanishes at every $t_0 \in \mathbb{R}$. But it proves that this function can be, in unique way, extended onto the set $\mathbb{R}$ (cf. [2], p.54). We put

$$(6) \quad f^{t_0}(x) := \lim_{t\to t_0, t\in Q} f^t(x) \quad \text{for} \quad t_0 \in \mathbb{R},\ x \in I.$$

We are going to prove that the function f^{t_0}, $t_0 \in \mathbb{R}$ is continuous.

Consider first an $x_0 \in I \setminus F[f]$.

The function f^{t_0} is increasing as a limit of increasing functions. Hence there exist

$$\lim_{x\to x_0^-} f^{t_0}(x) =: f^{t_0}(x_0^-), \quad \lim_{x\to x_0^+} f^{t_0}(x) =: f^{t_0}(x_0^+)$$

and

(7) $$f^{t_o}(x_o^-) \leqslant f^{t_o}(x_o) \leqslant f^{t_o}(x_o^+).$$

Suppose to prove by contradiction that

$$f^{t_o}(x_o^-) < f^{t_o}(x_o^+).$$

Assume that $f(x_o) < x_o$ (the considerations in the case $f(x_o) > x_o$ are analogous). The function $Q \ni t \to f^t(x)$ is decreasing for $x \in [a(x_o), b(x_o)]$ and so, in view of (6)

$$f^t(x) \leqslant f^{t_o}(x) \leqslant f^{\tau}(x) \quad \text{for} \quad x \in [a(x_o), b(x_o)],$$

$t, \tau \in Q$, $t > t_o$, $\tau < t_o$.

These inequalities yield the following

(8) $$f^t(x_o^-) \leqslant f^{t_o}(x_o^-) \leqslant f^{\tau}(x_o^-) \quad \text{for } t,\tau \in Q, t > t_o, \tau < t_o,$$

(9) $$f^t(x_o^+) \leqslant f^{t_o}(x_o^+) \leqslant f^{\tau}(x_o^+) \quad \text{for } t,\tau \in Q, t > t_o, \tau < t_o.$$

Making use of the continuity of the function f^t, $t \in Q$ (with respect to x) and combining together (7), (8) and (9) we get

$$f^t(x_o) = f^t(x_o^-) \leqslant f^{t_o}(x_o^-) < f^{t_o}(x_o^+) \leqslant f^{\tau}(x_o+) = f^{\tau}(x_o)$$

for $t, \tau \in Q$, $t > t_o$, $\tau < t_o$.

But this means that the set $\{f^t(x_o): t \in Q\}$ is not dense in $[a(x_o), b(x_o)]$, which contradicts Theorem 5. Hence $f^{t_o}(x_o^-) = f^{t_o}(x_o^+)$ and so, in consequence of (7) the function f^{t_o} is continuous at x_o.

Let now $x_o \in F[f]$. Denote by $\langle a,b \rangle$ a closed interval whose ends are a and b (it may be as well $a \leqslant b$ as $b \leqslant a$). Consider a $t_o \in \mathbb{R}$ and fix $t_1, t_2 \in Q$, $t_1 < t_o < t_2$.

In virtue of Theorem 2 we have then

(10) $$f^{t_0}(x) \in \langle f^{t_1}(x), f^{t_2}(x) \rangle \quad \text{for } x \in I.$$

Making use of continuity of the functions f^t, $t \in Q$ and applying Theorem 2 we get

$$\lim_{x \to x_0} f^{t_1}(x) = f^{t_1}(x_0) = x_0,$$

$$\lim_{x \to x_0} f^{t_2}(x) = f^{t_2}(x_0) = x_0.$$

These conditions together with (10) yield the relation

$$\lim_{x \to x_0} f^{t_0}(x) = x_0.$$

But $x_0 \in F[f]$ and therefore, in view of (6), $f^{t_0}(x_0) = x_0$. This completes the proof of the continuity of the function $f^{t_0}(x)$ (with respect to x).

We shall prove now that $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$ for $t_1, t_2 \in \mathbb{R}$. We have by (6)

(11) $$f^{t_1}(f^{t_2}(x)) = \lim_{t \to t_1, t \in Q} f^t\Big(\lim_{\tau \to t_2, \tau \in Q} f^{\tau}(x)\Big).$$

Making use of the continuity of the function f^t and of the function $\mathbb{R} \ni \tau \to f^{\tau}(x)$ we obtain

(12) $$\lim_{t \to t_1, t \in Q} f^t\Big(\lim_{\tau \to t_2, \tau \in Q} f^{\tau}(x)\Big) = \lim_{t \to t_1, t \in Q}\Big(\lim_{\tau \to t_2, \tau \in Q} f^t(f^{\tau}(x))\Big).$$

Since the function $\mathbb{R} \ni t \to f^t(x)$ is continuous we get

(13) $$\lim_{t \to t_1, \tau \to t_2, t, \tau \in Q} f^{t+\tau}(x) = f^{t_1+t_2}(x),$$

and consequently (cf. [3])

$$(14)\quad \lim_{t\to t_1, \tau\to t_2,\ t,\tau\in Q} f^{t+\tau}(x) = \lim_{t\to t_1, \tau\to t_2,\ t,\tau\in Q} f^{t}(f^{\tau}(x)) = \lim_{t\to t_1, t\in Q}\Big(\lim_{\tau\to t_2, \tau\in Q} f^{t}(f^{\tau}(x))\Big).$$

Gathering together (11), (12), (14) and (13) we have

$$f^{t_1}(f^{t_2}(x)) = f^{t_1+t_2}(x) \quad \text{for } t_1, t_2 \in \mathbb{R},\ x\in I. \square$$

5. At the XVIth International Symposium on Functional Equations (Graz 1978) C.M. Zdun posed the following question.

<u>Problem</u>. Does for every r.i.g. f^t, $t\in\mathbb{R}$ of f exist a c.r.i.g. $\bar{f}^t$, $t\in\mathbb{R}$ of this function and an additive function $\varphi:\mathbb{R}\to\mathbb{R}$ such that

$$(15)\qquad f^t = \bar{f}^{\varphi(t)} \quad \text{for all } t\in\mathbb{R}\ ?$$

We are going to show that the answer is "no". We first consider a non-singular case where $F[f]\neq I$. A singular case $F[f] = I$ will be considered next.

THEOREM 7. Let $F[f]\neq I$. Then a r.i.g. f^t, $t\in\mathbb{R}$ of f can be written in the form (15) iff the following conditions hold.

(a) The q.i.g. f^t, $t\in Q$ is continuous,

(b) $F[f]\subset F[f^t]$ for all $t\in\mathbb{R}$,

(c) There exists $x_0\in I\setminus F[f]$ such that for every $t_0, t_1\in\mathbb{R}$ the equality

$$\lim_{t \to t_0,\ t \in Q} f^t(x_0) = f^{t_1}(x_0)$$

implies the equality

$$\lim_{t \to t_0,\ t \in Q} f^t(x) = f^{t_1}(x) \quad \text{for all } x \in I.$$

P r o o f. We prove the sufficiency of conditions (a), (b), (c).

We show that the function $\mathbb{R} \ni t \to f^t(x_0)$, $x_0 \in I \setminus F[f]$ maps $\mathbb{R}$ into $(a(x_0), b(x_0))$. We have by (b)

$$f^t(a(x_0)) = a(x_0), \quad f^t(b(x_0)) = b(x_0) \quad \text{for all } t \in \mathbb{R}.$$

Moreover, as the function f^t, $t \in \mathbb{R}$ is invertible and continuous, it is strictly monotonic.

Thus $f^t(x_0) \in (a(x_0), b(x_0))$ for all $t \in \mathbb{R}$.

It follows from (a) and from Theorem 6 that the q.i.g. f^t, $t \in Q$ can be uniquely extended to a c.r.i.g. $\overline{f}^t$, $t \in \mathbb{R}$. This extension is of form (6). As we have shown in the proof of Theorem 6 the function $\mathbb{R} \ni t \to \overline{f}^t(x_0)$, $x_0 \in I \setminus F[f]$ is strictly monotonic. This is immediately seen from (6). Moreover, it follows from (6) and from the density of the set $\{f^t(x_0) : t \in Q\}$ in $(a(x_0), b(x_0))$ for $x_0 \in I \setminus F[f]$ that the function $\mathbb{R} \ni t \to \overline{f}^t(x_0)$, $x_0 \in I \setminus F[f]$ maps $\mathbb{R}$ onto $(a(x_0), b(x_0))$.

Let $x_0 \in I \setminus F[f]$ be an element satisfying condition (c). In view of the above considerations for every $t \in \mathbb{R}$ there exists exactly one element $\varphi(t) \in \mathbb{R}$ such that

$$f^t(x_0) = \overline{f}^{\varphi(t)}(x_0). \tag{16}$$

We are going to prove that

(17) $$f^t(x) = \bar{f}^{\varphi(t)}(x) \quad \text{for} \quad t \in \mathbb{R},\ x \in I.$$

Making use of (16) and of the continuity of the function

$$\mathbb{R} \ni t \longrightarrow \bar{f}^t(x_0)$$

we get

$$\lim_{\tau \to \varphi(t),\, \tau \in Q} f^{\tau}(x_0) = \lim_{\tau \to \varphi(t),\, \tau \in Q} \bar{f}^{\tau}(x_0) = \bar{f}^{\varphi(t)}(x_0) = f^t(x_0),$$

whence, by (c) we obtain

$$\lim_{\tau \to \varphi(t),\, \tau \in Q} f^{\tau}(x) = f^t(x) \quad \text{for} \quad x \in I.$$

Thus

$$\bar{f}^{\varphi(t)}(x) = \lim_{\tau \to \varphi(t),\, \tau \in Q} \bar{f}^{\tau}(x) = \lim_{\tau \to \varphi(t),\, \tau \in Q} f^{\tau}(x) = f^t(x).$$

We have to prove yet that φ is additive. Applying (16) and (17) we get

$$\bar{f}^{\varphi(t_1+t_2)}(x_0) = f^{t_1+t_2}(x_0) = f^{t_1}(f^{t_2}(x_0)) = \bar{f}^{\varphi(t_1)}(\bar{f}^{\varphi(t_2)}(x_0)) = \bar{f}^{\varphi(t_1)+\varphi(t_2)}(x_0),$$

which yields the equality

$$\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2).$$

Conversely, assume now that a r.i.g. f^t, $t \in \mathbb{R}$ can be written in the form (15). Consider an $x_0 \in I \setminus F[f]$. In virtue of Theorem 2 and Theorem 6 the function

$\mathbb{R} \ni t \to \bar{f}^t(x_0)$ is strictly monotonic (see also to [1]). Moreover,

$$\bar{f}^{\varphi(1)}(x_0) = f^1(x_0) = f(x_0) = \bar{f}^1(x_0).$$

Hence $\varphi(1) = 1$ and consequently $\varphi(t) = t$ for all $t \in Q$, whence we get

$$(18) \qquad \bar{f}^t(x) = f^t(x) \quad \text{for all } t \in Q,\ x \in I.$$

Condition (a) results immediately from (18). Consider an $x_1 \in F[f]$. In virtue of Theorem 2 and (18) we have

$$\bar{f}^t(x_1) = x_1 \quad \text{for all } t \in Q.$$

But the function $\mathbb{R} \ni t \to \bar{f}^t(x_1)$ is continuous. Hence

$$\bar{f}^t(x_1) = x_1 \quad \text{for all } t \in \mathbb{R},$$

which means that condition (b) is satisfied.

To prove (c) suppose that for given $t_0, t_1 \in \mathbb{R}$ there holds the equality

$$\lim_{t \to t_0,\ t \in Q} f^t(x_0) = f^{t_1}(x_0).$$

In view of (18) and (15) this equality can be rewritten as

$$\lim_{t \to t_0,\ t \in Q} f^t(x_0) = \bar{f}^{\varphi(t_1)}(x_0),$$

whence we get

$$\bar{f}^{t_0}(x_0) = \bar{f}^{\varphi(t_1)}(x_0).$$

But the function $\mathbb{R} \ni t \to \bar{f}^t(x_0)$ is strictly monotonic. Hence

$$(19) \qquad \varphi(t_1) = t_0 .$$

Making use of (18), of the continuity of the function $\mathbb{R} \ni t \to \bar{f}^t(x)$, (19) and (15), we obtain

$$\lim_{t \to t_0,\ t \in Q} f^t(x) = \lim_{t \to t_0,\ t \in Q} \bar{f}^t(x) =$$

$$= \bar{f}^{t_0}(x) = \bar{f}^{\varphi(t_1)}(x) = f^{t_1}(x). \square$$

Condition (c) in Theorem 7 can be replaced by a more visual condition.

THEOREM 8. Let $F[f] \neq I$. Then a r.i.g. f^t, $t \in \mathbb{R}$ of f can be written in the form (15) iff conditions (a) and (b) are satisfied and

(d) For every $x, y \in I \setminus F[f]$, $t_1, t_2 \in Q$, $t \in \mathbb{R}$ the following relation holds

$$f^t(x) \in \langle f^{t_1}(x), f^{t_2}(x) \rangle \text{ iff } f^t(y) \in \langle f^{t_1}(y), f^{t_2}(y) \rangle$$ [2].

P r o o f. Assume that conditions (a), (b) and (d) are valid. We shall prove condition (c). Suppose that for some $x_0 \in I \setminus F[f]$

(20) $$\lim_{t \to t_0,\ t \in Q} f^t(x_0) = f^{t_1}(x_0).$$

Since the function $Q \ni t \to f^t(x_0)$ is monotonic we get from (20)

$$f^{t_1}(x_0) \in \langle f^t(x_0), f^{\tau}(x_0) \rangle \quad \text{for } t, \tau \in Q,\ t \leq t_0 \leq \tau,$$

and hence, in consequence of condition (d)

$$f^{t_1}(x) \in \langle f^t(x), f^{\tau}(x) \rangle \quad \text{for } t, \tau \in Q,\ t \leq t_0 \leq \tau,\ x \in I.$$

On the other hand we obtain from (a) by applying Theorem 5 and Lemma 1

2/ $\langle a,b \rangle$ denotes as in the proof of Theorem 6 a closed onterval whose ends are a and b.

$$\lim_{t\to t_0^-,\ t\in Q} f^t(x) = \lim_{\tau\to t_0^+,\tau\in Q} f^\tau(x) \quad \text{for} \quad x\in I.$$

Thus we have

$$\lim_{t\to t_0^-,\ t\in Q} f^t(x) = f^{t_1}(x) = \lim_{\tau\to t_0^+,\ \tau\in Q} f^\tau(x) \text{ for } x\in I,$$

which proves (c).

Assume now that a r.i.g. f^t, $t\in \mathbb{R}$ can be represented in the form (15). Consider $x,y\in I\setminus F[f]$ and suppose that

$$f^t(x)\in \langle f^{t_1}(x),f^{t_2}(x)\rangle \text{ for some } t_1,t_2\in Q,\ t\in \mathbb{R}.$$

Making use of (15) we obtain

$$F^{\varphi(t)}(x)\in\langle F^{\varphi(t_1)}(x),F^{\varphi(t_2)}(x)\rangle$$

and consequently, as the function $\mathbb{R}\ni t\to F^t(x)$ is monotonic, we obtain

$$\varphi(t)\in\langle\varphi(t_1),\varphi(t_2)\rangle.$$

Applying once more the monotonicity of the function $\mathbb{R}\ni t\to F^t(y)$ we get

$$F^{\varphi(t)}(y)\in\langle F^{\varphi(t_1)}(y),F^{\varphi(t_2)}(y)\rangle$$

i.e., in view of (15)

$$f^t(y)\in\langle f^{t_1}(y),f^{t_2}(y)\rangle. \ \square$$

Now we are going to prove a theorem which is a generalization of Theorems 7 and 8. It contains also the singular case $F[f] = I$.

Let f^t, $t\in\mathbb{R}$ be a r.i.g. of f. Then every function f^t, $t\in\mathbb{R}$ is a monotonic bijection of I onto itself. Let us fix a $t_0\in\mathbb{R}$ and put

(21) $$g^t(x) = f^{t_0 t}(x) \quad \text{for } x \in I,\ t \in \mathbb{R}.$$

It is clear that then the family g^t, $t \in \mathbb{R}$ is a r.i.g. of the function $g = f^{t_0}$. The r.i.g. g^t, $t \in \mathbb{R}$ is continuous iff the r.i.g. f^t, $t \in \mathbb{R}$ is continuous.

THEOREM 9. A r.i.g. f^t, $t \in \mathbb{R}$ of f can be written in the form (15) iff

(22) $$f^t(x) = x \quad \text{for all } t \in \mathbb{R},\ x \in I,$$

or there exists $t_0 \in \mathbb{R}$ such that $F[f^{t_0}] \neq I$ and a r.i.g. g^t, $t \in \mathbb{R}$ defined by (21) satisfies conditions (a), (b), (c).

P r o o f. The case (22) is trivial. Assume that $F[f^{t_0}] \neq I$ and a r.i.g. g^t, $t \in \mathbb{R}$ defined by (21) satisfies conditions (a), (b), (c). Then, in virtue of Theorem 7 there exist a c.r.i.g. $\bar{g}^t$ of g and an additive function $\varphi_1 \colon \mathbb{R} \to \mathbb{R}$ such that

(23) $$g^t(x) = \bar{g}^{\varphi_1(t)}(x) \quad \text{for all } t \in \mathbb{R},\ x \in I.$$

Put

(24) $$\bar{f}^t = \bar{g}^{\frac{t}{t_0}}, \quad \varphi(t) = \varphi_1\left(\frac{t}{t_0}\right)t_0 \quad \text{for } t \in \mathbb{R}.$$

Gathering together (21), (23) and (24) we obtain for $t \in \mathbb{R}$, $x \in I$

$$f^t(x) = g^{\frac{t}{t_0}}(x) = \bar{g}^{\varphi_1\left(\frac{t}{t_0}\right)}(x) = \bar{g}^{\frac{\varphi(t)}{t_0}}(x) = \bar{f}^{\varphi(t)}(x).$$

Conversely, let us suppose that a r.i.g. f^t, $t \in \mathbb{R}$ can be written in the form (15) and there exists $t_0 \in \mathbb{R}$ such that $F[f^{t_0}] = I$. Substituting relations (21) and (24) into (15) w. get equality (23). Hence, in virtue of

Theorem 7 the r.i.g. f^t, $t \in \mathbb{R}$ satisfies conditions (a), (b), (c).

It follows from Theorems 7 and 8 that conditions (a), (b), (c) and conditions (a), (b), (d) are equivalent. Hence conditon (c) in Theorem 9 may be replaced by condition (d).

As a corollary from Theorems 4 and 7 we obtain the following statement.

THEOREM 10. If card $I > 1$ then there exists a r.i.g. f^t, $t \in \mathbb{R}$ of the function f, which cannot be expressed in the form (15).

P r o o f. Consider first the case where $F[f] \neq I$. Then, in virtue of Theorem 4 there exists a discontinuous q.i.g. f^t, $t \in Q$. Making use of Hamel basis we define an additive function $\varphi : \mathbb{R} \to Q$ such that $\varphi(1) = 1$. We put

$$\tilde{f}^t = f^{\varphi(t)} \quad \text{for} \quad t \in \mathbb{R}.$$

It is clear that $\tilde{f}^t$, $t \in \mathbb{R}$ forms a r.i.g. of the function f and $\tilde{f}^t = f^t$ for $t \in Q$.
Since the q.i.g. $\tilde{f}^t$, $t \in Q$ is discontinuous, the r.i.g. $\tilde{f}^t$, $t \in \mathbb{R}$ cannot be written in the form (15).

Suppose now that $F[f] = I$, i.e. $f(x) = x$ for all $x \in I$. Consider a strictly monotonic and continuous function $g: I \to I$, $g(I) = I$ such that $F[g] \neq I$. In view of Theorem 4 there exists a discontinuous q.i.g. g^t, $t \in \mathbb{R}$ of the function g. Fix a $t_0 \in \mathbb{R} \setminus Q$. By applying Hamel basis one can define an additive function $\varphi : \mathbb{R} \to Q$ such that $\varphi(1) = 1$, $\varphi\left(\frac{1}{t_o}\right) = 0$. Then a family $\tilde{g}^t$, $t \in \mathbb{R}$ defined as follows

$$\tilde{g}^t = g^{\varphi(t)} \quad \text{for} \quad t \in \mathbb{R}$$

is a r.i.g. of g which cannot be written in the form(15). It follows from Theorems 7 and 9 that a r.i.g. f^t, $t \in \mathbb{R}$ defined by (21) cannot be expressed in the form (15).

In the case where f has exactly two fixed points Theorems 7 and 8 can be essentially simplified, namely, conditions (b), (c), (d) may be omitted.

THEOREM 11. Assume that $I = [a,b]$ and $f(x) \neq x$ for $x \in (a,b)$. Then, a r.i.g. f^t $t \in \mathbb{R}$ of the function f can be written in the form (15) iff the q.i.g. f^t, $t \in Q$ is continuous.

P r o o f. As we know every function f^t, $t \in \mathbb{R}$ is continuous, strictly increasing and it maps $[a,b]$ onto itself. Hence

$$f^t(a) = a, \quad f^t(b) = b \quad \text{for all} \quad t \in \mathbb{R},$$

which proves that condition (b) is satisfied. We are going to prove that condition (d) is satisfied. Suppose, for the proof by contradiction, that there exist $x_1, x_2 \in (a,b)$, $t_o \in R$, $t_1, t_2 \in Q$ such that

$$f^{t_o}(x_1) \in \langle f^{t_1}(x_1), f^{t_2}(x_1) \rangle$$

but

$$f^{t_o}(x_2) \in \langle f^{t_1}(x_2), f^{t_2}(x_2) \rangle.$$

Since the functions f^t, $t \in \mathbb{R}$ are continuous, there exists $x_3 \in (x_1, x_2)$ such that

$$f^{t_o}(x_3) = f^{t_1}(x_3) \quad \text{or} \quad f^{t_o}(x_3) = f^{t_2}(x_3).$$

Assume that $f^{t_o}(x_3) = f^{t_1}(x_3)$. Then we have for all $t \in \mathbb{R}$

$$f^{t_o}(f^t(x_3)) = f^t(f^{t_o}(x_3)) = f^t(f^{t_1}(x_3)) = f^{t_1}(f^t(x_3)).$$

The set $\{f^t(x_3): t \in Q\}$ is dense in $[a,b]$ (see Theorem 5). Hence we get from the last equality

$$f^{t_o}(x) = f^{t_1}(x) \quad \text{for all } x \in [a,b].$$

Thus $f^{t_o}(x_2) = f^{t_1}(x_2)$, which contradicts to (25). Theorem 8 completes now the proof. □

6. Theorems 7 and 8 can be applied to prove sufficient conditions for a r.i.g. to be continuous.

THEOREM 12. Assume that a r.i.g. f^t, $t \in \mathbb{R}$ of the function f satisfies conditions (a), (b) and (c) or (d), and that there exists $x_o \in I \setminus F[f]$ such that the function $\mathbb{R} \ni t \to f^t(x_o)$ is measurable. Then f^t, $t \in \mathbb{R}$ is a c.r.i.g.

P r o o f. A r.i.g. f^t, $t \in \mathbb{R}$ can be written in the form (15). We are going to prove that the function φ occuring in (15) is measurable. We may assume that the function $\mathbb{R} \ni t \to \bar{f}^{\varphi(t)}(x)$ is strictly decreasing. Consider a $t_o \in R$. We have

$$\{t \in \mathbb{R}: \varphi(t) < t_o\} = \{t \in \mathbb{R}: \bar{f}^{\varphi(t)}(x_o) > \bar{f}^{t_o}(x_o)\} =$$

$$= \{t \in \mathbb{R}: f^t(x_o) > \bar{f}^{t_o}(x_o)\}.$$

The last set is measurable. It proves that φ is measurable and hence continuous (cf. [1]). But $\varphi(t) = t$ for $t \in Q$

(see (18)) . Thus $\varphi(t) = t$ for all $t \in Q$. This equality and (15) prove the assertion. ▭

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