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Antoni Chronowski

Ternary semigroups of linear mappings and matrices

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. Following the classical Green's equivalences we examine, by means of some equivalence relations, the structure of the ternary semigroup of linear mappings. The suitable results for the ternary semigroup of matrices are consequences of the considerations for the linear mappings.

1. Introduction

In this paper we introduce the notion of a ternary semigroup of linear mappings of two vector spaces. The ternary semigroup of linear mappings is a counterpart of the semigroup of endomorphisms of a vector space. By means of the ternary semigroup of linear mappings we can define a ternary (linear) algebra of linear mappings. The last one is isomorphic to a ternary (linear) algebra of matrices. The purpose of the present paper is to examine the structure of the ternary semigroup of linear mappings. To this end, we shall use the relations constructed after the pattern of Green's equivalences in the theory of semigroups. We shall show that the structure of the ternary semigroup of linear mappings is similar to that of the semigroup of endomorphisms of a vector space, but it is more varied. Moreover, we shall give a certain clear characterization of the structure of the inverses in the ternary semigroup of linear mappings.

The results concerning a ternary semigroup of matrices will be immediate consequences of those obtained for the ternary semigroup of linear mappings.

2. Some definitions and results on ternary semigroups

A ternary semigroup is a particular case of the m-semigroup (cf. [5], [6]). We will list some basic definitions and results concerning ternary semigroups which will be needed in this paper.

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Definition 2.1

A *ternary semigroup* is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \longrightarrow A$ is a ternary operation satisfying the following associativity law:

$$\begin{aligned} f(f(x_1, x_2, x_3), x_4, x_5) &= f(x_1, f(x_2, x_3, x_4), x_5) \\ &= f(x_1, x_2, f(x_3, x_4, x_5)) \end{aligned} \tag{2.1}$$

for all $x_1, \ldots, x_5 \in A$.

Because of (2.1) we may write $f(x_1, \ldots, x_5)$ for $x_1, \ldots, x_5 \in A$. If $X_i \subseteq A$ for $i = 1, 2, 3$, then we set

$$f(X_1, X_2, X_3) = \{f(x_1, x_2, x_3) \in A : \ x_i \in X_i \text{ for } i = 1, 2, 3\}.$$

For simplicity we will write $f(A^2, a, A^2) = f(A, A, a, A, A)$. Throughout this paper the letter f will be reserved to denote the ternary operation in a ternary semigroup.

Definition 2.2

Let (A, f) be a ternary semigroup. A nonempty subset $I \subseteq A$ is called:

(a) *a left ideal* if $f(A, A, I) \subseteq I$,

(b) *a right ideal* if $f(I, A, A) \subseteq I$,

(c) *a lateral ideal* if $f(A, I, A) \subseteq I$,

(d) *a two-sided ideal* if I is both a left and right ideal,

(e) *an ideal* if I is a left, right, and lateral ideal.

Let $a \in A$ be an arbitrary fixed element of a ternary semigroup (A, f). The symbols $I_l(a)$, $I_r(a)$, $I_c(a)$, $I_j(a)$, $I(a)$ denote the principal left ideal, right ideal, lateral ideal, two-sided ideal, and ideal generated by the element a, respectively.

A straightforward reasoning yields the following

Proposition 2.3

Let (A, f) be a ternary semigroup. Let a be an arbitrarily fixed element of A. Then

(a) $I_l(a) = a \cup f(A, A, a)$,

(b) $I_r(a) = a \cup f(a, A, A)$,

(c) $I_c(a) = a \cup f(A, a, A) \cup f(A^2, a, A^2)$,

(d) $I_j(a) = a \cup f(A, A, a) \cup f(a, A, A) \cup f(A^2, a, A^2)$,

(e) $I(a) = a \cup f(A, A, a) \cup f(a, A, A) \cup f(A, a, A) \cup f(A^2, a, A^2)$.

Definition 2.4

Let (A, f) be a ternary semigroup. We define the following relations on the set A:

(a) $aLb \Longleftrightarrow I_l(a) = I_l(b)$,

(b) $aRb \Longleftrightarrow I_r(a) = I_r(b)$,

(c) $aCb \Longleftrightarrow I_c(a) = I_c(b)$,

(d) $aJb \Longleftrightarrow I_j(a) = I_j(b)$,

(e) $aTb \Longleftrightarrow I(a) = I(b)$,

(f) $H = L \cap R$,

(g) $D = L \circ R$.

Applying a similar argument as in the theory of semigroups we can prove that $L \circ R = R \circ L$ and $L \subseteq J$, $R \subseteq J$, $H \subseteq J$, $D \subseteq J$. Thus, all the above relations are equivalences.

Definition 2.5 (cf. [6])

A ternary semigruop (A, f) is said to be *regular* if

$$\forall\, a \in A \;\exists\, x, y \in A \;\; [f(a, x, a, y, a) = a].$$

Let X and Y be nonempty sets. Let $T(X, Y)$ be the set of all mappings of X into Y. Put $T[X, Y] = T(X, Y) \times T(Y, X)$. Define the ternary operation $f : T[X, Y]^3 \longrightarrow T[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3) \qquad (*)$$

for all $(p_i, q_i) \in T[X, Y]$, where $i = 1, 2, 3$.

The algebraic structure $(T[X, Y], f)$ is a ternary semigroup.

Definition 2.6

The ternary semigroup $(T[X, Y], f)$ is called the *ternary semigroup of mappings of sets X and Y*. If $X \cap Y = \emptyset$, then $(T[X, Y], f)$ is called the *disjoint* ternary semigroup of mappings of sets X and Y.

It is easy to check that the ternary semigroups $(T[X, Y], f)$ and $(T[Y, X], f)$ are isomorphic.

A slightly modified argument applied in the proof of Theorem 3 in [5] yields the following theorem.

Theorem 2.7

Every ternary semigroup (A, f) is embeddable into a disjoint ternary semigroup $(T[X, Y], f)$ of mappings of sets X and Y.

In many areas of mathematics mutual connections between algebraic, ordered, topological structures and semigroups (groups) of some morphisms of these structures are studied. For characterizing two structures S_1 and S_2 by means of their morphisms we should consider morphisms from S_1 into S_2, and conversely. The ternary semigroups of morphisms of the structures S_1 and S_2 meet above requirements, and they are useful to achieve the desirable aim. For many structures (e.g. ordered sets, lattices, affine spaces, topological spaces) using ternary semigroups of morphisms we can obtain some clear information about a degree of characterization of these structures by means of their morphisms (cf. [1], [2], [3]). Taking into account the above justification and the remarks contained in the Introduction, it is well-founded to investigate the ternary semigroups of morphisms of various structures.

3. A ternary semigroup of linear mappings

The following two theorems concerning the linear mappings will be needed in this paper:

THEOREM 3.1

Let X and Y be vector spaces over a field K. Let $p : X \longrightarrow Y$ be a linear mapping. Then there exists a subspace X_0 of X such that:

(i) $\mathrm{Ker}\,(p) \oplus X_0 = X$,

(ii) $p|_{X_0} : X_0 \longrightarrow \mathrm{Im}\,(p)$ *is an isomorphism of the vector spaces* X_0 *and* $\mathrm{Im}\,(p)$.

THEOREM 3.2 (cf. [4], Th. 2, p. 83)

Let X and Y be vector spaces over a field K. Let X_0 be a subspace of the space X. Then every linear mapping $p_0 : X_0 \longrightarrow Y$ can be extended to a linear mapping $p : X \longrightarrow Y$, i.e. $p|_{X_0} = p_0$.

Let X and Y be vector spaces over a field K. Let $L(X,Y)$ be the set of all linear mappings of the space X into the space Y. Let us put $L[X,Y] = L(X,Y) \times L(Y,X)$. Define the ternary operation $f : L[X,Y]^3 \longrightarrow L[X,Y]$ by the formula $(*)$ for all $(p_i, q_i) \in L[X,Y]$, where $i = 1,2,3$.

The algebraic structure $(L[X,Y], f)$ is a ternary semigroup.

DEFINITION 3.3

The ternary semigroup $(L[X,Y], f)$ is called the *ternary semigroup of linear mappings of vector spaces X and Y over a field K.*

Throughout this paper we shall consider vector spaces over a field K. Suppose that $p \in L(X,Y)$. Put $r(p) = \dim \mathrm{Im}\,(p)$.

LEMMA 3.4

Let X and Y be vector spaces. For arbitrary $p, p' \in L(X,Y)$, $q \in L(Y,X)$ the following conditions are satisfied:

(a) $\operatorname{Im}(p) \subseteq \operatorname{Im}(p') \Longleftrightarrow \exists\, p_1 \in L(X,Y)\, \exists\, q_1 \in L(Y,X)\ [p = p' \circ q_1 \circ p_1]$;

(b) $\operatorname{Ker}(p) \subseteq \operatorname{Ker}(p') \Longleftrightarrow \exists\, p_1 \in L(X,Y)\, \exists\, q_1 \in L(Y,X)\ [p' = p_1 \circ q_1 \circ p]$;

(c) $r(p) \leqslant r(p') \Longleftrightarrow \exists\, p_1, p_2 \in L(X,Y)\, \exists\, q_1, q_2 \in L(Y,X)\ [p = p_1 \circ q_1 \circ p' \circ q_2 \circ p_2]$;

(d) $r(p) \leqslant r(q) \Longleftrightarrow \exists\, p_1, p_2 \in L(X,Y)\ [p = p_1 \circ q \circ p_2]$.

Proof. The implications $(\Longleftarrow)$ for equivalences (a)-(d) are evident. We shall prove the implications $(\Longrightarrow)$.

(a) By Theorem 3.1 it follows that there exists a subspace X_0 of the space X such that:

(i) $\operatorname{Ker}(p') \oplus X_0 = X$,

(ii) $p'|_{X_0} : X_0 \longrightarrow \operatorname{Im}(p')$ is an isomorphism.

Let q_1 be an extension onto Y of the isomophism $(p'|_{X_0})^{-1} : \operatorname{Im}(p') \longrightarrow X_0$ (see Th. 3.2). The implication $(\Longrightarrow)$ for condition (a) is a direct consequence of the equality $p = p' \circ q_1 \circ p$.

(b) Put $\operatorname{Ker}(p) \oplus X_0 = X$. Let $q_1 \in L(Y,X)$ be an extension onto Y of the isomorphism $(p|_{X_0})^{-1} : \operatorname{Im}(p) \longrightarrow X_0$. For every $x \in X$ we have $x = x' + x_0$, where $x' \in \operatorname{Ker}(p)$ and $x_0 \in X_0$. Let us notice that $p(x) = p(x_0)$. Since $\operatorname{Ker}(p) \subseteq \operatorname{Ker}(p')$, it follows that $p'(x) = p'(x_0)$. Then $(p' \circ q_1 \circ p)(x) = p'(q_1(p(x_0))) = p'(x_0) = p'(x)$, consequently $p' = p' \circ q_1 \circ p$. The implication $(\Longrightarrow)$ for condition (b) is satisfied.

(c) Since $r(p) \leqslant r(p')$, there exists a monomorphism $s : \operatorname{Im}(p) \longrightarrow \operatorname{Im}(p')$. Put $h = s \circ p$. Notice that $\operatorname{Ker}(h) = \operatorname{Ker}(p)$ and $\operatorname{Im}(h) \subseteq \operatorname{Im}(p')$. Applying conditions (a) and (b) we get $p = p_1 \circ q_1 \circ h$ and $h = p' \circ q_2 \circ p_2$ for some $p_1, p_2 \in L(X,Y)$ and $q_1, q_2 \in L(Y,X)$. Therefore $p = p_1 \circ q_1 \circ p' \circ q_2 \circ p_2$ for some $p_1, p_2 \in L(X,Y)$ and $q_1, q_2 \in L(Y,X)$.

(d) Put $\operatorname{Ker}(p) \oplus X_0 = X$ and $\operatorname{Ker}(q) \oplus Y_0 = Y$. In view of the inequality $r(p) \leqslant r(q)$ and Theorem 3.1(ii) we have $\dim X_0 \leqslant \dim Y_0$. Consider the following mappings:

π — the projection of X onto X_0;

η — a monomorphism of X_0 into Y_0;

$p_2 = \eta \circ \pi$;

σ — an extension of the isomorphism $(q \circ \eta)^{-1} : q(\eta(X_0)) \longrightarrow X_0$ onto X;

$p_1 = p \circ \sigma$.

Notice that $p = (p|_{X_0}) \circ \pi$ and $p_1, p_2 \in L(X,Y)$. Therefore we have:

$$\begin{aligned}p_1 \circ q \circ p_2 &= p \circ \sigma \circ q \circ \eta \circ \pi = (p|_{X_0}) \circ \pi \circ \sigma \circ q \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ (\pi|_{X_0}) \circ (\sigma|_{q(\eta(X_0))}) \circ (q|_{\eta(X_0)}) \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \eta^{-1} \circ (q|_{\eta(X_0)})^{-1} \circ (q|_{\eta(X_0)}) \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \eta^{-1} \circ \mathrm{id}_{\eta(X_0)} \circ \eta \circ \pi = (p|_{X_0}) \circ \eta^{-1} \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \mathrm{id}_{X_0} \circ \pi = (p|X_0) \circ \pi \\ &= p.\end{aligned}$$

Suppose that $(p,q),(p',q') \in L[X,Y]$. We set:

$\mathrm{Im}\,(p,q) = (\mathrm{Im}\,(p), \mathrm{Im}\,(q))$;

$\mathrm{Ker}\,(p,q) = (\mathrm{Ker}\,(p), \mathrm{Ker}\,(q))$;

$r(p,q) = (r(p), r(q))$;

$\mathrm{Im}\,(p,q) \subseteq \mathrm{Im}\,(p',q') \iff \mathrm{Im}\,(p) \subseteq \mathrm{Im}\,(p') \wedge \mathrm{Im}\,(q) \subseteq \mathrm{Im}\,(q')$;

$\mathrm{Ker}\,(p,q) \subseteq \mathrm{Ker}\,(p',q') \iff \mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p') \wedge \mathrm{Ker}\,(q) \subseteq \mathrm{Ker}\,(q')$;

$r(p,q) \leqslant r(p',q') \iff r(p) \leqslant r(p') \wedge r(q) \leqslant r(q')$;

$r(p,q) \leqslant^* r(p',q') \iff r(p) \leqslant r(q') \wedge r(q) \leqslant r(p')$.

According to Lemma 3.4 we have the following

THEOREM 3.5

Assume that $(p,q),(p',q') \in L[X,Y]$. Then:

(i) $\mathrm{Im}\,(p,q) \subseteq \mathrm{Im}\,(p',q') \iff \exists\,(p_1,q_1),(p_2,q_2) \in L[X,Y]$ $[(p,q) = f((p',q'),(p_1,q_1),(p_2,q_2))]$;

(ii) $\mathrm{Ker}\,(p',q') \subseteq \mathrm{Ker}\,(p,q) \iff \exists\,(p_1,q_1),(p_2,q_2) \in L[X,Y]$ $[(p,q) = f((p_1,q_1),(p_2,q_2),(p',q'))]$;

(iii) $r(p,q) \leqslant r(p',q') \iff \exists\,(p_i,q_i) \in L[X,Y]\ (i = 1,...,4)$ $[(p,q) = f((p_1,q_1),(p_2,q_2),(p',q'),(p_3,q_3),(p_4,q_4))]$;

(iv) $r(p,q) \leqslant^* r(p',q') \iff \exists\,(p_1,q_1),(p_2,q_2) \in L[X,Y]$ $[(p,q) = f((p_1,q_1),(p',q'),(p_2,q_2))]$.

In view of Theorem 3.5, Proposition 2.3, and Definition 2.4 we can formulate the following

COROLLARY 3.6

Assume that $(p,q),(p',q') \in L[X,Y]$. The following conditions are satisfied:

(i) $(p',q') \in I_r(p,q) \iff \mathrm{Im}\,(p',q') \subseteq \mathrm{Im}\,(p,q)$;

(ii) $I_r(p,q) = I_r(p',q') \iff \mathrm{Im}\,(p,q) = \mathrm{Im}\,(p',q')$;

(iii) $(p,q)\ R\ (p',q') \iff \mathrm{Im}\,(p,q) = \mathrm{Im}\,(p',q')$;

(iv) $(p',q') \in I_l(p,q) \iff \mathrm{Ker}\,(p,q)\ \subseteq\ \mathrm{Ker}\,(p',q')$;

(v) $I_l(p,q) = I_l(p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q')$;

(vi) $(p,q)\ L\ (p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q')$;

(vii) $(p,q)\ H\ (p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q') \wedge \mathrm{Im}\,(p,q) = \mathrm{Im}\,(p',q')$.

The following known fact concerning vector spaces will be useful in the proof of the next theorem.

Let X_1 and X_2 be subspaces of a space X such that $X_1 \subseteq X_2$. Then

$$\dim(X/X_2) \leqslant \dim(X/X_1). \tag{3.2}$$

THEOREM 3.7

Let $(p,q),(p',q') \in L[X,Y]$, then:

(i) $(p',q') \in I_j(p,q) \iff r(p',q') \leqslant r(p,q)$;

(ii) $I_j(p,q) = I_j(p',q') \iff r(p,q) = r(p',q')$;

(iii) $(p,q)\ J\ (p',q') \iff r(p,q) = r(p',q')$.

Proof. First we will prove (i). Assume that $(p',q') \in I_j(p,q)$. According to Proposition 2.3(d) we consider the following cases:

(a) If $(p',q') = (p,q)$, then $r(p',q') = r(p,q)$.

(b) Suppose that

$$(p',q') = f((p_1,q_1),(p_2,q_2),(p,q)) \text{ for some } (p_1,q_1),(p_2,q_2) \in L[X,Y].$$

It follows from Theorem 3.5(ii) that $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p')$ and $\mathrm{Ker}\,(q) \subseteq \mathrm{Ker}\,(q')$. By formula (3.2) we get

$$r(p') = \dim(X/\mathrm{Ker}\,(p')) \leqslant \dim(X/\mathrm{Ker}\,(p)) = r(p).$$

Similarly, $r(q') \leqslant r(q)$. Hence $r(p',q') \leqslant r(p,q)$.

(c) Suppose that

$$(p',q') = f((p,q),(p_1,q_1),(p_2,q_2)) \text{ for some } (p_1,q_1),(p_2,q_2) \in L[X,Y].$$

By Theorem 3.5(i), $\mathrm{Im}\,(p',q') \subseteq \mathrm{Im}\,(p,q)$, and therefore $r(p',q') \leqslant r(p,q)$.

(d) Suppose that

$$(p',q') = f((p_1,q_1),(p_2,q_2),(p,q),(p_3,q_3),(p_4,q_4)) \text{ for some } (p_i,q_i) \in L[X,Y],\ i = 1,...,4.$$

By Theorem 3.5(iii), $r(p',q') \leqslant r(p,q)$.

Conversely, assume that $r(p',q') \leqslant r(p,q)$. In view of Theorem 3.5(iii), $(p',q') \in I_j(p,q)$.

According to (i) and Definition 2.4(d) we get (ii) and (iii).

PROPOSITION 3.8

The relations D and J in the ternary semigroup $L[X,Y]$ are identical.

Proof. It is enough to prove that $J \subseteq D$. Suppose that $(p,q)J(p',q')$ for $(p,q),(p',q') \in L[X,Y]$. This means that $r(p,q) = r(p',q')$. Since $\dim(\mathrm{Im}\,(p)) = \dim(\mathrm{Im}\,(p'))$, there exists an isomorphism $b : \mathrm{Im}\,(p) \longrightarrow \mathrm{Im}\,(p')$. Put $p_1 = b \circ p$. Notice that $\mathrm{Ker}\,(p_1) = \mathrm{Ker}\,(p)$ and $\mathrm{Im}\,(p_1) = \mathrm{Im}\,(p')$. Similarly one can construct the linear mapping $q_1 \in L(Y,X)$ such that $\mathrm{Ker}\,(q_1) = \mathrm{Ker}\,(q)$ and $\mathrm{Im}\,(q_1) = \mathrm{Im}\,(q')$. Thus $(p,q\,)L\,(p_1,q_1)$ and $(p_1,q_1)\ R\ (p',q')$, and so $(p,q)\ D\ (p',q')$.

According to Proposition 3.8 and Theorem 3.7(iii) we obtain the following

COROLLARY 3.9

If $(p,q),(p',q') \in L[X,Y]$, then $(p,q)\ D\ (p',q')$ if and only if $r(p,q) = r(p',q')$.

The next result is an immediate consequence of Proposition 2.3, Theorems 3.5(iii) and 3.5(iv).

THEOREM 3.10

If $(p,q),(p',q') \in L[X,Y]$, then $(p',q') \in I_c(p,q)$ if and only if $r(p',q') \leqslant r(p,q)$ or $r(p',q') \leqslant^* r(p,q)$.

Assume that $(p,q),(p',q') \in L[X,Y]$. Notice that $r(p,q) \leqslant^* r(p',q')$ and $r(p',q') \leqslant^* r(p,q)$ iff $r(p) = r(q')$ and $r(q) = r(p')$. Therefore we set

$$r(p,q) \stackrel{*}{=} r(p',q') \Longleftrightarrow r(p) = r(q') \wedge r(q) = r(p').$$

LEMMA 3.11

If $r(p,q) \leqslant r(p',q')$ and $r(p',q') \leqslant^* r(p,q)$, then $r(p,q) = r(p',q')$ for $(p,q),(p',q') \in L[X,Y]$.

Proof. Since $r(p') \leqslant r(q) \leqslant r(q') \leqslant r(p)$ and $r(q') \leqslant r(p) \leqslant r(p') \leqslant r(q)$, it follows that $r(p',q') \leqslant r(p,q)$. Consequently $r(p,q) = r(p',q')$.

THEOREM 3.12

If $(p,q),(p',q') \in L[X,Y]$, then $I_c(p,q) = I_c(p',q')$ if and only if $r(p,q) = r(p',q')$ or $r(p,q) \stackrel{*}{=} r(p',q')$.

Proof. We have $I_c(p,q) = I_c(p',q')$ iff $(p,q) \in I_c(p',q')$ and $(p',q') \in I_c(p,q)$. In view of Theorem 3.10 and Lemma 3.11, applying a straightforward calculation we get the desired result.

The following corollary results from Definition 2.4(c) and Theorem 3.12.

COROLLARY 3.13

If $(p,q),(p',q') \in L[X,Y]$, then $(p,q)\ C\ (p',q')$ if and only if $r(p,q) = r(p',q')$ or $r(p,q) \stackrel{*}{=} r(p',q')$.

By Proposition 2.3(e) and Theorem 3.5(iv) applying an argument similar to that in the proof of Theorem 3.7(i) we get the following result:

THEOREM 3.14

If $(p,q),(p',q') \in L[X,Y]$, then $(p',q') \in I(p,q)$ if and only if $r(p',q') \leqslant r(p,q)$ or $r(p',q') \leqslant^* r(p,q)$.

PROPOSITION 3.15

If $(p,q) \in L[X,Y]$, then $I(p,q) = I_c(p,q)$.

The proof follows from Theorems 3.10 and 3.14.

By Proposition 3.15, Theorem 3.12, and Definition 2.4(e) the following corollaries hold.

COROLLARY 3.16

If $(p,q),(p',q') \in L[X,Y]$, then

(i) $I(p,q) = I(p',q')$ *iff* $r(p,q) = r(p',q')$ *or* $r(p,q) \stackrel{*}{=} r(p',q')$,

(ii) $(p,q)\ T\ (p',q')$ *iff* $r(p,q) = r(p',q')$ *or* $r(p,q) \stackrel{*}{=} r(p',q')$.

Corollaries 3.13 and 3.16 yield

COROLLARY 3.17

The relations C and T in the ternary semigroup $L[X,Y]$ are identical.

COROLLARY 3.18

The relations C and D in the ternary semigroup $L[X,Y]$ satisfy the set-inclusion $D \subseteq C$.

This statement follows from Corollaries 3.9 and 3.13.

Let S be an equivalence relation. The symbol $S(x)$ denotes the equivalence class of S containing x.

THEOREM 3.19

If $(p,q) \in L[X,Y]$ and $r(p) = r(q)$, then $C(p,q) = D(p,q)$.

Proof. By Corollary 3.18, $D(p,q) \subseteq C(p,q)$. Suppose that $(p',q') \in C(p,q)$. If $r(p',q') = r(p,q)$, then $(p',q') \in D(p,q)$. If $r(p',q') \stackrel{*}{=} r(p,q)$, then $r(p) = r(q) = r(p') = r(q')$, and so $r(p',q') = r(p,q)$. This means that $(p',q') \in D(p,q)$.

LEMMA 3.20

Assume that $(p,q) \in L[X,Y]$ and $r(p) \neq r(q)$. Then there exists a pair of linear mappings $(p',q') \in L[X,Y]$ such that:

(i) $r(p,q) \neq r(p',q')$,

(ii) $r(p,q) \stackrel{*}{=} r(p',q')$.

Proof. First we will construct $p' \in L(X,Y)$ such that $r(p') = r(q)$. Consider $\mathrm{Ker}\,(q) \oplus Y_0 = Y$ and put $g = q|_{Y_0}$. There exists an epimorphism $s : X \longrightarrow \mathrm{Im}\,(q)$. Put $p' = g^{-1} \circ s$. Thus $p' \in L(X,Y)$ and $r(p') = r(q)$. Similarly one can construct $q' \in L(Y,X)$ such that $r(q') = r(p)$. Therefore the conditions (i) and (ii) hold.

THEOREM 3.21

Assume that $(p,q) \in L[X,Y]$ and $r(p) \neq r(q)$. Then the C-class $C(p,q)$ is the union of the two distinct D-classes D_1 and D_2 defined by the formulas:

$$D_1 = \{(p',q') \in L[X,Y] : \; r(p',q') = r(p,q)\}, \tag{3.3}$$

$$D_2 = \{(p',q') \in L[X,Y] : \; r(p',q') \stackrel{*}{=} r(p,q)\}. \tag{3.4}$$

Proof. Since $r(p) \neq r(q)$, it follows from Lemma 3.20 and C orollary 3.18 that the C-class $C(p,q)$ contains at least two distinct D-classes. Suppose that the C-class $C(p,q)$ contains three pairwise distinct D-classes $D(p_1,q_1)$, $D(p_2,q_2), D(p_3,q_3)$. Thus $r(p_1,q_1) \stackrel{*}{=} r(p_2,q_2)$ and $r(p_2,q_2) \stackrel{*}{=} r(p_3,q_3)$. Consequently $r(p_1) = r(q_2)$, $r(q_1) = r(p_2)$, $r(p_2) = r(q_3)$, $r(q_2) = r(p_3)$, and so $r(p_1) = r(p_3)$ and $r(q_1) = r(q_3)$. Therefore $D(p_1,q_1) = D(p_3,q_3)$. This contradicts our assumption.

We can extend the notion of an inverse in a binary semigroup to the ternary semigroup $L[X,Y]$. A pair $(p',q') \in L[X,Y]$ is called an *inverse* of a pair $(p,q) \in L[X,Y]$ if

$$f((p,q),(p',q'),(p,q)) = (p,q) \quad \text{and} \quad f((p',q'),(p,q),(p',q')) = (p',q').$$

THEOREM 3.22

For every pair $(p,q) \in L[X,Y]$ there exists an inverse $(p',q') \in L[X,Y]$.

Proof. Let X_0, Y_0 be such that $\mathrm{Ker}\,(p) \oplus X_0 = X$ and $\mathrm{Ker}\,(q) \oplus Y_0 = Y$. The mappings $g_1 : X_0 \longrightarrow \mathrm{Im}\,(p)$ and $g_2 : Y_0 \longrightarrow \mathrm{Im}\,(q)$ such that $g_1 = p|_{X_0}$ and $g_2 = q|_{Y_0}$ are isomorphisms.Let $s_1 : X \longrightarrow \mathrm{Im}\,(q)$ and $s_2 : Y \longrightarrow \mathrm{Im}\,(p)$

be epimorphisms such that $s_1|_{\mathrm{Im}\,(q)} = \mathrm{id}_{\mathrm{Im}\,(q)}$ and $s_2|_{\mathrm{Im}\,(p)} = \mathrm{id}_{\mathrm{Im}\,(p)}$. Set $p' = g_2^{-1} \circ s_1$ and $q' = g_1^{-1} \circ s_2$. Evidently $(p', q') \in L[X, Y]$. First we will prove that $f((p, q), (p', q'), (p, q)) = (p, q)$. We have $f((p, q), (p', q'), (p, q))$ $= (p \circ q' \circ p,\ q \circ p' \circ q)$. Observe that $(p \circ q' \circ p)(x) = (p \circ g_1^{-1} \circ s_2 \circ p)(x) = p(x)$ for every $x \in X$. Similarly, $(q \circ p' \circ q)(y) = q(y)$ for every $y \in Y$. Next we will show that $f((p', q'), (p, q), (p', q')) = (p', q')$. We have $f((p', q'), (p, q), (p', q')) =$ $(p' \circ q \circ p',\ q' \circ p \circ q')$. Notice that

$$\begin{aligned}(p' \circ q \circ p')(x) &= (p' \circ q \circ g_2^{-1} \circ s_1)(x) = p'(s_1(x)) \\ &= g_2^{-1}(s_1(s_1(x))) = g_2^{-1}(s_1(x)) \\ &= p'(x)\end{aligned}$$

for every $x \in X$. Similarly, $(q' \circ p \circ q')(y) = q'(y)$ for every $y \in Y$. Therefore (p', q') is an inverse of (p, q) in $L[X, Y]$.

From Definition 2.5 and Theorem 3.22 it follows

COROLLARY 3.23

The ternary semigroup $L[X, Y]$ is regular.

PROPOSITION 3.24

If $(p', q') \in L[X, Y]$ is an inverse of $(p, q) \in L[X, Y]$, then $r(p, q) \overset{*}{=} r(p', q')$.

This fact follows immediately from Theorem 3.5(iv).

Taking into account Corollary 3.13 and Proposition 3.24 we get

COROLLARY 3.25

If $(p', q') \in L[X, Y]$ is an inverse of $(p, q) \in L[X, Y]$, then $(p, q)\ C\ (p', q')$.

Assume that $E = \{(p, q) \in L[X, Y] : r(p) = r(q)\}$ and $E^* = L[X, Y] \setminus E$. From Corollary 3.9 it follows that $D(p, q) \subseteq E$ for every $(p, q) \in E$. Therefore $E = \bigcup\{D(p, q) : (p, q) \in E\}$.

PROPOSITION 3.26

For every C-class $C_0 \subseteq L[X, Y]$ precisely one of the following two conditions holds:

(i) $C_0 \subseteq E$,

(ii) $C_0 \subseteq E^*$.

Proof. Suppose that there exists a C-class $C_0 \subseteq L[X, Y]$ such that (p_1, q_1), $(p_2, q_2) \in C_0$, $(p_1, q_1) \in E$, and $(p_2, q_2) \in E^*$ for some $(p_1, q_1), (p_2, q_2) \in L[X, Y]$. From the foregoing and Theorem 3.19 it follows that $C_0 = C(p_1, q_1) = D(p_1, q_1) \subseteq E$. We have obtained a contradiction.

Summarizing we get the following theorem.

THEOREM 3.27

Given the ternary semigroup $L[X,Y]$.

(A) *Assume that a C-class $C_0 \subseteq E$. Then every inverse (p',q') of $(p,q) \in C_0$ is an element of the C-class C_0 (C_0 is a D-class).*

(B) *Assume that a C-class $C_0 \subseteq E^*$. Then $C_0 = D_1 \cup D_2$, where the D-classes D_1 and D_2 are defined by the formulas (3.3) and (3.4). Every inverse (p',q') of $(p,q) \in D_1$ is an element of D_2. Every inverse (p',q') of $(p,q) \in D_2$ is an element of D_1.*

Proof. The condition (A) is an immediate consequence of Corollary 3.25. To prove (B), assume that $C_0 = C(p_0,q_0)$. Therefore

$$D_1 = \{(p,q) \in L[X,Y] : r(p,q) = r(p_0,q_0)\}$$

and

$$D_2 = \{(p,q) \in L[X,Y] : r(p,q) \stackrel{*}{=} r(p_0,q_0)\}.$$

Suppose that $(p,q) \in D_1$ and (p',q') is an inverse of (p,q). In view of Proposition 3.24 we get $r(p,q) \stackrel{*}{=} r(p',q')$, and so $r(p_0,q_0) \stackrel{*}{=} r(p',q')$. Consequently $(p',q') \in D_2$. Suppose that $(p,q) \in D_2$ and (p',q') is an inverse of (p,q). By Proposition 3.24, $r(p,q) \stackrel{*}{=} r(p',q')$, and so $r(p_0,q_0) = r(p',q')$. Consequently $(p',q') \in D_1$.

4. A ternary semigroup of matrices

Let K be a field. Let $M(m,n)$ denote the set of all $m \times n$ matrices over K. Put $M[m,n] = M(m,n) \times M(n,m)$. Define the ternary operation $f : M[m,n]^3 \longrightarrow M[m,n]$ by the formula:

$$f((A_1,B_1),(A_2,B_2),(A_3,B_3)) = (A_1B_2A_3,\ B_1A_2B_3)$$

for all $(A_i,B_i) \in M[m,n]$, where $i = 1,2,3$.

The algebraic structure $(M[m,n],f)$ is a ternary semigroup.

DEFINITION 4.1

The ternary semigroup $(M[m,n],f)$ is called the *ternary semigroup of $m \times n$ matrices over a field K.*

Assume that $A \in M(m,n)$. Let $I(A)$ denote the subspace of the vector space K^m spanned by all the columns of the matrix A. Consider the homogeneous system of linear equations

$$AX = 0. \tag{4.5}$$

Let $K(A)$ denote the subspace of the vector space K^n of all the solutions of the system (4.5). Consider the linear mapping $p_A \in L(K^n,K^m)$ determined by the matrix A with respect to the canonical bases $(e_1,...,e_n)$ and $(\hat{e}_1,...,\hat{e}_m)$

in the vector spaces K^n and K^m, respectively. It is easy to notice that $K(A) = \mathrm{Ker}\,(p_A)$ and $I(A) = \mathrm{Im}\,(p_A)$. The rank $r(A)$ of the matrix A is identical with the rank of the linear mapping p_A, i.e. $r(A) = r(p_A)$. Assume that $(A,B) \in M[m,n]$. We set $K(A,B) = (K(A),K(B))$, $I(A,B) = (I(A),I(B))$, $r(A,B) = (r(A),r(B))$. The pair of matrices $(A,B) \in M[m,n]$ represents the pair of linear mappings $(p_A, q_B) \in L[K^n, K^m]$. Consider the pairs of matrices $(A_i, B_i) \in M[m,n]$, where $i = 1,2,3$. Then the pair of matrices $(A,B) = f((A_1,B_1),(A_2,B_2),(A_3,B_3))$ represents the pair of linear mappings $(p_A, p_B) = f((p_{A_1}, q_{B_1}),(p_{A_2}, q_{B_2}),(p_{A_3}, q_{B_3}))$.

Taking into account the foregoing considerations we can formulate all the results obtained for the ternary semigroup of linear mappings to get the similar results for the ternary semigroup of matrices.

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General continuous solution of a nonlinear functional inequality

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. In this paper we present theorems on the existence of continuous solutions of the functional inequality (1) in the case where the continuous solution of the corresponding functional equation (2) is not unique.

In the present paper we shall deal with the problem of existence of continuous solutions ψ of the functional inequality

$$\psi[f(x)] \leqslant G(x, \psi(x)), \tag{1}$$

in the case where the continuous solution φ of the corresponding functional equation

$$\varphi[f(x)] = G(x, \varphi(x)) \tag{2}$$

depends on an arbitrary function.

Some problems connected with continuous solutions of nonlinear functional inequalities have been investigated by D. Brydak in [3], [4], [5] and also by K. Baron in [2]. But these results concerned the case of uniqueness of continuous solutions of (2).

1. Let $I = (\xi, a)$, where $\xi < a \leqslant \infty$. We assume that

(i) The function $f : I \longrightarrow \mathbb{R}$ is continuous and strictly increasing in I. Moreover

$$\xi < f(x) < x, \quad x \in I.$$

REMARK 1

Hypothesis (i) implies that $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$ (cf. [6], p. 21). Here f^n denotes the n-th iterate of the function f.

As to the function G we assume

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(ii) $G : \Omega \longrightarrow \mathbb{R}$ is continuous, where $\Omega \subset I \times \mathbb{R}$ is an open set.

(iii) For every $x \in I$ the set

$$\Omega_x := \{y : \ (x, y) \in \Omega\} \tag{3}$$

is a nonempty open interval and

$$G(x, \Omega_x) \subset \Omega_{f(x)}. \tag{4}$$

Let $J \subset I$ be an interval such that $\xi \in \operatorname{cl} J$. We shall consider the solutions ψ of inequality (1) and those φ of equation (2) such that their graphs lie in Ω, i.e.

$$\psi(x), \varphi(x) \in \Omega_x \quad \text{for } x \in J \subset I.$$

The class of this solutions we shall denote by $\boldsymbol{\Psi}(J)$ and $\boldsymbol{\Phi}(J)$ respectively. Moreover

$$I_k := [f^{k+1}(x_0), f^k(x_0)] \quad \text{for } x_0 \in I,\ k \in \mathbb{N} \cup \{0\}.$$

At first we shall prove an important property of the set Ω which is implied by the above condition.

Lemma 1

Let us assume that the open set $\Omega \subset I \times \mathbb{R}$ is such that (iii) *holds. Then for two arbitrary points $(x_i, y_i) \in \Omega$, $i = 1, 2$ such that $x_1 < x_2$ there exists a continuous function φ defined in $[x_1, x_2]$ such that $\varphi(x) \in \Omega_x$ for $x \in [x_1, x_2]$ and $\varphi(x_i) = y_i$ for $i = 1, 2$.*

Proof. The lemma results from known facts from the theory of multivalued functions, cf. Propositions 3 and 2 on p. 81 of [1]:

The multifunction $F : I \longrightarrow n(\mathbb{R})$ (the family of all nonempty subsets of $\mathbb{R}$) which has the open graph admits a local selection, whence so does the function $\Phi : [x_1, x_2] \longrightarrow n(\mathbb{R})$ defined by

$$\Phi(x) = \begin{cases} F(x), & x \in (x_1, x_2), \\ \{y_1\}, & x = x_1, \\ \{y_2\}, & x = x_2. \end{cases}$$

Thus there exists a continuous selection $\varphi : [x_1, x_2] \longrightarrow \mathbb{R}$, having the properties stated in the lemma.

It is known (see [6], p. 68) that if the given functions f and G fulfil hypotheses (i)-(iii), then the continuous solution of equation (2) depends on an arbitrary function. It means that for an arbitrary $x_0 \in I$ and an arbitrary continuous function $\varphi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions:

$$\varphi_0(x) \in \Omega_x, \tag{5}$$

$$\varphi_0[f(x_0)] = G(x_0, \varphi_0(x_0)) \tag{6}$$

there exists exactly one continuous solution $\varphi \in \mathbf{\Phi}((\xi, x_0])$ of equation (2) such that

$$\varphi(x) = \varphi_0(x) \quad \text{for } x \in I_0. \tag{7}$$

If we assume additionally that

(iv) For every $x \in I$ the function G is invertible with respect to the second variable,

(v) The function f fulfils condition $f(I) = I$,

(vi) For every $x \in I$ the following condition is fulfilled

$$G(x, \Omega_x) = \Omega_{f(x)}, \tag{8}$$

where Ω_x is defined by (3),

then for an arbitrary $x_0 \in I$ every continuous function $\varphi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (5) and (6) may be extended to a continuous solution $\varphi \in \mathbf{\Phi}(I)$ of (2) (see Theorem 3.1 of [6]).

We are going to present corresponding results for inequality (1).

2. Let us assume (i)-(iv). Hypothesis (iv) guarantees the existence of the function $G^{-1}(x, \cdot)$ inverse to the function G with respect to the second variable.

We introduce the sequence $\{g_k\}$ defined on Ω by the formula

$$\begin{cases} g_0(x,y) = y, \\ g_{k+1}(x,y) = G(f^k(x), g_k(x,y)), \quad k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{9}$$

If we assume (v) and (vi) additionally, then we may put

$$g_{-k-1}(x,y) = G^{-1}(f^{-k-1}(x), g_{-k}(x,y)), \quad k \in \mathbb{N} \cup \{0\}.$$

It is obvious (by virtue of (4) and (8)) that the above sequences are well defined. By induction we see that

$$g_k(x,y) \in \Omega_{f^k(x)}, \quad k \in \mathbb{Z}.$$

Moreover, if φ is a solution of equation (2) then

$$\varphi[f^k(x)] = g_k(x, \varphi(x)), \quad k \in \mathbb{N} \cup \{0\}. \tag{10}$$

We omit elementary proofs of the above properties.

Now, we shall prove the following:

THEOREM 1

Let hypotheses (i) - (iii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions*

$$\psi_0[f(x_0)] \leqslant G(x_0, \psi_0(x_0)), \tag{11}$$

$$\psi_0(x) \in \Omega_x, \quad x \in I_0 \tag{12}$$

there exists a continuous solution $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ *of inequality* (1) *such that*

$$\psi(x) = \psi_0(x), \quad x \in I_0, \tag{13}$$

This solution is given by the formula

$$\psi[f^k(x)] = \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \quad \textit{for } x \in I_0,\ k \in \mathbb{N} \cup \{0\} \tag{14}$$

where $\lambda_k : I_k \longrightarrow \mathbb{R}$ *are arbitrary continuous functions fulfilling the following conditions*

$$\lambda_0(x) = 0, \quad x \in I_0, \tag{15}$$

$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}, \quad x \in I_0,\ k \in \mathbb{N} \cup \{0\}, \tag{16}$$

$$\begin{aligned} \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \leqslant G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] \\ + g_{k-1}(x, \psi_0(x))), \quad x \in I_0,\ k \in \mathbb{N}, \end{aligned} \tag{17}$$

$$\begin{aligned} \lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) = \lambda_{k-1}[f^k(x_0)] \\ + g_{k-1}(f(x_0), \psi_0[f(x_0)]), \quad k \in \mathbb{N}. \end{aligned} \tag{18}$$

Moreover, all continuous solutions $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Proof. Let us fix $x_0 \in I$ and an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions (11) and (12). Moreover let us fix an arbitrarily chosen sequence of continuous functions $\lambda_k : I_k \longrightarrow \mathbb{R}$ fulfilling conditions (15) - (18)[1]. If we define the function ψ by formula (14), then we have (13) from (15) and $\psi(x) \in \Omega_x$, for $x \in (\xi, x_0]$ by virtue of (16).

Now, let $x \in (\xi, f(x_0))$. If $k \in \mathbb{N}$ and $t \in I_0$ are such that $x = f^k(t)$, then (17) implies

$$\begin{aligned} \psi[f(x)] &= \psi[f^{k+1}(t)] = \lambda_{k+1}[f^{k+1}(t)] + g_{k+1}(t, \psi_0(t)) \\ &\leqslant G(f^k(t), \lambda_k[f^k(t)] + g_k(t, \psi_0(t))) \\ &= G(x, \psi(x)). \end{aligned}$$

Consequently formula (14) defines a solution of (1) in $(\xi, x_0]$. Now, we shall prove that ψ is continuous in $(\xi, x_0]$.

The function ψ is continuous in every interval $(f^{i+1}(x_0), f^i(x_0))$, $i \in \mathbb{N}$. By (13), (14), (18) and the continuity of the functions f, G, λ_k it follows that

$$\lim_{x \to f^i(x_0)} \psi(x) = \psi[f^i(x_0)], \quad i \in \mathbb{N}. \tag{19}$$

Indeed, we have

[1]As to the construction of $\{\lambda_k\}$, cf. the Remark 2

$$\begin{aligned}\lim_{x\to f^{k+1}(x_0)^+}\psi(x) &= \lim_{x\to f(x_0)^+}\psi[f^k(x)]\\ &= \lambda_k[f^{k+1}(x_0)] + g_k(f(x_0),\psi_0[f(x_0)])\\ &= \lambda_{k+1}[f^{k+1}(x_0)] + g_{k+1}(x_0,\psi_0(x_0))\\ &= \psi[f^{k+1}(x_0)],\end{aligned}$$

$$\begin{aligned}\lim_{x\to f^{k+1}(x_0)^-}\psi(x) &= \lim_{x\to x_0^-}\psi[f^k(x)]\\ &= \lambda_{k+1}[f^{k+1}(x_0)] + g_{k+1}(x_0,\psi_0(x_0))\\ &= \psi[f^{k+1}(x_0)].\end{aligned}$$

This completes the proof of (19).

Let us now assume that $\psi\in\mathbf{\Psi}((\xi,x_0])$ fulfils (1). It is sufficient to put

$$\psi_0(x) := \psi(x) \quad \text{for } x\in I_0, \tag{20}$$

$$\lambda_k[f^k(x)] := \psi[f^k(x)] - g_k(x,\psi(x)) \quad \text{for } x\in I_0,\ k\in\mathbb{N}\cup\{0\}, \tag{21}$$

to see that conditions (11), (12), (15)-(18) hold and that the solution ψ is represented by formula (14). This ends the proof of the theorem.

If we assume (iv)-(vi) additionally, then we may prove the following:

THEOREM 2

Let hypotheses (i)-(vi) *be fulfilled. Then for any $x_0\in I$ and for an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling* (11) *and* (12)*, there exists a continuous solution $\psi\in\mathbf{\Psi}(I)$ of inequality* (1) *such that* (13) *holds.*

This solution is given by formulas (14) *and*

$$\psi[f^{-k}(x)] = l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x)) \quad \textit{for } x\in I_0,\ k\in\mathbb{N} \tag{22}$$

where $\lambda_k : I_k \longrightarrow \mathbb{R}$, $l_k : I_{-k} \longrightarrow \mathbb{R}$ are arbitrarily chosen continuous functions fulfilling conditions (15)-(18) *and moreover the conditions*

$$l_0(x) = 0 \quad \textit{for } x\in I_0, \tag{23}$$

$$l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x)) \in \Omega_{f^{-k}(x)}, \quad x\in I_0,\ k\in\mathbb{N}, \tag{24}$$

$$\begin{aligned} &l_{k+1}[f^{-k-1}(x)] + g_{-k-1}(x,\psi_0(x))\\ &\quad\leqslant G(f^{-k}(x), l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x))), \quad x\in I_0,\ k\in\mathbb{N},\end{aligned} \tag{25}$$

$$\begin{aligned} &l_{k+1}[f^{-k-1}(x_0)] + g_{-k-1}(x_0,\psi_0(x_0))\\ &\quad= l_k[f^{-k-1}(x_0)] + g_{-k}(f(x_0),\psi_0[f(x_0)]), \quad k\in\mathbb{N}.\end{aligned} \tag{26}$$

Moreover, all continuous solutions $\psi\in\mathbf{\Psi}(I)$ of inequality (1) *may be obtained in this way.*

The proof of the above theorem runs analogously to that of Theorem 1 and will be omitted here.

Remark 2

Contrary to the situation with continuous solutions of equation (2) in I, a continuous function ψ_0 fulfilling (11), (12) cannot be extended uniquely to a continuous solution of inequality (1) in I. This follows from the fact that the sequences of continuous functions $\{\lambda_k\}$, $\{l_k\}$ fulfilling (15)-(18) and (23)-(26) may be chosen in various ways.

We show a construction of a sequence of continuous functions $\lambda_k : I_k \longrightarrow \mathbb{R}$ such that conditions (15)-(18) hold.

Let us take a continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (11) and (12). We put

$$y_{1,0} := \psi_0[f(x_0)] - G(x_0, \psi_0(x_0)).$$

Let us fix a $y_{1,1} \leqslant 0$ fulfilling additionally the condition

$$y_{1,1} + G(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^2(x_0)}.$$

It is possible since $\Omega_{f^2(x_0)}$ is a nonempty open interval and

$$G(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^2(x_0)}.$$

Thus we may take (by virtue of Lemma 1) a continuous function $\mu_1 : I_1 \to \mathbb{R}$ such that the conditions

$$\mu_1[f(x_0)] = \psi_0[f(x_0)], \qquad \mu_1[f^2(x_0)] = y_{1,1} + G(f(x_0), \psi_0[f(x_0)]),$$
$$\mu_1[f(x)] \in \Omega_{f(x)}, \quad x \in I_0$$

hold. For the function

$$\lambda[f(x)] := \mu_1[f(x)] - G(x, \psi_0(x)), \quad x \in I_0$$

we now put

$$\lambda_1(x) := \frac{1}{2}\left(\lambda(x) - |\lambda(x)|\right), \quad x \in I_1 .$$

It is obvious that $\lambda_1 : I_1 \longrightarrow \mathbb{R}$ is a continuous function such that the conditions

$$\lambda_1[f(x_0)] = y_{1,0}, \quad \lambda_1[f^2(x_0)] = y_{1,1}, \quad \lambda_1(x) \leqslant 0, \quad x \in I_1,$$
$$\lambda_1[f(x)] + G(x, \psi_0(x)) \in \Omega_{f(x)}, \quad x \in I_0$$

hold. If we assume that we have continuous functions $\lambda_0, \ldots, \lambda_{k-1}$ defined on I_j, $j = 0, \ldots, k-1$, respectively, and fulfilling

$$\lambda_i[f^i(x)] + g_i(x, \psi_0(x)) \leqslant G(f^{i-1}(x), \lambda_{i-1}[f^{i-1}(x)] + g_{i-1}(x, \psi_0(x))),$$
$$x \in I_0, i = 1, \ldots, k-1$$

$$\lambda_i[f^i(x)] + g_i(x, \psi_0(x)) \in \Omega_{f^i(x)}, \quad x \in I_0\,,\ i = 1, \ldots, k-1$$

$$\lambda_i[f^i(x_0)] + g_i(x_0, \psi_0(x_0)) = \lambda_{i-1}[f^i(x_0)] + g_{i-1}(f(x_0), \psi_0[f(x_0)]), \qquad i = 1, \ldots, k-1$$

then it is sufficient to put

$$y_{k,0} := \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]) - g_k(x_0, \psi_0(x_0))$$

and fix a $y_{k,1}$,

$$y_{k,1} \leqslant G(f^k(x_0), \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)])) - g_k(f(x_0), \psi_0[f(x_0)])$$

fulfilling the condition

$$y_{k,1} + g_k(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^{k+1}(x_0)}.$$

It is possible because of the relations

$$G(f^k(x_0), \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)])) \in \Omega_{f^{k+1}(x_0)},$$

$$g_k(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^{k+1}(x_0)}.$$

Thus we may take, again by Lemma 1, a continuous function $\mu_k : I_k \longrightarrow \mathbb{R}$ such that the condition

$$\mu_k[f^k(x_0)] = y_{k,0} + g_k(x_0, \psi_0(x_0)), \quad \mu_k[f^{k+1}(x_0)] = y_{k,1} + g_k(f(x_0), \psi_0[f(x_0)]),$$

$$\mu_k[f^k(x)] \in \Omega_{f^k(x)}, \quad x \in I_0$$

Now, for the functions γ, H defined by formulas:

$$\gamma[f^k(x)] := \mu_k[f^k(x)] - g_k(x, \psi_0(x)), \quad x \in I_0,$$

$$H[f^k(x)] := G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))) - g_k(x, \psi_0(x)), \ x \in I_0,$$

we put

$$\lambda_k(x) := \frac{1}{2}\left(\gamma(x) + H(x) - |\gamma(x) - H(x)|\right), \quad x \in I_k.$$

It is easy to notice that $\lambda_k : I_k \longrightarrow \mathbb{R}$ is a continuous function such that the conditions

$$\lambda_k[f^k(x_0)] = y_{k,0}, \quad \lambda_k[f^{k+1}(x_0)] = y_{k,1},$$

$$\lambda_k[f^k(x)] \leqslant G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))) - g_k(x, \psi_0(x)), \quad x \in I_0,$$

$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}, \quad x \in I_0.$$

hold. This ends the inductive construction of the sequence $\{\lambda_k\}$. In a similar way we may construct a sequence $\{l_k\}$ fulfilling (23)-(26).

3. Let us assume (i)-(iii) again. We will consider the following assumption stronger than (iv):

(vii) For every $x \in I$ the function G is strictly increasing with respect to the second variable.

In this section we shall characterize continuous solutions ψ of inequality (1) which fulfil additionaly the following condition

$$L_k^{\psi}[f(x)] \in G(x, \Omega_x), \quad x \in I, \ k \in \mathbb{N}. \tag{27}$$

where the sequence $\{L_k^{\psi}\}$ is defined by the reccurence formula

$$\begin{cases} L_0^{\psi}(x) = \psi(x), \\ L_{k+1}^{\psi}(x) = G^{-1}(x, L_k^{\psi}[f(x)]), \quad k \in \mathbb{N}. \end{cases} \tag{28}$$

It is easy to notice that (cf. (vii)) the sequence $\{L_k^{\psi}\}$ is decreasing and if φ is a solution of (2), then $L_k^{\varphi}(x) = \varphi(x)$, $k \in N$.

Moreover if ψ is a solution of inequality (1) then we obtain by induction that

$$\psi[f^k(x)] \leqslant g_k(x, \psi(x)), \quad k \in \mathbb{N}. \tag{29}$$

and that the function $g_k(x, \cdot)$ is also strictly increasing.

Now, we may formulate the following

THEOREM 3

Let hypotheses (i)-(iii), (vii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling* (11), (12) *and, moreover, the condition*

$$\psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}) \tag{30}$$

there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality (1) *fulfilling* (13) *and such that*

$$L_k^{\psi}[f(x)] \in G(x, \Omega_x), \quad x \in (\xi, x_0], \ k \in \mathbb{N} \cup \{0\}. \tag{31}$$

This solution is given by the formula

$$\psi[f^k(x)] = g_k(x, \gamma_k(x) + \psi_0(x)) \quad \textit{for } x \in I_0, \ k \in \mathbb{N} \cup \{0\}, \tag{32}$$

where γ_k are arbitrary continuous functions defined in I_0 and fulfilling the conditions:

$$\gamma_0(x) = 0 \quad \textit{for } x \in I_0, \tag{33}$$

$$\textit{the sequence } \{\gamma_k\} \textit{ is decreasing in } I_0, \tag{34}$$

$$\gamma_k(x) + \psi_0(x) \in \Omega_x \quad \textit{for } x \in (f(x_0), x_0], \ k \in \mathbb{N} \cup \{0\}, \tag{35}$$

$$\gamma_k[f(x_0)] + \psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}), \quad k \in \mathbb{N}, \tag{36}$$

$$g_k(x_0, \gamma_k(x_0) + \psi_0(x_0)) = g_{k-1}(f(x_0), \gamma_{k-1}[f(x_0)] + \psi_0[f(x_0)]), \quad k \in \mathbb{N}. \tag{37}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ of inequality (1)*, fulfilling* (31) *may be obtained in this manner.*

Proof. Let us fix $x_0 \in I$ and an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (11), (12) and (30). Moreover let us fix an arbitrarily chosen sequence of continuous functions $\{\gamma_k\}$ defined in I_0 and fulfilling conditions (33)-(37) [2]

If we define the function ψ by formula (32) then we have (13) from (33) and we obtain (31) from (30), (32), (35), (36). Indeed, let $x \in (\xi, x_0]$. If $k \in \mathbb{N} \cup \{0\}$ and $t \in I_0$ are such that $x = f^k(t)$, then formulas (28), (32) imply

$$\begin{aligned} L_0^{\psi}[f(x)] &= \psi[f(x)] = \psi[f^{k+1}(t)] = g_{k+1}(t, \gamma_{k+1}(t) + \psi_0(t)) \in g_{k+1}(t, \Omega_t) \\ &= G(f^k(t), g_k(t, \Omega_t)) \\ &\subset G(f^k(t), \Omega_{f^k(t)}) \\ &= G(x, \Omega_x). \end{aligned}$$

Thus (31) holds for $k = 0$. Now, we introduce the sequence $\{h_k\}$ defined by the formula

$$\begin{cases} h_0(x,y) = y, \\ h_{k+1}(x,y) = G^{-1}(f(x), h_k(f(x), y)), \quad k \in \mathbb{N}. \end{cases} \tag{38}$$

It is easy to prove (by induction) that

$$L_k^{\psi}[f(x)] = h_k(x, \psi[f^{k+1}(x)]), \quad x \in (\xi, x_0],\ k \in \mathbb{N} \cup \{0\} \tag{39}$$

and (9) with (38) imply

$$h_k(x, g_{k+1}(x, \Omega_x)) = G(x, \Omega_x), \quad x \in (\xi, x_0],\ k \in \mathbb{N} \cup \{0\}. \tag{40}$$

Let us fix a $k \in \mathbb{N}$. From (39) and (40) we have

$$L_k^{\psi}[f(x)] = h_k(x, \psi[f^{k+1}(x)]) \in h_k(x, g_{k+1}(x, \Omega_x)) = G(x, \Omega_x), \quad x \in (\xi, x_0].$$

Consequently condition (31) holds.

Now, let $x \in (\xi, f(x_0))$. If $k \in \mathbb{N}$ and $t \in I_0$ are such that $x = f^k(t)$ then the monotonicity of $g_k(x, \cdot)$ and (34) imply

$$\begin{aligned} \psi[f(x)] &= \psi[f^{k+1}(t)] = g_{k+1}(t, \gamma_{k+1}(t) + \psi_0(t)) \\ &\leqslant g_{k+1}(t, \gamma_k(t) + \psi_0(t)) \\ &= G(f^k(t), g_k(t, \gamma_k(t) + \psi_0(t))) \\ &= G(x, \psi(x)). \end{aligned}$$

Consequently, formula (32) defines the solution of (1) in $(\xi, x_0]$. The function ψ is continuous in every interval $(f^{i+1}(x_0), f^i(x_0))$, $i \in \mathbb{N}$, and it is sufficient to show that (19) holds. The proof of (19) runs analogously as that in the proof of Theorem 1 and it will be omitted here.

Now, let us assume that $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ is a continuous solution of (1) fulfilling (31). It is sufficient to define ψ_0 by (20) and to put

[2] see Remark 3

$$\gamma_k(x) := L_k^{\psi}(x) - \psi_0(x) \quad \text{for } x \in I_0,\ k \in \mathbb{N} \cup \{0\}. \tag{41}$$

Let us notice that (33), (35)-(37) hold. Inequality (1) implies also condition (34). We may prove by induction on p that:

$$L_{k-p}^{\psi}[f^p(x)] = g_p(x, L_k^{\psi}(x)), \quad \textit{for } x \in I_0,\ p = 0, 1, \ldots, k,\ k \in \mathbb{N}.$$

This implies that ψ may be represented by formula (32) and ends the proof of the theorem.

We have also the following theorem corresponding to Theorem 2. Its simple proof is omitted.

THEOREM 4

Let hypotheses (i)-(iii), (v)-(vii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0 : I_0 \longrightarrow \mathbb{R}$ *fulfilling* (11) *and* (12), *there exists a continuous solution* $\psi \in \mathbf{\Psi}(I)$ *of inequality* (1) *such that* (13) *and* (27) *holds. This solution is given by formula* (32) *and*

$$\psi[f^{-k}(x)] = g_{-k}(x, \eta_k(x) + \psi_0(x)), \quad x \in I_0,\ k \in \mathbb{N}$$

where γ_k, η_k *are arbitrary chosen sequences of continuous functions defined in* I_0 *such that* (33)-(37) *and, moreover, the conditions*

$$\eta_0(x) = 0 \quad \textit{for } x \in I_0,$$

$$\textit{the sequence } \{\eta_k\} \textit{ is decreasing in } I_0,$$

$$\eta_k(x) + \psi_0(x) \in \Omega_x \quad \textit{for } x \in I_0,\ k \in \mathbb{N},$$

$$g_{-k}(f(x_0), \eta_k[f(x_0)] + \psi_0[f(x_0)]) = g_{-k+1}(x_0, \eta_k(x_0) + \psi_0(x_0)), \quad k \in \mathbb{N},$$

are satisfied. Moreover, all continuous solutions $\psi \in \mathbf{\Psi}(I)$ *of inequality* (1) *fulfilling* (27) *may be obtained in this manner.*

REMARK 3

We may construct a sequence $\{\gamma_k\}$ fulfilling conditions (33)-(37) in a similar way as in Remark 2. However, having taken a solution $\psi \in \mathbf{\Psi}((\xi, x_0])$ of (1) defined by formula (14) and fulfilling (31) we may also define a sequence $\{\gamma_k\}$ by (41), cf. (28).

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D'Alembert's functional equation and Chebyshev polynomials

Abstract. We consider D'Alembert's functional equation (1) where the domain of the function f is the additive group of the integers and the codomain is an arbitrary commutative ring with identity. We show that if $f(0) = 1$ then $f(n)$ is the value of the Chebyshev polynomial $T_{|n|}$ evaluated at $f(1)$.

1. Introduction

In 1750 d'Alembert introduced the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{1}$$

This arose in modelling the motion of a stretched string and in the foundations of mechanics. See Aczél & Dhombres [1; Chs. 1 & 8] for more details. This equation is also called the cosine equation since the cosine certainly satisfies it.

To see how one can "find" the cosine in (1) assume that the not identically zero solution f is twice continuously differentiable. Then from

$$\frac{f(x+y) + f(x-y) - 2f(x)}{y^2} = f(x)\,\frac{f(y) + f(-y) - 2f(0)}{y^2}, \tag{2}$$

using $f(-y) = f(y)$ and $f(0) = 1$ which follow from (1), and taking the limit as y tends to 0, one obtains

$$f''(x) = f(x)f''(0). \tag{3}$$

Hence

$$f(x) = \begin{cases} \cos(cx) & \text{if } f''(0) \leqslant 0 \\ \cosh(cx) & \text{if } f''(0) > 0 \end{cases} \tag{4}$$

where $c := \sqrt{|\, f''(0)\,|}$.

It is worth remarking that d'Alembert was among those calling for a theory of limits that would justify the argument just given. It is also worth remarking that the technique of reducing a functional equation (such as (1))to a differential

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equation (such as (3)) has been a mainstay for the past 250 years. Indeed Hilbert [1; p. 375], in proposing his fifth problem at the beginning of the 20^{th} century, said

> Specifically, we come to the broad and not uninteresting field of functional equations, hitherto largely investigated by assuming differentiability of the occurring functions. Equations treated in the literature, particularly the functional equations treated by Abel with such incisiveness, show no intrinsic characteristics that require the assumption of differentiability of the occurring functions...

Indeed Kannappan [3] solved (1) in great generality, in particular where x, y are elements of an additive abelian group and $f(x)$ is a complex number, without assuming any regularity (e.g. continuity) in the function. Kannappan proved that given a solution of d'Alembert's functional equation with $f(0) = 1$ there is a function $e : \mathrm{dom}\,(f) \to \mathbb{C}$ such that $e(0) = 1$ and $e(x+y) = e(x)e(y)$ and $2f(x) = e(x) + e(-x)$ for all $x \in \mathrm{dom}\,(f)$. In the classical cases $e(x) = e^{icx}$ for $\cos(cx)$ and $e(x) = e^{cx}$ for $\cosh(cx)$.

In this paper equation (1) (d'Alembert's equation) is solved when the domain of f is the additive group of the integers and the codomain of f is a commutative ring R. It is here that, perhaps surprisingly, the Chebyshev polynomials show up.

THEOREM

Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. Then

$$f(m+n) + f(m-n) = 2f(m)f(n); \quad (m,n) \in \mathbb{Z}^2, \tag{5}$$

if, and only if

$$f(n) = T_{|n|}(f(1)); \quad n \in \mathbb{Z}. \tag{6}$$

DEFINITION 1

$T_m \in \mathbb{Z}[X]$ is given by, for $m \in \mathbb{N}_o$,

$$T_m(X) = \sum_{k=0}^{q} \binom{m}{2k} X^{m-2k} \left(X^2 - 1\right)^k, \tag{7}$$

where q is the largest integer with $2q \leqslant m$. For equation (7) see Temme [4] eq. (6.39).

If $p \in \mathbb{Z}[X]$, say

$$p(X) = p_0 + p_1 X + \cdots + p_d X^d$$

and if $r \in R$ then, as usual,

$$p(r) := p_0 + p_1 r + \cdots + p_d r^d.$$

The general reference for Chebyshev polynomials (Tchebycheff — hence T) is

Rivlin [3]. The occurrence of T_n here is really as a polynomial not a polynomial function as in Rivlin generally.

The necessity ((5) implies (6)) is proved in Proposition 2. The sufficiency is proved in Proposition 3. Both use Proposition 1 that reduces the d'Alembert equation to a second order linear difference equation.

The identically zero function satisfies (5) but is not of the form $T_{(n)}(f(1))$: this is why $f(0) = 1$ is a constant assumption.

It is important to note that the domain of an equation must always be made clear: the equations

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (x, y) \in \mathbb{R}^2$$

and

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (x, y) \in \mathbb{R}_+^2 \ \ (\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\})$$

are different even though for both of them dom $(f) = \mathbb{R}$.

2. Reduction to a difference equation

The result below shows that the two variables m, n in (5) can, over $\mathbb{Z}$, be replaced by a single variable equation.

PROPOSITION 1

Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. Then f satisfies equation (5) if, and only if,

$$f(n+2) + f(n) = 2f(1)f(n+1) \quad n \in \mathbb{Z}. \tag{8}$$

Proof. Assume f satisfies equation (5). Then

$$f(n+1+1) + f(n+1-1) = 2f(n+1)f(1);$$

this is equation (8).

Assume conversely that f satisfies equation (8). Then f is even — that is

$$f(-n) = f(n) \quad n \in \mathbb{Z}. \tag{9}$$

It clearly suffices to prove (9) for all $n \in \mathbb{N}_0$. This is proved by induction on $n \in \mathbb{N}_0$. It is trivially true for $n = 0$. Also $f(-1) = f(1)$ since $f(0) = 1$. Assume it is true for all $n \in \mathbb{N}_0$ with $0 \leqslant n \leqslant N$, where $N \geqslant 1$. Then it is true for $n = N + 1$: using (8)

$$f(N+1) + f(N-1) = 2f(1)f(N).$$

Since $f(N-1) = f(-(N-1))$ and $f(N) = f(-N)$ by the induction hypothesis

$$f(N+1) + f(1-N) = 2f(1)f(-N). \tag{10}$$

But again using equation (8) with $n = -N - 1$

$$f(-N+1)+f(-N-1)=2f(1)f(-N). \tag{11}$$

Comparing (10) and (11) yields $f(-N-1)=f(N+1)$. Thus (9) is true for all $n \in \mathbb{N}_0$ by induction.

To show that f satisfies equation (5) the function $F:\mathbb{Z}^2 \to R$ defined next must be identically zero:

$$F(k,\ell):=f(k+\ell)+f(k-\ell)-2f(k)f(\ell) \quad (k,\ell)\in\mathbb{Z}^2. \tag{12}$$

Now since f, assumed to satisfy equation (8), has been shown to be even

$$F(k,\ell)=F(\ell,k)=F(-k,\ell) \quad (k,\ell)\in\mathbb{Z}^2. \tag{13}$$

So iterating the involutions $(k,\ell)\to(\ell,k)$ and $(k,\ell)\to(-k,\ell)$ it follows that F is identically zero on $\mathbb{Z}^2$ if, and only if, F is zero on

$$X:=\left\{(k,\ell)\in\mathbb{Z}^2:\ 0\leqslant\ell\leqslant k\right\}. \tag{14}$$

Using equation (8) to express $f(k+\ell)$ in terms of $f(k+\ell-1)$ and $f(k+\ell-2)$ and similarly $f(k-\ell)$ in terms of $f(k-\ell-1)$ and $f(k-\ell-2)$ it follows that

$$F(k,\ell)=2f(1)F(k-1,\ell)-F(k-2,\ell); \quad (k,\ell)\in\mathbb{Z}^2. \tag{15}$$

The 'size' of $(k,\ell)\in X$ is $k+\ell$ — the taxicab distance from $(0,0)$ to (k,ℓ) in X. By induction on the 'size' of $(k,\ell)\in X$ it is easy to show, using equation (15) that F is zero on X. [$F(0,0)=0$ is true since $f(0)=1$, as is $F(1,0)=0$. $F(1,1)=f(2)+f(0)-2f(1)f(1)=0$, since f satisfies (8)].

This completes the proof that (8) implies (5).

Corollary 1

Let $f:\mathbb{Z}\to R$ with $f(0)=1$. Then f satisfies equation (8) if, and only if, it is even (equation (9)) and

$$f(n+2)+f(n)=2f(1)f(n+1); \quad n\in\mathbb{N}_0. \tag{16}$$

Proof. Assume f satisfies (8). Then as above f must be even. Clearly f satisfies (16) as the domain of equation (8) includes the domain of equation (16). So this direction is proved.

Assume, conversely, that f satisfies (16) and is even. Then

$$\begin{aligned} f(-1)+f(1) &= f(1)+f(1) \ (f(-1)=f(1)) \\ &= 2f(1)f(0) \quad (f(0)=1). \end{aligned}$$

So f satisfies equation (8) for $n=-1$. Now let $n\in\mathbb{Z}$ with $n\leqslant -2$. Then

$$\begin{aligned} f(n+2)+f(n) &= f(n)+f(n+2) \\ &= f(-n)+f(-n-2) \quad (-n-2,\ -n\in\mathbb{N}_0) \\ &= 2f(1)f(-n-1) \quad (-n-1\in\mathbb{N}_0) \\ &= 2f(1)f(n+1). \end{aligned}$$

So f satisfies equation (8) for all integers $n \leqslant -1$. Thus

$$f(n+2)+f(n)=2f(1)f(n+1); \quad n \in \mathbb{Z}$$

as claimed.

Equation (16) is a linear difference equation of the second order: since $f(0)=1$ if $f(1)$ is given then $f(2)$, and recursively $f(3), f(4)\dots$ are determined.

3. The universal solution

DEFINITION 2

$T : \mathbb{Z} \to \mathbb{Z}[X]$ is given by $T(0)=1$, $T(1)=X$, $T(-n)=T(n)$; $n \in \mathbb{N}_0$ and

$$T(n+2)+T(n)=2XT(n+1); \quad n \in \mathbb{N}_0. \tag{17}$$

It is customary to write $T(n)$ as $T_n(X)$. Thus $T_2(X)=2X^2-1$, $T_3(X)=4X^3-3X$, $T_4(X)=8X^4-8X^2+1$ follow immediately from equation (17). By Proposition 1 $T : \mathbb{Z} \to \mathbb{Z}(X)$ is a solution of d'Alembert's functional equation (5). Indeed more is true!

PROPOSITION 2

Let $f : \mathbb{Z} \to R$ with $f(0)=1$. If f satisfies equation (5) then

$$f(n)=T_n(f(1)); \quad n \in \mathbb{Z}. \tag{18}$$

where $n \longmapsto T_n(X)$ is the family of polynomials from Definition 2 above.

Proof. Since both f and T are even (one by virtue of satisfying equation (5), the other by definition), it suffices to prove (18) for all $n \in \mathbb{N}_0$. Now $f(0)=1=T_0(f(1))$, and $f(1)=T_1(f(1))$. So assume it has been shown that $f(n)=T_n(f(1))$ for all $n \in \mathbb{N}_0$ with $n \leqslant N$ where $N \in \mathbb{N}_0$ and $N \geqslant 1$. Then

$$\begin{aligned} f(N+1) &= 2f(1)f(N)-f(N-1) \quad (f \text{ satisfies } (5)) \\ &= 2T_1(f(1))T_N(f(1))-T_{N-1}(f(1)) \quad (\text{induction hypothesis}) \\ &= T_{N+1}(f(1)) \quad (T \text{ satisfies } (17)). \end{aligned}$$

Thus the result is true for $n \leqslant N+1$. So the result follows for all $n \in \mathbb{N}_0$.

Note that what makes the preceding proof work is that for each $r \in R$ the evaluation ev_r of $p \in \mathbb{Z}(X)$ at r is a homomorphism from $\mathbb{Z}(X)$ to R:

$$ev_r(p+q)=ev_r(p)+ev_r(q) \ \ [(p+q)(r)=p(r)+q(r)]$$

$$ev_r(p \cdot q)=ev_r(p)+ev_r(q) \ \ [(pq)(r)=p(r)q(r)]\,.$$

Given f satisfying equation (5) there is a unique homomorphism (of commutative rings) $ev_{f(1)}$ such that $f=ev_{f(1)} \circ T$. Also $ev_{f(1)}$ is completely specified by

$$ev_{f(1)}(1) = 1 \quad \text{and} \quad ev_{f(1)}(X) = f(1):$$

there is one, and only one, ring homomorphism that sends 1 (of $\mathbb{Z}$) to 1 (of R), and X of $\mathbb{Z}[X]$ to $r \in R$.

4. Identification of the universal solution

The difference equation for T is

$$T(n+2) - 2XT(n+1) + T(n) = 0; \quad n \in \mathbb{N}_0. \tag{19}$$

The (quadratic) indicial equation for this is

$$\lambda^2 - 2X\lambda + 1 = 0. \tag{20}$$

This has roots

$$\lambda_1 = X + \sqrt{X^2 - 1},\ \lambda_2 = X - \sqrt{X^2 - 1}. \tag{21}$$

These roots lie in the quadratic extension A of $\mathbb{Z}[X]$ where

$$A = \left\{ \begin{bmatrix} p & q \\ (X^2-1)\,q & p \end{bmatrix} : p, q \in \mathbb{Z}[X] \right\}$$

so that, since $\begin{bmatrix} 0 & 1 \\ (X^2-1) & 0 \end{bmatrix}^2 = \begin{bmatrix} X^2-1 & 0 \\ 0 & X^2-1 \end{bmatrix}$,

$$\lambda_1 = \begin{bmatrix} X & 1 \\ X^2-1 & X \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} X & -1 \\ 1-X^2 & X \end{bmatrix}.$$

[Note that the characteristic polynomial of λ_1 is $t^2 - 2Xt + 1$.]

Hence, for some α_1 and $\alpha_2 \in A$

$$2T(n) = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n; \quad n \in \mathbb{N}_0. \tag{22}$$

From the initial conditions $T(0) = 1$, $T(1) = X$ and so $\alpha_1 = 1$ and $\alpha_2 = 1$. Thus

$$\begin{aligned} 2T_n(X) &= \sum_{j=0}^{n} \binom{n}{j} X^{n-j} \left[\left(\sqrt{X^2-1}\right)^j + (-1)^j \left(\sqrt{X^2-1}\right)^j \right] \\ &= \sum_{k=0}^{q} \binom{n}{2k} X^{n-2k}\, 2\left(X^2-1\right)^k \end{aligned}$$

where $j = 2k$ since for j odd $1 + (-1)^j = 0$. Hence the following result has been proved.

PROPOSITION 3

Suppose $T : \mathbb{Z} \to \mathbb{Z}[X]$ is given by Definition 2 (basically equation (17). Then $T(n)(= T_n(X))$ given by Definition 1. In other words: the universal *solution to the d'Alembert equation over $\mathbb{Z}$ is given by the family of Chebyshev polynomials.*

Since

$$\lambda_2 = \lambda_1^{-1}, \quad \left(\begin{bmatrix} X & 1 \\ X^2 - 1 & X \end{bmatrix}^{-1} = \begin{bmatrix} X & -1 \\ 1 - X^2 & X \end{bmatrix} \right)$$

the solution is seen to agree with Kannappan's general description. Define $E(n) := \lambda_1^n$. Then $E(m+n) = E(m)E(n)$, and $E(0) = 1$ and

$$2T(n) = E(n) + E(-n). \tag{23}$$

5. Concluding remarks

One direction of the Theorem in section 1 says: if $f : \mathbb{Z} \to R$ satisfies d'Alembert's equation then $f(n) = T_{|n|}(f(1))$ for all integers n. This has been proved: Proposition 2 gives this for the universal T, but Proposition 3 identifies the universal T as the Chebyshev family.

The other direction of the Theorem is just as easy now: the Chebyshev family is given by $(E(n) + E(-n))/2$ and so satisfies d'Alembert's equation, and consequently so does any homomorphic image via evaluation maps $T_n(X) \to T_n(f(1))$.

Thus the Theorem has been proved.

Finally, the well-known definition of the Chebyshev polynomial [see Rivlin [3; eq 1.2]] is a consequence of the Theorem: define for $\theta \in \mathbb{R}$ the function $f : \mathbb{Z} \to \mathbb{R}$ by $n \mapsto \cos(n\theta)$. Then f satisfies d'Alembert's functional equation as was noted in the introduction. Hence, by the Theorem

$$\cos(n\theta) = f(n) = T_n(f(1)) = T_n(\cos\theta). \tag{24}$$

In a similar way it can be shown that

$$\cosh(nt) = T_n(\cosh t) \quad n \in \mathbb{Z},\ t \in \mathbb{R}. \tag{25}$$

Equations (22) and (23) can be subsumed under the general result

$$\frac{X^n + X^{-n}}{2} = T_n\left(\frac{X + X^{-1}}{2}\right) \quad n \in \mathbb{Z}. \tag{26}$$

where the theorem has been applied to the function

$$n \mapsto \frac{X^n + X^{-n}}{2} \in \mathbb{Q}\left[X, X^{-1}\right].$$

So equation (22) follows from (24) by evaluating X at $e^{i\theta}$, as does (23) with evaluation at e^t.

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On a Banach space automorphism and its connections to functional equations and continuous nowhere differentiable functions

Abstract. Denote by $\mathcal{H}$ the Banach space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even. It turns out that $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

is a Banach space automorphism. Important properties of F are closely related to a de Rham type functional equation for $F[\varphi]$.

Many continuous nowhere differentiable functions are of the form $F[\varphi]$. A large part of them can be identified by simple properties of the generating function φ.

1. Introduction

The set $\mathcal{H}$ of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even, equipped with the uniform norm on $\mathbb{R}$, is a Banach space. Several prominent continuous but nowhere differentiable (cnd) functions can be generated from functions $\varphi \in \mathcal{H}$ via the linear operator $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x). \tag{1}$$

As examples we mention the Takagi function $T : \mathbb{R} \to \mathbb{R}$, given by

$$T(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} D(2^k x), \quad D(y) := \operatorname{dist}(y, \mathbb{Z}) \tag{2}$$

and the Weierstrass type function $W : \mathbb{R} \to \mathbb{R}$ given by

$$W(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} C(2^k x), \quad C(y) := \cos 2\pi y. \tag{3}$$

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The *cnd* property of W has been proved by Hardy [4] 1916, the *cnd* property of T by Takagi [11] 1903 and later by many other authors. The *cnd* property of functions defined by general series of type (1) has been investigated by Knopp [8] 1918, Behrend [1] 1949, Mikolás[9] 1956 and Girgensohn [2] 1993, [3] 1994. These authors stated properties of the generating function $\varphi : \mathbb{R} \to \mathbb{R}$ which imply the nondifferentiability of $F[\varphi]$. We mention two results which can be deduced from [1] respectively [9]:

THEOREM 1 (Behrend)
Assume that $\varphi \in \mathcal{H}$ is polygonal with a finite number of vertices in $[0,1]$ all of which have rational abscissae with $\varphi'_{+}(0) \neq 0$. Then $F[\varphi]$ is cnd.

THEOREM 2 (Mikolás)
Assume that $\varphi \in \mathcal{H}$ is convex on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$ (or concave on both intervals) with $\varphi(0) \neq \varphi(\frac{1}{2})$. Then $F[\varphi]$ is cnd.

These results show that there is an ample supply of *cnd* functions in $F[\mathcal{H}]$. In this note we pay special attention to the operator $F : \mathcal{H} \to \mathcal{H}$ given by (1). A detailed analysis of F is the subject of Section 2. It turns out that $F : \mathcal{H} \to \mathcal{H}$ is a Banach space automorphism. In proving this and other properties of F, a simple functional equation for $F[\varphi]$ is very useful. It can be shown in a few lines: for any $x \in \mathbb{R}$ we have (with $\psi := F[\varphi]$) $\frac{1}{2}\psi(2x) = \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \cdots$, hence

$$\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x). \tag{4}$$

Equation (4) has been investigated by de Rham [10] 1957 for $\varphi = D$, the distance function defined in (2). The general case and other functional equations for $F[\varphi]$ have been discussed by Kairies [5] 1997, [6] 1998, [7] 1999.

In Section 3 we derive the Fourier expansion of $F[\varphi]$. The Fourier coefficients of $F[\varphi]$ are connected to the Fourier coefficients of φ by means of a recursion formula which follows from (4) and can be interpreted in part as a discrete analog of (4). As an application we compute the Fourier series of Takagi's function T.

2. The Banach space automorphism F

It is straightforward to check that

$$\mathcal{H} := \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ continuous, } 1-\text{periodic and even}\},$$

equipped with the uniform norm $\|\ldots\|_u$ on $\mathbb{R}$, is a real Banach space and that the operator F, given by (1): $F[\varphi](x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$, is linear and maps $\mathcal{H}$ into $\mathcal{H}$.

In the following statement we describe the interaction of F with the functional equation (4): $\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x)$.

PROPOSITION 1

Let $\varphi \in \mathcal{H}$. Then $\psi = F[\varphi]$ iff ψ is a bounded solution of (4) on $\mathbb{R}$.

Proof. $\psi(x) = \varphi(x) + \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \cdots$ and $\varphi \in \mathcal{H}$ imply the boundedness of ψ and because of

$$\frac{1}{2}\psi(2x) = \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \cdots,$$

we get $\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x)$ for every $x \in \mathbb{R}$.

On the other hand, $\varphi(x) = \psi(x) - \frac{1}{2}\psi(2x)$ implies

$$\begin{aligned}\psi(x) &= \frac{1}{2}\psi(2x) + \varphi(x)\\ &= \frac{1}{2}\{\frac{1}{2}\psi(4x) + \varphi(2x)\} + \varphi(x)\\ &= \frac{1}{4}\{\frac{1}{2}\psi(8x) + \varphi(4x)\} + \frac{1}{2}\varphi(2x) + \varphi(x)\\ &\vdots\\ &= \frac{1}{2^m}\psi(2^m x) + \sum_{k=0}^{m-1} \frac{1}{2^k}\varphi(2^k x) \quad \text{for every } x \in \mathbb{R},\ m \in \mathbb{N}.\end{aligned}$$

As ψ is bounded,

$$\psi(x) = \lim_{m\to\infty} \psi(x) = \sum_{k=0}^{\infty} \frac{1}{2^k}\varphi(2^k x) = F[\varphi](x).$$

Now we shall list some important properties of the operator F. As usual, $\| F \| := \sup\{\| F[\varphi] \|_u;\ \varphi \in \mathcal{H},\ \| \varphi \|_u \leqslant 1\}$ denotes the operator norm of F.

THEOREM 3

$F : \mathcal{H} \to \mathcal{H}$ is a continuous Banach space automorphism with $\| F \| = 2$. The inverse operator F^{-1} is given by

$$F^{-1}[\psi](x) = \psi(x) - \frac{1}{2}\psi(2x)$$

and is continuous as well with $\| F^{-1} \| = \frac{3}{2}$.

Proof. The linearity of F was already stated. As we shall see, the bijectivity is an immediate consequence of Proposition 1.

Namely, to prove injectivity, observe that if $\varphi \in \mathcal{H}$ and $F[\varphi] = \mathbf{o}$ (the zero function) then, by Proposition 1, necessarily $\varphi = \mathbf{o}$.

To prove surjectivity, let $\psi \in \mathcal{H}$. Define $\varphi(x) := \psi(x) - \frac{1}{2}\psi(2x)$ for $x \in \mathbb{R}$. Then clearly $\varphi \in \mathcal{H}$. By Proposition 1, $\psi(x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x) = F[\varphi](x)$, hence $\psi = F[\varphi]$ for some $\varphi \in \mathcal{H}$.

For $\varphi \in \mathcal{H}$ with $\| \varphi \|_u \leqslant 1$ we obtain

$$\| F[\varphi] \|_u = \sup \{ | \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)|;\ x \in \mathbb{R}\} \leqslant \sum_{k=0}^{\infty} 2^{-k} \cdot 1 = 2,$$

hence $\| F \| \leqslant 2$. On the other hand, for the constant function $\mathbf{1}$ ($\mathbf{1}(x) = 1$) we have $\mathbf{1} \in \mathcal{H}, \| \mathbf{1} \|_u = 1$ and $\| F[\mathbf{1}] \|_u = \sum\limits_{k=0}^{\infty} 2^{-k} \cdot 1 = 2$, hence $\| F \| \geqslant 2$.

The inverse operator F^{-1} can be explicitely given: By Proposition 1 it follows that $\psi = F[\varphi]$ iff $\varphi(x) = F^{-1}[\psi](x) = \psi(x) - \frac{1}{2}\psi(2x)$ for every $x \in \mathbb{R}$.

Consequently, for $\| \psi \|_u \leqslant 1$ we have $\| F^{-1}[\psi] \|_u = \sup\{|\psi(x) - \frac{1}{2}\psi(2x)|;\ x \in \mathbb{R}\} \leqslant 3/2$. On the other hand, let $\psi_0(x) := 4D(x) - 1$, i.e., $\psi_0 \in \mathcal{H}$ with $\psi_0(x) = 4x - 1$ for $0 \leqslant x \leqslant \frac{1}{2}$. Then $\| \psi_0 \|_u = 1$ and $\| F^{-1}[\psi_0] \|_u = \sup\{|\psi_0(x) - \frac{1}{2}\psi_0(2x)|;\ x \in \mathbb{R}\} \geqslant \psi_0(1/2) - \frac{1}{2}\psi_0(1) = 3/2$.

REMARK 1

a) Let $\mathcal{A} := \{\varphi \in \mathcal{H};\ \varphi$ real analytic on $\mathbb{R}\}$.

Clearly $\mathcal{A}$ and $F[\mathcal{A}] = \{F[\varphi]; \varphi \in \mathcal{A}\}$ are subspaces of $\mathcal{H}$.

The examples $\mathbf{1} \in \mathcal{A}$ and $C \in \mathcal{A}$ ($C(x) = \cos 2\pi x$) show that F does not preserve this kind of regularity: $F[\mathbf{1}] = 2 \cdot \mathbf{1}$ is again analytic whereas $F[C] = W$ is *cnd* (this is in our context the worst possible regularity property which can occur).

In severe contrast, the operator F^{-1} obviously maps $\mathcal{A}$ into $\mathcal{A}$ and preserves similar types of regularity as well, e.g., differentiability of order $n \in \mathbb{N}$.

b) Let $\mathcal{B} := \{\varphi \in \mathcal{H};\ \varphi$ nowhere differentiable$\}$. The last observation in a) shows that F does not map any $\varphi \in \mathcal{B}$ to some $F[\varphi] \in \mathcal{H} \cap C^n(\mathbb{R})$ with $n \in \mathbb{N}$ or even to some $F[\varphi] \in \mathcal{H} \cap \mathrm{BV}\ [0,1]$.

c) The operator equation $F[\varphi] = \psi$ has for any given $\psi \in \mathcal{H}$ exactly one solution: $\varphi = F^{-1}[\psi], \varphi(x) = \psi(x) - \frac{1}{2}\psi(2x)$.

Similarly, $F^2[\varphi] = \psi$ if and only if $\varphi(x) = \psi(x) - \psi(2x) + \frac{1}{4}\psi(4x)$.

In this manner, $F^n[\varphi] = \psi$ can be explicitely solved in terms of the given $\psi \in \mathcal{H}$ for every $n \in \mathbb{N}$.

3. Fourier series of $F[\varphi]$

First we fix some notations. The Fourier series of a function $g \in L^1[0,1]$ will be denoted by $S[g]$. Throughout this section we assume $\varphi \in \mathcal{H}$ and write

$$S[\varphi](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi kx, \quad a_k = 2\int_0^1 \varphi(t) \cos 2\pi kt\ dt$$

and

$$S[F[\varphi]](x) = \frac{u_0}{2} + \sum_{k=1}^{\infty} u_k \cos 2\pi kx, \quad u_k = 2\int_0^1 F[\varphi](t)\cos 2\pi kt\, dt, \quad k \in \mathbb{N}_0.$$

REMARK 2

a) For $\varphi \in \mathcal{H}$ the Fourier coefficients a_k and u_k exist and we have

$$u_k = 2\int_0^1 \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n t)\cos 2\pi kt\, dt = 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(2^n t)\cos 2\pi kt\, dt,$$

in particular,

$$\begin{aligned} u_0 &= 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(2^n t)\, dt = 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^{2^n}\varphi(\tau_n)\frac{1}{2^n}\, d\tau_n \quad (\tau_n = 2^n t),\\ &= 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(\tau)\, d\tau = \sum_{n=0}^{\infty}\frac{1}{2^n}\cdot a_0\\ &= 2\cdot a_0. \end{aligned}$$

b) In general it is not true that $\varphi \in \mathcal{H}$ coincides with its Fourier series $S[\varphi]$: Fejér's famous example of a continuous function γ whose Fourier series is divergent at the point zero can be modified in such a way that the new function $\widetilde{\gamma}$ belongs to $\mathcal{H}$ and $S[\widetilde{\gamma}](0)$ diverges. However, if $(c_k) \in \ell^1$ and $\varphi(x) := \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos 2\pi kx$, then clearly $\varphi \in \mathcal{H}$ and $S[\varphi] = \varphi$.

THEOREM 4

a) *Let* $\varphi \in \mathcal{H}$. *Then*

$$u_{2k} - \frac{1}{2}u_k = a_{2k} \text{ and } u_{2k+1} = a_{2k+1} \text{ for every } k \in \mathbb{N}_0. \tag{5}$$

b) *The recursive system* (5) *has, for any given sequence* (a_k) *of real numbers, a unique solution* (u_k), *namely*

$$u_0 = 2a_0,\ u_k = u_{2^m(2j+1)} = \sum_{\lambda=0}^{m} \frac{1}{2^{m-\lambda}} a_{2^\lambda(2j+1)} \text{ for } k \in \mathbb{N}. \tag{6}$$

Proof. a) By Proposition 1 we have $F[\varphi](x) = \frac{1}{2}F[\varphi](2x) + \varphi(x)$. This implies

$$2\int_0^1 F[\varphi](x)\cos 2\pi kx\; dx = 2\int_0^1 \left\{\frac{1}{2}F[\varphi](2x)\cos 2\pi kx + \varphi(x)\cos 2\pi kx\right\}\, dx,$$

hence $(2x = t)$

$$u_k = \frac{1}{2}\int_0^2 F[\varphi](t)\cos \pi kt\; dt + a_k \quad (k \in \mathbb{N}_0).$$

In particular,

$$\begin{aligned} u_{2k} &= \frac{1}{2}\int_0^2 F[\varphi](t)\cos 2\pi kt\; dt + a_{2k} \\ &= \int_0^1 F[\varphi](t)\cos 2\pi kt\; dt + a_{2k} \\ &= \frac{1}{2}u_k + a_{2k}, \end{aligned}$$

because the integrand has period 1 and

$$u_{2k+1} = \frac{1}{2}\int_0^2 F[\varphi](t)\cos \pi(2k+1)t\; dt + a_{2k+1} = a_{2k+1},$$

because the integrand is odd with respect to $1/2$ in $[0,1]$ and odd with respect to $3/2$ in $[1,2]$.

b) $u_0 = 2a_0$ follows immediately from (5).

Every $k \in \mathbb{N}$ has a unique representation $2^m(2j+1)$ with some $m, j \in \mathbb{N}_0$. For $m = 0$, the second equation of (5) gives $u_{2j+1} = a_{2j+1}$ for every $j \in \mathbb{N}_0$. For $m \geqslant 1$, by repeated use of the first equation of (5), we get

$$\begin{aligned} u_k = u_{2^m(2j+1)} &= \frac{1}{2}u_{2^{m-1}(2j+1)} + a_{2^m(2j+1)} \\ &= \frac{1}{4}u_{2^{m-2}(2j+1)} + \frac{1}{2}a_{2^{m-1}(2j+1)} + a_{2^m(2j+1)} \\ &\;\vdots \\ &= \frac{1}{2^m}u_{2^0(2j+1)} + \sum_{\lambda=1}^{m}\frac{1}{2^{m-\lambda}}a_{2^\lambda(2j+1)} \\ &= \sum_{\lambda=0}^{m}\frac{1}{2^{m-\lambda}}a_{2^\lambda(2j+1)}. \end{aligned}$$

On the other hand, any sequence (u_k) given by (6) satisfies in fact (5): The case $k = 0$ is trivial. For $k = 2^m(2j+1)$ and $m = 0$ we get immediately $u_{2j+1} = a_{2j+1}$, whereas for $m \geqslant 1$ we obtain

$$\begin{aligned} u_{2k} - \frac{1}{2}u_k &= u_{2^{m+1}(2j+1)} - \frac{1}{2}u_{2^m(2j+1)} \\ &= \frac{1}{2}\sum_{\lambda=0}^{m+1} \frac{1}{2^{m-\lambda}}\, a_{2^\lambda(2j+1)} - \frac{1}{2}\sum_{\lambda=0}^{m} \frac{1}{2^{m-\lambda}}\, a_{2^\lambda(2j+1)} \\ &= \frac{1}{2}\,\frac{1}{2^{-1}}\, a_{2^{m+1}(2j+1)} = a_{2k}. \end{aligned}$$

As a first useful consequence of Theorem 2 we note

PROPOSITION 2
Assume that $(c_k) \in \ell^1$ *and that*

$$\varphi(x) := \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \cos 2\pi kx.$$

Then $\varphi \in \mathcal{H}$, $S[\varphi] = \varphi$ *and with*

$$S[F[\varphi]](x) = \frac{1}{2}v_0 + \sum_{k=1}^{\infty} v_k \cos 2\pi kx$$

we have $(v_k) \in \ell^1$ *and* $S[F[\varphi]] = F[\varphi]$.

Proof. Clearly φ is continuous, 1-periodic and even, hence $\varphi \in \mathcal{H}$. The uniform convergence of the series representing φ implies that φ coincides with its Fourier series $S[\varphi]$. By Theorem 2 we have

$$|v_0| = 2|c_0| \text{ and } |v_{2^m(2j+1)}| \leqslant \sum_{\lambda=0}^{m} 2^{\lambda-m}|c_{2^\lambda(2j+1)}| \text{ for } m, j \in \mathbb{N}_0.$$

Consequently, for every $j \in \mathbb{N}_0$,

$$\begin{aligned} \sum_{m=0}^{\infty} |v_{2^m(2j+1)}| &\leqslant \sum_{m=0}^{\infty}\sum_{\lambda=0}^{m} 2^{\lambda-m}|c_{2^\lambda(2j+1)}| \\ &= |c_{2j+1}| + \big(\frac{1}{2}|c_{2j+1}| + |c_{2(2j+1)}|\big) \\ &\quad + \big(\frac{1}{4}|c_{2j+1}| + \frac{1}{2}|c_{2(2j+1)}| + |c_{2^2(2j+1)}|\big) + \cdots \\ &= 2\sum_{m=0}^{\infty} |c_{2^m(2j+1)}|. \end{aligned}$$

Moreover,

$$\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}|v_{2^m(2j+1)}| \leqslant \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}|2c_{2^m(2j+1)}| = 2\sum_{k=1}^{\infty}|c_k| < \infty$$

because of $(c_k) \in \ell^1$. By the main rearrangement theorem, $\sum_{k=1}^{\infty}|v_k| < \infty$, i.e., $(v_k) \in \ell^1$ as well.

This implies the convergence of $S[F[\varphi]](x)$ for every $x \in \mathbb{R}$, thus by Fejér's theorem $F[\varphi]$ coincides with its Fourier series $S[F[\varphi]]$.

As a second consequence of Theorem 2 we derive the Fourier series of Takagi's function $T = F[D]$.

PROPOSITION 3
If T is given by (2), then

$$S[T](x) = \frac{1}{2} + \sum_{k=1}^{\infty} u_k \cos 2\pi kx$$

with

$$u_k = u_{2^m(2j+1)} = \frac{-1}{2^{m-1}\pi^2(2j+1)^2} \quad \textit{for } m, j \in \mathbb{N}_0.$$

This series is absolutely and uniformly convergent on $\mathbb{R}$ and $S[T] = T$.

Proof. It is well known that the distance function D has the Fourier series

$$S[D](x) = \frac{1}{4} - \frac{2}{\pi^2}\Big\{\frac{1}{1^2}\cos 2\pi x + \frac{1}{3^2}\cos(3\cdot 2\pi x) + \frac{1}{5^2}\cos(5\cdot 2\pi x) + \cdots\Big\}.$$

Hence $a_0 = \frac{1}{2}$, $a_{2k} = 0$ for $k \in \mathbb{N}$ and $a_{2j+1} = \frac{-2}{\pi^2(2j+1)^2}$ for $j \in \mathbb{N}_0$. Clearly $(a_n) \in \ell^1$.

By Theorem 2 we have $u_0 = 2a_0 = 1, u_{2j+1} = a_{2j+1} = \frac{-2}{\pi^2(2j+1)^2}$ for $j \in \mathbb{N}_0$ and, because of $a_{2k} = 0$,

$$u_{2^m(2j+1)} = \frac{1}{2^m}a_{2j+1} + \sum_{\lambda=1}^{m} a_{2^\lambda(2j+1)} = \frac{-1}{2^{m-1}\pi^2(2j+1)^2} \text{ for } j \in \mathbb{N}_0,\ m \in \mathbb{N}.$$

By Proposition 2, T coincides with its Fourier series $S[T]$ and $(u_k) \in \ell^1$.

Therefore we have the following representation of Takagi's function by an absolutely and uniformly (on $\mathbb{R}$) convergent trigonometric series:

$$
\begin{aligned}
T(x) = \frac{1}{2} - \frac{1}{\pi^2} \Big\{ & \frac{2}{1^2}\cos(1\cdot 2\pi x) + \frac{1}{1^2}\cos(2\cdot 2\pi x) + \frac{2}{3^2}\cos(3\cdot 2\pi x) \\
&+ \frac{1}{2}\cos(4\cdot 2\pi x) + \frac{2}{5^2}\cos(5\cdot 2\pi x) + \frac{1}{3^2}\cos(6\cdot 2\pi x) \\
&+ \frac{2}{7^2}\cos(7\cdot 2\pi x) + \frac{1}{2^2}\cos(8\cdot 2\pi x) + \frac{2}{9^2}\cos(9\cdot 2\pi x) \\
&+ \frac{1}{5^2}\cos(10\cdot 2\pi x) + \frac{2}{11^2}\cos(11\cdot 2\pi x) + \frac{1}{2\cdot 3^2}\cos(12\cdot 2\pi x) \\
&+ \frac{2}{13^2}\cos(13\cdot 2\pi x) + \frac{1}{7^2}\cos(14\cdot 2\pi x) + \frac{2}{15^2}\cos(15\cdot 2\pi x) \\
&+ \frac{1}{2^3}\cos(16\cdot 2\pi x) + \cdots \Big\}.
\end{aligned}
$$

Note that in our approach we did not need an explicit calculation of the rather unpleasant series $u_k = 2\sum_{n=0}^{\infty} \frac{1}{2^n}\int_0^1 D(2^n x)\cos 2\pi k x\, dx,\ k \in \mathbb{N}_0$.

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On the stability of the generalized cosine functional equations

Abstract. The aim of this paper is to study the stability problem of the generalized cosine functional equations for complex and vector valued functions.

1. Introduction

One of the trigonometric functional equations studied extensively is the equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad \text{for } x,y \in G \tag{C}$$

(Wilson [8], Kannappan [6], Dacić [3]) known as the *cosine* or *d'Alembert's functional equation* where $f : G \to \mathbb{C}$, G, a group (not necessarily Abelian) and $\mathbb{C}$, the set of complex numbers. It is known (see [6]) that if f satisfies (C) and the condition

$$f(x+y+z)=f(x+z+y), \quad \text{for } x,y,z \in G, \tag{K}$$

then there is a homomorphism $m : G \to \mathbb{C}^*$ $(\mathbb{C}^* = \mathbb{C}\backslash\{0\})$

$$m(x+y)=m(x)m(y), \quad \text{for } x,y \in G, \tag{1}$$

such that f has the form

$$f(x)=\frac{1}{2}(m(x)+m(-x)), \quad \text{for } x \in G. \tag{2}$$

Ever since Ulam [7] in 1940 raised the stability problem of the Cauchy equation $f(x+y)=f(x)+f(y)$, many authors (see Hyers [5], Ger [4], etc.) treated the stability problem for many other functional equations. Baker [2] proved the result:

Let $\varepsilon \geqslant 0$ be a given number and let $G(+)$ be an Abelian group. Let $f : G \to \mathbb{C}$ be such that

$$|f(x+y)+f(x-y)-2f(x)f(y)| \leqslant \varepsilon, \quad \text{for } x,y \in G.$$

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Then either $|f(x)| \leqslant \frac{1}{2}(1+\sqrt{1+2\varepsilon})$ for all $x \in G$ or f is a solution of the cosine equation (C).

Badora [1] presented a new, short proof of Baker's result.

The aim of this paper is to investigate the stability problem of the functional equations

$$f(x+y)+f(x-y)=2g(x)f(y), \quad x,y \in G \tag{3}$$

and

$$f(x+y)+f(x-y)=2f(x)g(y), \quad x,y \in G \tag{4}$$

in the next two sections modelled after [1], where G is a group.

2. Stability of (3) and (4) for complex functions

In this section we will consider the stability of (3) and (4) and their variants. First we will take up (3) and prove the following theorem.

THEOREM 1

Let $\varepsilon \geqslant 0$ and $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x+y)+f(x-y)-2g(x)f(y)| \leqslant \varepsilon \tag{3$'$}$$

with f satisfying the (K) *condition, where $G(+)$ is a group. Then either f and g are bounded or g satisfies* (C) *and f and g satisfy* (3) *and* (4)*. Further, in the latter case there exists a homomorphism $m : G \to \mathbb{C}^*$ satisfying* (1) *such that*

$$f(x)=\frac{b}{2}(m(x)+m(-x)) \quad \text{and} \quad g(x)=\frac{1}{2}(m(x)+m(-x)), \tag{5}$$

for $x \in G$, where b is a constant.

Proof. We will consider only the nontrivial f (that is, $f \neq 0$). Put $y=0$ in (3)$'$ to get

$$|f(x)-g(x)f(0)| < \frac{\varepsilon}{2}, \quad \text{for } x \in G. \tag{6}$$

If g is bounded, then using (6), we have

$$\begin{aligned} |f(x)| &= |f(x)-g(x)f(0)+g(x)f(0)| \\ &\leqslant \frac{\varepsilon}{2}+|g(x)f(0)|, \end{aligned}$$

which shows that f is also bounded. On the other hand if f is bounded, choose y_0 such that $f(y_0) \neq 0$ and then use (3)$'$,

$$|g(x)| - \left|\frac{f(x+y_0)+f(x-y_0)}{2f(y_0)}\right| \leqslant \left|\frac{f(x+y_0)+f(x-y_0)}{2f(y_0)} - g(x)\right| \leqslant \frac{\varepsilon}{2|f(y_0)|}$$

to get that g is also bounded on G.

It follows easily now that if f (or g) is unbounded, then so is g (or f). Let f and g be unbounded. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in G such that $g(x_n) \neq 0$, $|g(x_n)| \to \infty$ as $n \to \infty$ and $f(y_n) \neq 0$, $\lim_n |f(y_n)| = \infty$.

First we will show that g indeed satisfies (C).

From $(3)'$ with $y = y_n$ we obtain

$$\left| \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} - g(x) \right| \leqslant \frac{\varepsilon}{2|f(y_n)|},$$

that is,

$$\lim_n \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} = g(x). \tag{7}$$

Using $(3)'$ again and (K) we have

$$\begin{aligned} &|f(x+(y+y_n)) + f(x-(y+y_n)) - 2g(x)f(y+y_n) \\ &\quad + f(x+(y-y_n)) + f(x-(y-y_n)) - 2g(x)f(y-y_n)| \leqslant 2\varepsilon \end{aligned}$$

so that

$$\begin{aligned} &\left| \frac{f((x+y)+y_n) + f((x+y)-y_n)}{2f(y_n)} + \frac{f((x-y)+y_n) + f((x-y)-y_n)}{2f(y_n)} \right. \\ &\quad \left. -2g(x)\frac{f(y+y_n) + f(y-y_n)}{2f(y_n)} \right| \leqslant \frac{\varepsilon}{|f(y_n)|} \quad \text{for } x, y \in G, \end{aligned}$$

which with the use of (7) implies that

$$|g(x+y) + g(x-y) - 2g(x)g(y)| \leqslant 0,$$

that is g is a solution of (C).

As before applying $(3)'$ twice and the (K) condition, first we have

$$\lim_n \frac{f(x_n+y) + f(x_n-y)}{2g(x_n)} = f(y), \tag{8}$$

and then

$$\begin{aligned} &|f((x_n+x)+y) + f((x_n+x)-y) - 2g(x_n+x)f(y) \\ &\quad + f((x_n-x)+y) + f((x_n-x)-y) - 2g(x_n-x)f(y)| \leqslant 2\varepsilon \end{aligned}$$

so that

$$\begin{aligned} &\left| \frac{f(x_n+(x+y)) + f(x_n-(x+y))}{2g(x_n)} + \frac{f(x_n+(x-y)) + f(x_n-(x-y))}{2g(x_n)} \right. \\ &\quad \left. - 2 \cdot \frac{g(x_n+x) + g(x_n-x)}{2g(x_n)} f(y) \right| \leqslant \frac{\varepsilon}{|g(x_n)|}. \end{aligned}$$

From (8) and g satisfying (C), it follows that

$$|f(x+y)+f(x-y)-2g(x)f(y)| \leqslant 0,$$

that is, f and g are solutions of (3).

Choose y_0 such that $f(y_0) \neq 0$. Then (3) gives

$$g(x) = \frac{f(x+y_0)+f(x-y_0)}{2f(y_0)}$$

so that g also satisfies the condition (K). Since g satisfies (K), from [6] we see that there exists a homomorphism $m : G \to \mathbb{C}^*$ satisfying the second part of (5).

Finally, applying (3)′, (7) and (K), we get

$$\begin{aligned}|f((x_n+y)+x)+f((x_n+y)-x)-2g(x_n+y)f(x)+f((x_n-y)+x)\\ +f((x_n-y)-x)-2g(x_n-y)f(x)| \leqslant 2\varepsilon\end{aligned}$$

and that

$$\begin{aligned}\Bigg|\frac{f(x_n+(x+y))+f(x_n-(x+y))}{2g(x_n)}+\frac{f(x_n+(x-y))+f(x_n-(x-y))}{2g(x_n)}\\ -2f(x)\cdot\frac{g(x_n+y)+g(x_n-y)}{2g(x_n)}\Bigg| \leqslant \frac{\varepsilon}{|g(x_n)|}\end{aligned}$$

resulting to (4). From (3) and (4), it is easy to see that $f(x) = bg(x)$, for some constant b.

This proves the theorem.

We now consider a slight variation of (3)′.

Corollary 2

Let $\varepsilon \geqslant 0$. Let $f_n : G \to \mathbb{C}$ (where G is a group) be a sequence of functions converging uniformly to f on G. Suppose $f, g, f_n : G \to \mathbb{C}$ be such that

$$|f(x+y)+f(x-y)-2g(x)f_n(y)| \leqslant \varepsilon, \quad \text{for } x,y \in G, \qquad (3)''$$

with f satisfying (K). Then either f is bounded or g satisfies (C) and f and g satisfy (3) and (4).

Proof. Since $\{f_n\}$ is uniformly convergent to f, taking the limit with respect to n in (3)″, we obtain (3)′. The result now follows from Theorem 1.

Now we take up the stability of (4). We prove the following theorems

Theorem 3

Let $\varepsilon \geqslant 0$ and G be a group. Suppose $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x+y)+f(x-y)-2f(x)g(y)| \leqslant \varepsilon, \quad \text{for } x,y \in G \qquad (4)'$$

with f even (that is, $f(-x) = f(x)$) and f satisfies (K). *Then either f and g are bounded or f and g are unbounded and g satisfies* (C) *and f and g are solutions of* (4) *and* (3).

Proof. We consider only nontrivial f, that is, $f \neq 0$, When f is bounded, choose x_0 such that $f(x_0) \neq 0$ and use $(4)'$ to get

$$|g(y)| - \frac{|f(x_0+y)+f(x_0-y)|}{2|f(x_0)|} \leqslant \left| \frac{f(x_0+y)+f(x_0-y)}{2f(x_0)} - g(y) \right| \leqslant \frac{\varepsilon}{2|f(x_0)|},$$

which shows that g is also bounded.

Suppose f is unbounded. Choose $x = 0$ in $(4)'$ to have $|f(y) + f(-y) - 2f(0)g(y)| \leqslant \varepsilon$, that is, $|f(y) - f(0)g(y)| \leqslant \frac{\varepsilon}{2}$ (this is the only place we use that f is even). Since f is unbounded, $f(0) \neq 0$. Hence g is also unbounded.

Let f and so g be unbounded. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in G such that $f(x_n) \neq 0$, $|f(x_n)| \to \infty$, $g(y_n) \neq 0$, $|g(y_n)| \to \infty$.

Applying twice the inequality (4) and using (K) for f twice, first we get

$$\left| \frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} - g(y) \right| \leqslant \frac{\varepsilon}{2|f(x_n)|}$$

that is,

$$\lim_n \frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} = g(y), \quad \text{for } y \in G, \tag{9}$$

and then we obtain

$$\begin{aligned} |f((x_n+x)+y) &+ f((x_n+x)-y) - 2f(x_n+x)g(y) + f((x_n-x)+y) \\ &+ f((x_n-x)-y) - 2f(x_n-x)g(y)| \leqslant 2\varepsilon, \end{aligned}$$

that is,

$$\begin{aligned} \Bigg| \frac{f(x_n+(x+y))+f(x_n-(x+y)}{2f(x_n)} &+ \frac{f(x_n+(x-y))+f(x_n-(x-y))}{2f(x_n)} \\ - 2g(y)\, \frac{f(x_n+x)+f(x_n-x)}{2f(x_n)} \Bigg| &\leqslant \frac{\varepsilon}{|f(x_n)|} \end{aligned}$$

which by (9) leads to $|g(x+y) + g(x-y) - 2g(y)g(x)| \leqslant 0$, so that g satisfies (C).

Again applying the inequality $(4)'$ twice and using (K) condition for f twice, first we have

$$\left| \frac{f(x+y_n)+f(x-y_n)}{2g(y_n)} - f(x) \right| \leqslant \frac{\varepsilon}{2|g(y_n)|} \tag{10}$$

and then we get

$$\begin{aligned}&|f(x+(y_n+y))+f(x-(y_n+y))-2f(x)g(y_n+y)+f(x+(y_n-y))\\&\qquad+f(x-(y_n-y)-2f(x)g(y_n-y)|\leqslant 2\varepsilon\end{aligned}$$

that is,

$$\begin{aligned}&\left|\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)}\right.\\&\qquad\left.-2f(x)\frac{g(y_n+y)+g(y_n-y)}{2g(y_n)}\right|\leqslant\frac{\varepsilon}{|g(y_n)|}\end{aligned}$$

which by (10) and (C) yields

$$|f(x+y)+f(x-y)-2f(x)g(y)|\leqslant 0,$$

so that f and g are solutions of (4).

Consider the inequality

$$\begin{aligned}&|f((y_n+x)+y)+f(y_n+x)-y)-2g(y_n+x)f(y)+f(y_n-x)+y)\\&\qquad+f((y_n-x)-y)-2g(y_n-x)f(y)|\leqslant 2\varepsilon.\end{aligned}$$

As before using (K), (10), evenness of f and (C) and the division by $2g(y_n)$ yields

$$\begin{aligned}&\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)}\\&\qquad-2f(y)\frac{g(y_n+x)+g(y_n-x)}{2g(y_n)}\to 0\quad\text{as } n\to\infty\end{aligned}$$

so that f and g are solutions of (3).

This completes the proof of this theorem.

Note that the evenness of f is used to prove that g is unbounded when f is and nowhere else.

COROLLARY 4

Let $\varepsilon\geqslant 0$. Let $f_n: G\to\mathbb{C}$ (where G is a group) be a sequence of functions converging uniformly to f on G. Suppose $f, f_n, g: G\to\mathbb{C}$ be such that

$$|f(x+y)+f(x-y)-2f_n(x)g(y)|\leqslant\varepsilon,\quad \textit{for } x,y\in G, n\in\mathbb{N},\qquad (4)''$$

where f is even and it satisfies (K). *Then either f is bounded or g satisfies* (C) *and f and g are solutions of* (4) *and* (3).

Proof. Since $\{f_n\}$ is uniformly convergent to f, taking the limit with respect to n in $(4)''$, we get $(4)'$. The result now follows from Theorem 3.

3. Stability of (3) and (4) for vector valued functions

In [1] Badora gave a counter-example to illustrate the failure of the superstability of the cosine functional equation (C) in the case of the vector valued mappings. Here consider the following example. Let f and g be unbounded solution of (3) (or (4)) where $f, g : G \to \mathbb{C}$. Define $f_1, g_1 : G \to M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) by

$$f_1(x) = \begin{pmatrix} f(x) & 0 \\ 0 & c_1 \end{pmatrix}, \quad g_1(x) = \begin{pmatrix} g(x) & 0 \\ 0 & c_2 \end{pmatrix}$$

for $x \in G$ where $c_1 \neq 0$, $c_2 \neq 1$. Then

$$\|f_1(x+y) + f_1(x-y) - 2f_1(y)g_1(x)\| = \text{ constant } > 0$$

(or $\|f_1(x+y) + f_1(x-y) - 2f_1(x)g_1(y)\| = \text{ constant } > 0$) for $x, y \in G$. This f_1 and g_1 are neither bounded nor satisfy (C).

Therefore there is a need to consider the vector valued functions separately. We prove the following two theorems in this section. Let G be a group and A be a complex normed algebra with identity.

THEOREM 5

Suppose $f, g : G \to A$ satisfy the inequality

$$\|f(x+y) + f(x-y) - 2g(x)f(y)\| \leqslant \varepsilon, \tag{3'''}$$

for $x, y \in G$ with f satisfying (K) and

$$\|f(x) - f(-x)\| \leqslant \eta, \quad \text{for } x \in G, \tag{11}$$

for some $\varepsilon, \eta \geqslant 0$. Suppose there is a $z_0 \in G$ such that $g(z_0)^{-1}$ exists and $\|f(x)g(z_0)\|$ is bounded for $x \in G$. Then there is an $m : G \to A$ such that

$$\|m(x+y) - m(x)m(y)\| \leqslant a_1, \quad \text{for } x, y \in G \tag{12}$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| \leqslant a_2, \quad \text{for } x \in G \tag{13}$$

for some constants a_1 and a_2.

Proof. Let $M := \sup_{x \in G} \|f(x)g(z_0)\|$. Then using (3)''' and (11), we get by using (K)

$$\begin{aligned} \|f(x)g(-z_0)\| &\leqslant \|f(-x)g(z_0)\| + \|f(x)g(-z_0) - f(-x)g(z_0)\| \\ &\leqslant M + \tfrac{1}{2}\|f(z_0 - x) + f(z_0 + x) - 2g(z_0)f(-x) \\ &\quad - (f(-z_0 + x) + f(-z_0 - x) - 2g(-z_0)f(x)) \\ &\quad - (f(z_0 + x) - f(-z_0 - x) + f(z_0 - x) - f(-z_0 + x))\| \\ &\leqslant M + \varepsilon + \eta. \end{aligned}$$

Define a function $h : G \to A$ by the formula

$$h(x) = \frac{1}{2}(f(x) + f(-x)), \quad \text{for } x \in G.$$

Then h is even, that is, $h(-x) = h(x)$,

$$\|h(x) - f(x)\| \leqslant \frac{\eta}{2} \quad \text{for } x \in G, \quad \|h(x)g(z_0)\| \leqslant M. \tag{14}$$

Define a function $m : G \to A$ by

$$m(x) = h(x) + ig(z_0), \quad \text{for } x \in G.$$

Utilizing (14), we get (using first commutativity in A)

$$\begin{aligned}
\|m(x+y) - m(x)m(y)\| &= \|h(x+y) + ig(z_0) - h(x)h(y) \\
&\quad + i(h(x) + h(y))g(z_0) + g(z_0)^2\| \\
&\leqslant \|h(x+y)\| + \|h(x)h(y)\| \\
&\quad + \|(h(x) + h(y))g(z_0)\| + \|g(z_0)\| + \|g(z_0)\|^2 \\
&\leqslant \|h(x+y) - f(x+y)\| + \|f(x+y)\| \\
&\quad + \|h(x)h(y)g(z_0)^2 \cdot g(z_0)^{-2}\| + \|h(x)g(z_0)\| \\
&\quad + \|h(y)g(z_0)\| + \|g(z_0)\| + \|g(z_0)\|^2 \\
&\leqslant \frac{\eta}{2} + M\|g(z_0)\|^{-1} + M^2\|g(z_0)\|^{-2} \\
&\quad + 2M + \|g(z_0)\| + \|g(z_0)\|^2 \\
&= a_1
\end{aligned}$$

(say) which is (12). Finally by (14), we have

$$\begin{aligned}
\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| &= \Big\| f(x) - h(x) + h(x) \\
&\quad - \frac{1}{2}(h(x) + h(-x)) - ig(z_0) \Big\| \\
&\leqslant \frac{\eta}{2} + \|g(z_0)\| = a_2
\end{aligned}$$

(say), which is (13). This proves the theorem.

Lastly we prove the following theorem.

THEOREM 6

Let $f, g : G \to A$ satisfy the inequality

$$\|f(x+y) + f(x-y) - 2f(x)g(y)\| \leqslant \varepsilon, \quad x, y \in G, \tag{4)'''$$

with f satisfying (K) *and*

$$\|f(x) - f(-x)\| \leqslant \eta, \quad \textit{for } x \in G,$$

for some nonnegative ε *and* η. *Suppose there exists a* $z_0 \in G$ *such that* $g(z_0)^{-1}$ *exists and* $\|f(x)g(z_0)\|$ *is bounded over* G. *Then there exists a mapping* $m : G \to A$ *such that*

$$\|m(x+y) - m(x)m(y)\| \leqslant a_1, \quad \text{for } x, y \in G$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| \leqslant a_2, \quad \text{for } x \in G,$$

for some constants a_1 *and* a_2.

The proof runs parallel to that of Theorem 5.

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Generalized Hosszú functional equations

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. Pexiderizations of the Hosszú functional equation

$$f(xy) + f(x + y - xy) = f(x) + f(y)$$

are considered on a variety of domains.

1. Introduction

The functional equation

$$f(xy) + f(x + y - xy) = f(x) + f(y) \tag{H}$$

was first considered by M. Hosszú.

The general solution for real functions was given by Blanuša [1] and Daróczy [3]. They proved the following

Theorem B-D

The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (H) *for all $x, y \in \mathbb{R}$ if and only if*

$$f(x) = A(x) + C, \quad x \in \mathbb{R}, \tag{1}$$

where A is an additive function on $\mathbb{R}^2$ and $C \in \mathbb{R}$ is an arbitrary constant.

Equation (H) was also studied on other structures (see [2], [5], [6], [7], [9], [10], [14], [15], [16]).

In [12] and [13] we studied the functional equation

$$f(xy) + g(x + y - xy) = f(x) + f(y) \tag{GH1}$$

and we proved the following theorems.

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THEOREM L1

If the functions $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (GH1) *for all* $x, y \in \mathbb{R}$ *then*

$$f(x) = g(x) = A(x) + C, \quad x \in \mathbb{R},$$

where $A : \mathbb{R} \to \mathbb{R}$ *is an additive function on* $\mathbb{R}^2$ *and* $C \in \mathbb{R}$ *is an arbitrary constant.*

THEOREM L2

The functions $f : \mathbb{R}_0 \to \mathbb{R}$ $(\mathbb{R}_0 = \mathbb{R} \setminus \{0\})$ *and* $g : \mathbb{R} \to \mathbb{R}$ *satisfy the equation* (GH1) *for all* $x, y \in \mathbb{R}_0$ *if and only if*

$$f(x) = A_1(x) + A_2(\log|x|) + C, \quad x \in \mathbb{R}_0, \tag{2}$$

$$g(x) = A_1(x) + C, \quad x \in \mathbb{R}, \tag{3}$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ *are additive functions on* $\mathbb{R}^2$ *and* C *is a real constant.*

In this paper we shall deal with the following problems.

PROBLEM A

Let $f, g : (0,1) \to \mathbb{R}$ be real functions which satisfy the functional equation (GH1) for all $x, y \in (0,1)$. What is the general solution of (GH1) on the interval $(0,1)$?

PROBLEM B

Let $f, h : \mathbb{R}_0$ (or $\mathbb{R}_1 = \mathbb{R} \setminus \{1\}$) $\to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be real functions satisfying the functional equation

$$f(xy) + g(x + y - xy) = h(x) + h(y) \tag{GH2}$$

for all $x, y \in \mathbb{R}_0$ or $x, y \in \mathbb{R}_1$ respectively. Find the general solution of (GH2).

PROBLEM C

Let $f, g, h : (0,1) \to \mathbb{R}$ be real functions satisfying (GH2) for all $x, y \in (0,1)$. Find the general measurable solution of (GH2).

2. Problem A

Here we shall use the following result by Z. Daróczy and L. Losonczi (see [4]).

THEOREM D-L

If f *is additive on an open connected domain of* $\mathbb{R}^2$, *then* f *has one and only one quasi-extension.*

THEOREM 1

The functions $f, g : (0,1) \to \mathbb{R}$ *satisfy the generalized Hosszú equation* (GH1) *for all* $x, y \in (0,1)$ *if and only if*

$$f(x) = A_1(x) + A_2(\log x) + C, \quad x \in (0,1), \tag{4}$$

$$g(x) = A_1(x) + C, \quad x \in (0,1), \tag{5}$$

where A_1 and A_2 are additive functions on $\mathbb{R}^2$ and C is an arbitrary real constant.

Proof. First we follow the idea used in [14] for the proof of Lemma 2.
The function

$$F(x,y) = f(x) + f(y) - f(xy)$$

satisfies the equation

$$F(xy,z) + F(x,y) = F(x,yz) + F(y,z) \tag{6}$$

for all $x, y, z \in (0,1)$. On the other hand we have

$$F(x,y) = g(x + y - xy).$$

Putting this into (6), we obtain the equation

$$g(xy + z - xyz) + g(x + y - xy) = g(x + yz - xyz) + g(y + z - yz) \tag{7}$$

for all $x, y, z \in (0,1)$.

By the substitution

$$\left.\begin{aligned} xy + z - xyz &= t + \frac{1}{2} \\ x + y - xy &= s + \frac{1}{2} \\ y + z - yz &= \frac{1}{2} \end{aligned}\right\} \tag{8}$$

we obtain from (7) the functional equation

$$g\left(t + \frac{1}{2}\right) + g\left(s + \frac{1}{2}\right) = g\left(t + s + \frac{1}{2}\right) + g\left(\frac{1}{2}\right)$$

on the domain (t,s) defined by (8), i.e. on

$$D = \left\{(t,s) \mid -\frac{1}{2} < t < 0,\ \frac{1}{2} + \frac{1}{2t-1} < s < \frac{1}{2} + t\right\}.$$

Thus the function

$$A^* : \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbb{R}, \quad A^*(t) = g\left(t + \frac{1}{2}\right) - g\left(\frac{1}{2}\right) \tag{9}$$

is additive on the open connected domain D. So, by Theorem D-L, A^* has one and only one quasi-extension A_1 with $A^*(x) = A_1(x)$ for all $x \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$. Now, using (9), we have

$$g(x) = A_1\left(x - \frac{1}{2}\right) + g\left(\frac{1}{2}\right), \quad x \in (0,1).$$

This implies (5) with an arbitrary real constant $C = g\left(\frac{1}{2}\right) - A_1\left(\frac{1}{2}\right)$.

Substituting (5) in (GH1), we get that the function φ defined by

$$\varphi(x) = f(x) - A_1(x) - C, \quad x \in (0,1) \tag{10}$$

satisfies the functional equation

$$\varphi(xy) = \varphi(x) + \varphi(y), \quad x, y \in (0,1).$$

On setting

$$x = e^{-t},\ y = e^{-s} \quad (t, s > 0), \quad B(t) = \varphi(e^{-t}), \tag{11}$$

this is transformed into

$$B(t+s) = B(t) + B(s), \quad t, s > 0.$$

So, using again Theorem D-L, B has one and only one quasi-extension A with $B(t) = A(t)$ for all $t \in \mathbb{R}_+$. This, together with (11) implies

$$\varphi(x) = A_2(\log x), \quad x \in (0,1), \tag{12}$$

where $A_2 = -A$ is an additive function on $\mathbb{R}^2$.

Finally, from (10) and (12), (4) follows for the function f.

It is easy to see that (4) and (5) indeed satisfy (GH1).

3. Problem B

THEOREM 2

The functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation (GH2) *for all $x, y \in \mathbb{R}$ if and only if*

$$f(x) = A(x) + C_2, \quad x \in \mathbb{R}, \tag{13}$$
$$g(x) = A(x) + C_3, \quad x \in \mathbb{R}, \tag{14}$$
$$h(x) = A(x) + C_1, \quad x \in \mathbb{R}, \tag{15}$$

where A is an additive function on $\mathbb{R}^2$ and $C_i \in \mathbb{R}$ $(i = 1, 2, 3)$ are arbitrary constants with $2C_1 = C_2 + C_3$.

Proof. Putting into (GH2) $y = 0$ or $y = 1$, one gets

$$g(x) = h(x) + h(0) - f(0), \quad x \in \mathbb{R} \tag{16}$$

and

$$f(x) = h(x) + h(1) - g(1), \quad x \in \mathbb{R} \tag{17}$$

respectively. Substituting these into (GH2) we have

$$h(xy) + h(x + y - xy) = h(x) + h(y), \quad x, y \in \mathbb{R}.$$

This is the Hosszú functional equation. So, by Theorem B-D h is of the form (15). Taking (16), (17) and (15) into consideration also, we have proved (13) and (14).

One can easily to see that (13), (14) and (15) satisfy (GH2) if $2C_1 = C_2 + C_3$.

THEOREM 3

The functions $f, h : \mathbb{R}_0 \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in \mathbb{R}_0$ *if and only if*

$$f(x) = A_1(x) + A_2(\log|x|) + C_3, \quad x \in \mathbb{R}_0, \tag{18}$$

$$g(x) = A_1(x) + C_2, \quad x \in \mathbb{R}, \tag{19}$$

$$h(x) = A_1(x) + A_2(\log|x|) + C_1, \quad x \in \mathbb{R}_0, \tag{20}$$

where A_1, A_2 *are additive functions on* $\mathbb{R}^2$ *and* $C_i \in \mathbb{R}$ $(i = 1, 2, 3)$ *are arbitrary constants with* $2C_1 = C_2 + C_3$.

Proof. Setting $y = 1$ in (GH2) we obtain the identity

$$f(x) = h(x) + h(1) - g(1), \quad x \in \mathbb{R}_0 \tag{21}$$

and putting this into (GH2) we get

$$h(xy) + g(x + y - xy) + h(1) - g(1) = h(x) + h(y),$$

where $x, y \in \mathbb{R}_0$. This is an instance of the generalized Hosszú equation (GH1). Thus, by Theorem L1, h and g are of the forms (20) and (19) respectively, where $C_2 = C_1 + g(1) - h(1)$ is arbitrary constant. Finally, by (19), (20) and (21), we get (18) for f.

An easy calculation shows that the functions (18), (19) and (20) satisfy (GH2) if $2C_1 = C_2 + C_3$.

THEOREM 4

The functions $f : \mathbb{R} \to \mathbb{R}$ *and* $g, h : \mathbb{R}_1 \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in \mathbb{R}_1$ *if and only if*

$$f(x) = A_1(x) + C_3, \quad x \in \mathbb{R}, \tag{22}$$

$$g(x) = A_1(x) + A_2(\log|1 - x|) + C_2, \quad x \in \mathbb{R}_1, \tag{23}$$

$$h(x) = A_1(x) + A_2(\log|1 - x|) + C_1, \quad x \in \mathbb{R}_1, \tag{24}$$

where A_1 *and* A_2 *are additive functions on* $\mathbb{R}^2$ *and* $C_i \in \mathbb{R}$ $(i = 1, 2, 3)$ *are arbitrary constants with* $2C_1 = C_2 + C_3$.

Proof. Letting $y = 0$ in (GH2), we obtain

$$g(x) = h(x) + h(0) - f(0), \quad x \in \mathbb{R}_1. \tag{25}$$

Using (25) in (GH2), we get

$$f(xy) + h(x + y - xy) + h(0) - f(0) = h(x) + h(y), \quad x, y \in \mathbb{R}.$$

Replacing here x, y by $1 - x$, $1 - y$ respectively, we have

$$f(1 - (x + y - xy)) + h(0) - f(0) + h(1 - xy) = h(1 - x) + h(1 - y)$$

for all $x, y \in \mathbb{R}_0$, which implies that the functions f^* and h^* defined by

$$\begin{aligned} f^*(x) &= f(1-x) + h(0) - f(0), \quad x \in \mathbb{R}, \\ h^*(x) &= h(1-x), \quad x \in \mathbb{R}_0 \end{aligned} \tag{26}$$

satisfy the functional equation

$$f^*(x+y-xy) + h^*(xy) = h^*(x) + h^*(y), \quad x, y \in \mathbb{R}_0.$$

This is (GH1). Now, using Theorem L2, we have

$$h^*(x) = A_1^*(x) + A_2(\log|x|) + C_1^*, \quad x \in \mathbb{R}_0, \tag{27}$$

$$f^*(x) = A_1^*(x) + C_3^*, \quad x \in \mathbb{R}. \tag{28}$$

From (26), (27) and (28), we obtain (22) and (24) with $A_1 = -A_1^*$, $C_1 = A_1^*(1) + C_1^*$ and $C_3 = A_1^*(1) + C_3^* + f(0) - h(0)$. Finally (24) and (25) imply (23).

The functions (22), (23) and (24) indeed satisfy (GH2).

4. Problem C

We need the following result of A. Járai ([11] Theorem 2.7.2).

THEOREM J

Let $\mathcal{T}$ be a locally compact metric space, let Z_0 be a metric space, and let Z_i $(i = 1, 2, \ldots, n)$ be separable metric spaces. Suppose that D is an open subset of $\mathcal{T} \times \mathbb{R}^k$ and $X_i \subset \mathbb{R}^k$ for $i = 1, 2, \ldots, n$. Let $f_0 : \mathcal{T} \to Z_0$, $f_i : X_i \to Z_i$, $g_i : D \to X_i$, $H : D \times Z_1 \times Z_2 \times \ldots \times Z_n \to Z_0$ be functions. Suppose, that the following conditions hold:

(1) *For every $(t, y) \in D$*

$$f_0(t) = H(t, y, f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))).$$

(2) *f_i is Lebesgue measurable over X_i for $i = 1, 2, \ldots, n$.*

(3) *H is continuous on compact sets.*

(4) *For $i = 1, 2, \ldots, n$, g_i is continuous, and for every fixed $t \in \mathcal{T}$ the mapping $y \to g_i(t, y)$ is differentiable with the derivative $D_2 g_i(t, y)$ and with the Jacobian $J_2 g_i(t, y)$; moreover, the mapping $(t, y) \to D_2 g_i(t, y)$ is continuous on D and for every $t \in \mathcal{T}$ there exists a $(t, y) \in D$ so that*

$$J_2 g_i(t, y) \neq 0 \quad \textit{for } i = 1, 2, \ldots, n.$$

Then f_0 is continuous on $\mathcal{T}$.

LEMMA 1

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in (0,1)$ *then the functions* $f, g, h : (0,1) \to \mathbb{R}$ *are continuous.*

Proof. First we prove the continuity of f. From (GH2), with $t = xy$, we obtain

$$f(t) = h\left(\frac{t}{y}\right) + h(y) - g\left(1 - \frac{t}{y} - y + t\right), \quad (0 < t < y < 1). \tag{29}$$

Let $\mathcal{T} = (0,1)$, $n = 3$, $Z_0 = Z_1 = Z_2 = Z_3 = \mathbb{R}$, $X_1, X_2, X_3 = (0,1)$, $D = \{(t,y) \in \mathbb{R}^2 \mid 0 < t < y < 1\}$. Define the functions g_i on D by $g_1(t,y) = \frac{t}{y}$, $g_2(t,y) = y$, $g_3(t,y) = 1 - \frac{t}{y} - y + t$ and let $H(t, y, z_1, z_2, z_3) = z_1 + z_2 - z_3$. It follows from (29) that the functions f_i $(i = 0,1,2,3)$ given by

$$f_0 = f, \quad f_1 = f_2 = h, \quad f_3 = g$$

satisfy the functional equation occurring in (1) of Theorem J for all $(t,y) \in D$ and f_i $(i = 0,1,2,3)$ is measurable by the conditions of our lemma. H is clearly continuous and condition (4) of Theorem J holds too, since calculating $D_2 g_i$ one can see that for every $t \in \mathcal{T} = (0,1)$

$$D_2 g_i(t,y) \neq 0 \quad \text{for } i = 1,2,3 \text{ if } y \neq \sqrt{t}.$$

Thus, by Theorem J, $f = f_0$ is continuous on $(0,1)$.

The continuity of g can be proved by making the substitutions $x \to 1 - x$, $y \to 1 - y$ in (GH2) and repeating the above argument.

Substituting $y = \frac{1}{2}$ in (GH2) and solving the equation obtained for h we get

$$h(x) = g\left(\frac{x+1}{2}\right) + f\left(\frac{1}{2}x\right) - h\left(\frac{1}{2}\right), \quad x \in (0,1), \tag{30}$$

whence, by the continuity of f, g it follows that h is continuous as well.

LEMMA 2

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ *satisfy the functional equation* (GH2) *then they are differentiable infinitely many times on* $(0,1)$.

Proof. Write (GH2) in the form (29) and let $[\alpha, \beta] \subset (0,1)$ be arbitrary and choose the interval $[\lambda, \mu]$ such that $\sqrt{\beta} < \lambda < \mu < 1$ (then $[\alpha, \beta] \times [\lambda, \mu] \subset D = \{(t,y) \mid 0 < t < y < 1\}$ holds). Integrating (29) with respect to y on $[\lambda, \mu]$ we obtain

$$(\mu - \lambda) f(t) = \int_\lambda^\mu h\left(\frac{t}{y}\right) dy + \int_\lambda^\mu h(y)\, dy - \int_\lambda^\mu g\left(1 - \frac{t}{y} - y + t\right) dy.$$

We use the substitutions $g_1(t,y) = \frac{t}{y} = u$ and $g_3(t,y) = 1 - \frac{t}{y} - y + t = u$ in the first and third integral respectively. It is easy to check that these equations can uniquely be solved for y if $t \in [\alpha, \beta]$. In the case of $\frac{t}{y} = u$ this is clear. In the case of $1 - \frac{t}{y} - y + t = u$ this uniqueness is ensured by the assumption $\sqrt{\beta} < \lambda$, namely, by this condition, the derivative of the function $y \to g_3(t,y)$:

$$D_2 g_3(t,y) = \frac{t}{y^2} - 1$$

is negative on $[\alpha, \beta] \times [\lambda, \mu]$ hence our function is strictly decreasing. The solutions

$$y = \frac{t}{u} = \gamma_1(t,u) \quad \text{and} \quad y = \frac{1 + t - u + \sqrt{(1+t-u)^2 - 4t}}{2} = \gamma_2(t,u)$$

are infinitely many times differentiable functions of t and u. Performing the substitutions we have for $t \in [\alpha, \beta]$

$$f(t) = \frac{1}{\mu - \lambda} \left[\int_{\frac{t}{\lambda}}^{\frac{t}{\mu}} h(u) D_2\gamma_1(t,u)\, du - \int_{1-\frac{t}{\lambda}-\lambda+t}^{1-\frac{t}{\mu}-\mu+t} g(u) D_2\gamma_2(t,u)\, du + C \right],$$

where $C = \int_\lambda^\mu h(y)\, dy$. The functions h, g are at least continuous. Hence, by repeated application of the theorem concerning the differentiation of parametric integrals (see e.g. Dieudonné [8]), the sum on the right hand side is differentiable infinitely many times on $[\alpha, \beta]$. Since $[\alpha, \beta]$ is an arbitrary subinterval of $(0,1)$, we have that f is differentiable infinitely many times on $(0,1)$. The differentiability of g can be obtained similarly. Finally, from (30), we can deduce that h is also differentiable infinitely many times on $(0,1)$.

Lemma 3

If the functions $f, g, h : (0,1) \to \mathbb{R}$ *satisfy the functional equation* (GH2) *and they are twice differentiable in* $(0,1)$, *then there exist constants* $\gamma, C_i, \delta_i \in \mathbb{R}$ $(i = 1,2)$ *such that*

$$f(x) = C_1 \ln x + \gamma x + \delta_1, \quad x \in (0,1), \tag{31}$$

$$g(x) = C_2 \ln(1-x) + \gamma x + \delta_2, \quad x \in (0,1), \tag{32}$$

$$h(x) = C_1 \ln x + C_2 \ln(1-x) + \gamma x + \frac{\delta_1 + \delta_2}{2}, \quad x \in (0,1). \tag{33}$$

Proof. Differentiating (GH2) with respect to x, then the resulting equation with respect to y, we have

$$f'(xy) + xyf''(xy) - g'(x+y-xy) + (1-x)(1-y)g''(x+y-xy) = 0, \qquad x,y \in (0,1).$$

This can hold if and only if

$$tf''(t) + f'(t) = \gamma = (s-1)g''(s) + g'(s), \quad t, s \in (0,1)$$

for some constant γ.

The general solutions of the differential equations

$$tf''(t) + f'(t) = \gamma, \quad t \in (0,1)$$

and

$$(s-1)g''(s) + g'(s) = \gamma, \quad s \in (0,1)$$

have the following forms

$$f(t) = C_1 \ln t + \gamma t + \delta_1, \quad t \in (0,1),$$
$$g(s) = C_2 \ln(1-s) + \gamma s + \delta_2, \quad s \in (0,1).$$

Then, from (30), (31) and (32), we get (33) for h.

Thus we have proved our lemma.

We may sum up the results of Lemmas 1, 2 and 3 in the following theorem.

THEOREM 5

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ satisfy the functional equation (GH2) *for all $x, y \in (0,1)$, then there exist constants $\gamma, C_i, \delta_i \in \mathbb{R}$ $(i = 1,2)$ such that f, g, h have the forms* (31), (32) *and* (33) *respectively.*

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Christian Mira

Dynamical interpretation of Schröder's equation. Its consequences

Dedicated to Professor Zenon Moszner
on his 70th birthday

Abstract. This text deals with the domain of existence of the solution of Schröder's equation, related to a two-dimensional real iteration process, defined by functions which do not satisfy the Cauchy-Riemann conditions. Its purpose is limited to the identification of the difficulties generated by the determination of this domain. When the Cauchy-Riemann conditions are verified the answer to this problem was given by Fatou at the beginning of the 20th century. The qualitative theory of dynamical systems permits to identity the difficulties which may be met, from the notion of immediate basin of an attractor (stable fixed point in our case), and the singular set generated by the iteration associated with Schröder's equation.

1. Introduction

An evident link exists between *autonomous recurrences* (equivalent denominations depending on the mathematical field: *iterations, maps*), and some functional equations like those of Schröder, or Böttcher, or Abel, or Perron-Frobenius in some special case, or the equation of automorphic functions. This paper is essentially devoted to Schröder's functional equation. It is well known that the first set of studies on this equation appeared from the end of the 19th century in n-dimensional problems. The fundamental contributions are those of Grevy, Leau, Koenigs, Lattés whose papers concern a "*local*" study of the solution, i.e. its determination inside a sufficiently small neighborhood of a fixed point, or a cycle. A first "*global*" study is due to Julia (1918) [1], and Fatou (1919) [2] [3]. It concerns one-dimensional "rational" iterations with a complex variable, i.e. equivalently, two-dimensional iterations defined by functions with real variables satisfying the *Cauchy-Riemann conditions*. In particular Fatou's results suppose the existence of a stable fixed point with a non-zero multiplier, the boundary of its basin (domain of convergence toward

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this point) being what is called now a "*Julia set*". In this case it was stated that the fundamental solution of the Schröder's equation is a holomorphic function inside the ***immediate basin*** of the fixed point, the basin boundary belonging to the set of the ***essential singular points*** of this solution. As far as I know, this question has remained unexplored after these results for Schröder's equations related to classes of two-dimensional real iterations which do not satisfy the Cauchy-Riemann conditions. In this last more general case, it is a question of knowing if the notion of immediate basin can play the same role. In fact for noninvertible maps, and maps with canceling denominators, it appears that it is prudent to consider only a part of the immediate basin as domain of existence of the solution of the Schröder's equation associated with a stable fixed point of the iteration.

This paper does not pretend to deal with a close mathematical presentation of an extension of the Fatou's results in the case of *two-dimensional iterations with real variables*. Such a presentation would imply very long developments related to the convergence of series expansions, or infinite products, with the inherent difficulties induced by the boundaries of the domains of convergence. The aim is more modest. Indeed this text only tries to show how the dynamical approach permits to outline an extension of the Fatou's results. For the mathematicians specialists of functional equations this might give some first indications about the "landscape" of this question and its difficulties, from a point of view external to their field of study. For such a limited purpose, in the framework of the *qualitative theory of Dynamics*, it is sufficient to expose with commentaries a summarized presentation of certain results, obtained since some 30 years, on the basins structure generated by two-dimensional iterations with real variables. About the qualitative methods, it is well known that the solutions of equations of nonlinear dynamic systems are in general not classical transcendental functions of the Mathematical Analysis, which are very complex. So analytical methods generally failing, the "qualitative strategy" is of the same type as the one used for the characterization of a function of the complex variable by its singularities: zeros, poles, essential singularities. Here, for *two-dimensional maps with real variables* (topic of the paper) the complex transcendental functions are defined by the ***singularities*** of continuous (or discrete) dynamic systems such as:

— stationary states which are equilibrium points (fixed points), or periodical solutions (cycles); which can be stable, or unstable;
— trajectories (invariant curves) passing through *saddle* singularities of two dimensional systems;
— stable and unstable manifold for a dimension greater than two;
— boundary, or separatrix, of the influence domain of a stable (attractive) stationary state, called domain of attraction, or *basin*;
— *homoclinic*, or *heteroclinic singularities*;

— or more complex singularities of *fractal*, or nonfractal type.

The qualitative methods consist in the identification of two spaces associated with the map (iteration, recurrence relationship). The first space, called ***phase space*** (defined by the map variables), is related to the nature of the above singularities. The second space, called ***parameter space***, characterizes the singularities evolution when the system parameters vary, or in presence of a continuous structure modification of the system (definition of a ***function space***), by identification of the ***bifurcation sets***, loci of points boundary between two different qualitative changes. In the dynamics framework an *iteration* (equivalent denominations: recurrence relationship, map) is considered as a mathematical model of a discrete dynamical system. Since 1960, the important development of the computer means has given a large extension to the numerical approach of the problems of dynamic systems. Such an approach constitutes a powerful tool, when it is associated with the qualitative, or analytical, methods. In particular such a "mixed" approach has permitted to understand the complex structure of basins, and their bifurcations, that is the change of their qualitative properties in presence of parameter variations, cf. [7].

The paper is limited to ***two-dimensional Schröder's equations with real variables*** considered in the framework of the qualitative approach. This implies to define different classes of problems associated with basin boundaries (singular sets) of different nature. So problems involving ***invertible iterations***, ***noninvertible ones***, ***iterations defined by functions with a vanishing denominator*** must be differentiated.

The first part is a reminder of the Julia-Fatou's results. It is followed by the presentation of the matter related to two-dimensional maps not satisfying the Cauchy-Riemann conditions. The considered maps are firstly invertible, then noninvertible, without vanishing denominators in these two cases. The case of a vanishing denominator is dealt with in the last part.

2. Reminder of the Julia-Fatou's results

Let

$$z' = R(z) \tag{1}$$

be a one-dimensional iteration (or map, or recurrence relationship, or substitution), $R(z)$ being a rational function of the complex variable z, supposed not being of "fundamental circle" type. For simplicity sake it is assumed that the map has a unique attracting stable fixed point O, $S = R'(O)$ is its multiplier, $|S| < 1$, $S \neq 0$.

Let E be the set of all the unstable cycles generated by the iteration. Julia and Fatou [1]-[3] proved that the derived set E' of E contains E and is perfect. They showed that ***la structure de E' est la même dans toutes ses parties***, which means that the E' structure is self-similar, called *fractal* from

1976. The set E' can be either continuous, or discontinuous, it constitutes the set of essential singularities for any function limit of functions, extracted from an iterated sequence. It is also the set, the iterates of which do not form a *normal sequence* in the Montel sense. E' is the basin boundary of O, i.e., the boundary of the open domain of convergence toward O. It contains the whole set of the increasing rank preimages of the points of E. When it is a continuous set, the basin is generally disconnected, and made up of infinitely many disjoint parts. Then the part D_0 containing O is called the *immediate basin*, it includes a *critical point* (image of the point at which $\frac{dR}{dz} = 0$) of $R(z)$, and may be multiply connected with infinitely many holes.

The Fatou's contribution to the Schröder equation,

$$\gamma[R(z)] = S\gamma(z), \tag{2}$$

constitutes a particular case of more general functional equations considered in Chap. 7 of [3]. The main result states:

> The fundamental solution of the functional Schröder equation is a holomorphic function inside the immediate basin D_0 of O. Inside this domain it has infinitely many zeros having as limit points all the points of the immediate basin boundary ∂D_0. In the neighborhood of each of these boundary points, the function is completely indeterminated and takes all the values except infinity. Then the points of the immediate basin boundary are essential singular points of $\gamma(z)$.

The domain of existence of the function $\gamma(z)$ coincides with the connected domain of convergence (containing O) of the infinite product which permits to define $\gamma(z)$. The total basin of O may be disconnected. Then it is made up of the immediate basin D_0 and infinitely many domains which are the "arborescent" infinite sequences of its increasing rank preimages. Let D_1 be a rank-one preimage of D_0, different from D_0. So inside D_1 a function $\gamma_1(z)$ is defined. The variable z being in D_1, $R(z)$ is in D_0 and one has:

$$S\gamma_1(z) = \gamma[R(z)].$$

When z is inside D_0, the function $\gamma(z)$ satisfies (2). A generalization of a process of analytic continuation would give $\gamma_1(z)$ as the continuation of $\gamma(z)$ inside D_1. But in general D_1 and D_0 have no common points. So it would be necessary to find some lines out of the total basin, having contacts with the boundary, and leading along such lines to a uniform convergence of the expressions defining $\gamma(z)$. Until now it seems that this process has not been realized. So the functional equation defines infinitely many analytical functions having different bounded domains of existence. In the framework of the qualitative theory of dynamical systems, the solution $\gamma(z)$ of the Schröder equation is defined by the singular set made up of the zeros of $\gamma(z)$, which are the successive preimages of O in infinite number inside D_0, and the points of the boundary $\partial D_0 \subseteq E'$ of the immediate basin of O.

It is worth noting that the one-dimensional map (1) of the complex variable $z = x + jy$, $j^2 = -1$, is equivalent to the two-dimensional map with real variables:

$$x' = f(x,y), \quad y' = g(x,y), \tag{3}$$

the functions $f(x,y)$, $g(x,y)$ satisfying the conditions of Cauchy-Riemann:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}. \tag{4}$$

An example illustrating the properties of the complex set ∂D_0 is given by the map:

$$\begin{gathered} z' = \frac{3z - z^3}{2}; \\ f(x,y) = \frac{3x}{2} - \frac{x^3 - 3xy^2}{2}, \quad g(x,y) = \frac{3y}{2} + \frac{y^3 - 3x^2y}{2}. \end{gathered} \tag{5}$$

The origin is an unstable fixed point. This map has two stable fixed points ($x = \pm 1$, $y = 0$). The set E' made up of double points is everywhere dense. It is formed by the union of infinitely many closed simple Jordan curves, every points of one of these curves being the limit points of similar curves out of the one considered, their sizes tending toward zero, cf. [1]. The whole fractal set E' is symmetric with respect to the two axes. Figure 1 (see p. 74) represents the basin of each of the two stable fixed points from two different grey shades, and an enlargement of a basins part. The domain of existence of the solution of the Schröder equation related to one of the stable fixed point is its immediate basin.

3. Two-dimensional maps not satisfying the Cauchy-Riemann conditions

3.1. Difficulties generated by the problem

Consider the two-dimensional map (3) with real variables, the functions $f(x,y)$, $g(x,y)$ being analytic, and not satisfying conditions (4). Denote this map by T, and put $X = [x,y]$. The map (3) can be written in the form $X' = TX$. Let $O(0;0)$ be a stable fixed point of T, i.e., with multipliers $0 <\mid S_i \mid< 1$, $i = 1,2$. Consider the corresponding Schröder's equation:

$$\gamma_i(x',y') = S_i\gamma_i(x,y)], \quad \text{or} \quad \Gamma(TX) = S\Gamma(X), \tag{6}$$

with $\Gamma = [\gamma_1, \gamma_2]$, $S = [S_1, S_2]$. In the case of a stable cycle of period (or order) k, i.e., made up of k consequent points verifying: $T^k(X) = X$, $T^m(X) \neq X$, $0 < m < k$ ($k = 1$ gives a fixed point), the conclusions will be the same by considering T^k in (6).

An outline of extension of the Fatou's results would be given remarking that, if X varies in the whole immediate basin D_0 of the fixed point $O(0;0)$, it would seem reasonable to conjecture that the infinite products which define

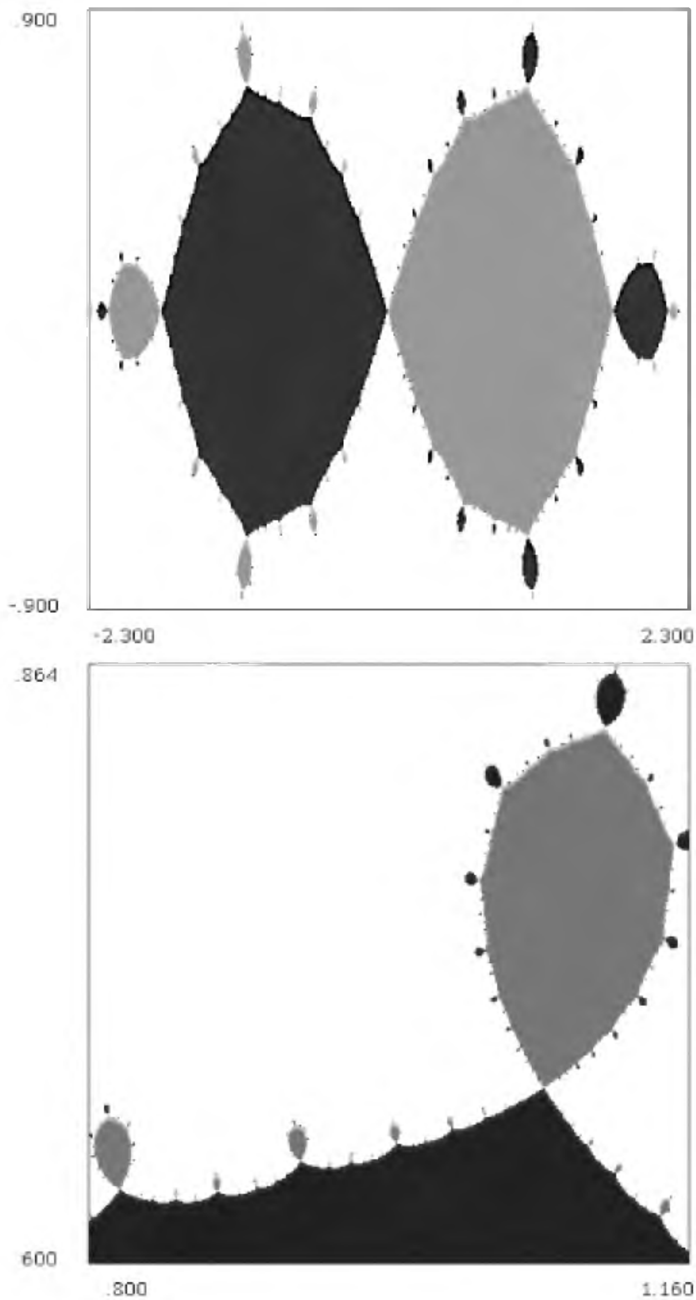

Fig. 1. Map (5). Basins of the fixed points ($x = +1$; $y = 0$; clear grey) and ($x = -1$; $y = 0$; dark gray)

$\Gamma(X)$ are uniformly convergent inside all close domains fully interior to D_0. Then $\Gamma(X)$ would be analytic inside D_0 and would satisfy (6) for each of its points. In this case the boundary ∂D_0 of the immediate basin would belong to the singular set related to $\Gamma(X)$. Therefore, in the framework of a qualitative approach, the problem boils down to study the structure of ∂D_0 and its qualitative modifications in presence of parameters variations. I think that such a conjecture is not true for all the iteration (or map) forms. It depends on the nature of the map, which in particular implies the consideration of the following classes of problems:

(a) T is a *diffeomorphism* defined by functions without canceling denominator,

(b) T is a *noninvertible map* defined by functions without vanishing denominator,

(c) For each of the two last cases T is defined by *functions with vanishing denominator.*

For the two-dimensional maps considered now it is important to note that ∂D_0 loses the properties of the perfect set E' mentioned in Sec. 2. Generally the new situations also present difficulties explained as follows. In the dynamics approach the knowledge of cells, giving the same qualitative behavior of solutions in the parameter space, is of prime importance for the analysis and the synthesis of continuous, or discrete mathematical models. On the boundary (*bifurcation set*) of a cell, a dynamic system is *structurally unstable.* In order to identify the difficulties, it is necessary to remind that the study of ordinary differential equations can be made via a Poincaré section leading to a map, the effective dimension of which is smaller. So a three-dimensional *flow* (vector field, or autonomous ordinary differential equation) leads to the formulation of a two-dimensional invertible map. In 1966 Smale showed that *for n-dimensional vector fields, $n > 2$, structurally stable systems are generally not dense in the function space*, which does not occur for $n = 2$. This means that p-dimensional maps, $p \geqslant 2$, have the same properties. So it appears that, with an increase of the problem dimension, one has an increase of complexity of the parameter (or function) space. This complexity appears for flows from the case $n = 3$, or for maps from the dimension $p = 2$. It results that the boundaries of the cells defined in the phase space (basins), as well as in the parameter space, have in general a complex structure which may be fractal (self-similarity properties) for n-dimensional vector fields, $n > 2$, and for p-dimensional maps with $p \geqslant 2$.

In 1979 Newhouse stated that in any neighborhood of a C^r-smooth ($r \geqslant 2$) dynamical system, in the space of dynamical systems, there exist regions for which systems with homoclinic tangencies (then with structurally unstable, or nonrough, homoclinic orbits) are dense. Domains having this property are called *Newhouse regions.* This result, as completed in [4], asserts that systems

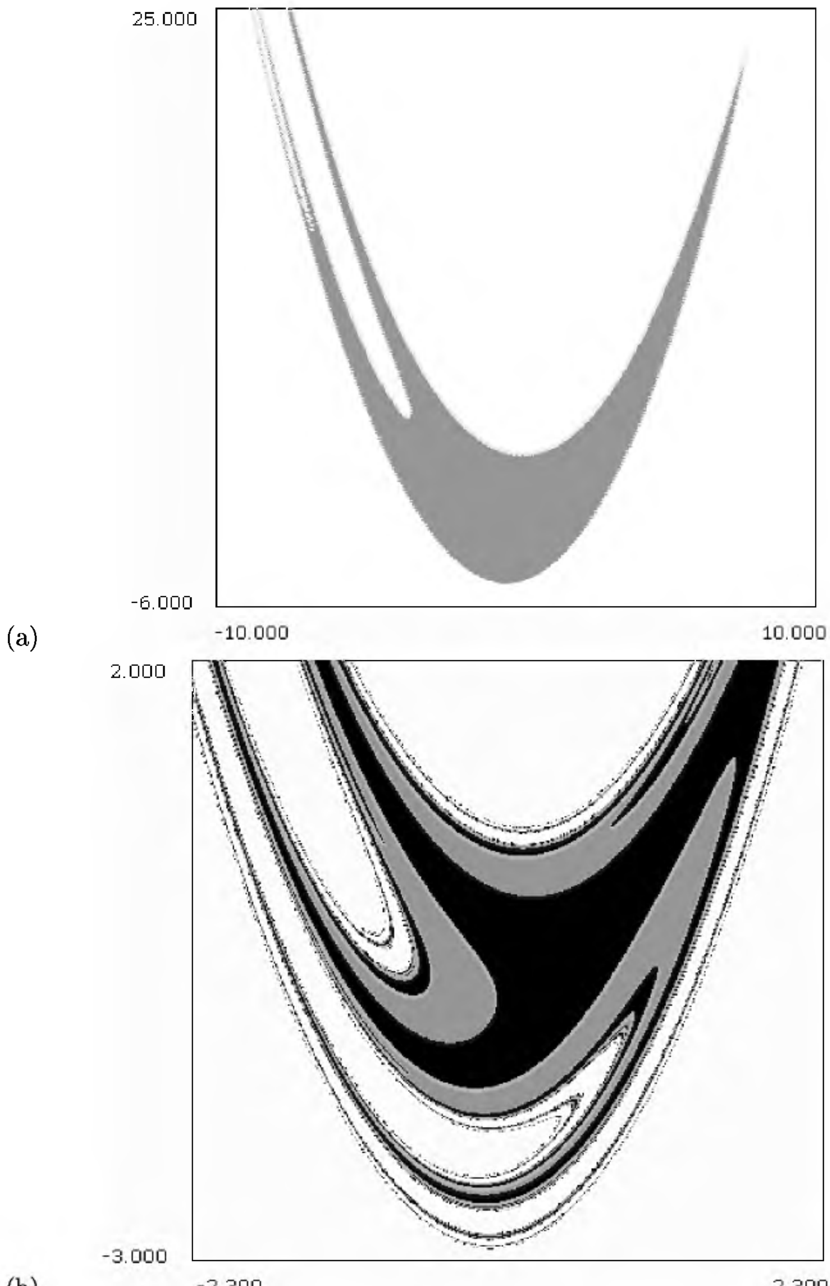

Fig. 2. Map (7). Basin of the fixed point q_2 (grey), and basin of the period 3 cycle (black). (a) $a = 0.4$, $b = 0.6$. (b) $a = 0.92$, $b = 0.7$.

with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has the important consequence concerning the dynamical properties:

Systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible.

More particularly, many of the attractors obtained numerically contain a “large” hyperbolic subset in presence of a finite, or an infinite number of stable periodic solutions. Generally such stable solutions have large periods, and narrow “oscillating” tangled basins, which are very difficult to display numerically. So it is only possible to consider some of the characteristic properties of the system, their interest depending on the nature of the problem nature, cf. [5]. The general problem of defining globally, and not locally, the solution of the functional equation (6) suffers from such limitations, due to the complex structure of a basin boundary. In the case of two-dimensional non-invertible maps this complexity increases, due to the introduction of another type of singularity: the *critical curve*, locus of points having two coincident rank-one preimages.

3.2. Diffeomorphisms without canceling denominator

Such maps, being invertible, have the important property: *the basin D of an attractor is always simply connected*, that is the immediate basin coincides with the total basin and contains no hole. In the simplest case, i.e., in absence of homoclinic and heteroclinic points, in general the boundary ∂D of D belongs to the *stable manifold* of some saddle cycles of period k ($k = 1$ corresponding to a fixed point). Locally this stable manifold can be defined by the series expansion given by S. Lattés. Its global determination is obtained by using the terms of the series until a given rank as a “germ” in a numerical method, for constructing ∂D, which belongs to the singular set of the solution of (6). Another numerical method consists in a scanning of the phase plane (x, y), which checks the convergence of the iterated sequence generated by each pixel of this plane as an initial condition. For each of these two methods it is possible to control the precision of the result.

Figure 2 (see p. 76) shows a type of basin (with fractal properties) obtained from the map:

$$x' = 1 - ax^2 + y, \quad y' = bx \tag{7}$$

which has two fixed points, $q_1(x_1, bx_1)$, and $q_2(x_2, bx_2)$, where

$$x_1 := \frac{1}{2a}\left(b - 1 - \sqrt{\Delta}\right), \quad x_2 := \frac{1}{2a}\left(b - 1 + \sqrt{\Delta}\right); \quad \Delta := (1 - b)^2 + 4a.$$

For $a = 0.4$, $b = -0.6$, q_1 is a saddle point, and q_2 is asymptotically stable. The basin D of q_2 is given by the grey marked region of Fig. 2a, the white one being the domain of divergence. It is bounded by the stable manifold of the saddle q_1, $W^S(q_1) \equiv \partial D$, a branch going to infinity. Such a parameter value of the plane (a, b) belongs to a region of the parameter plane, called *Morse-Smale region*, for which a unique attractor exists, with absence of homoclinic points, cf. [6]. It results a simple structure of the basin boundary, and then a "simple" singular set related to the solution of the Schröder equation (6).

Fig. 2b corresponds to $a = 0.92$, $b = -0.7$, a parameter point out of the Morse-Smale region, leading to the presence of homoclinic and heteroclinic points. The grey part is the basin of the stable fixed point q_2, the dark one is the basin of a stable period three cycle, the white region gives rise to divergence. The basins of q_2 and that of the stable period three cycle are separated by the stable manifold of its "satellite" period three saddle (i.e., the two period three cycles come from the same fold bifurcation). The two basins present infinitely many more and more narrow oscillating parts, tangled with the domain of divergence. A section of such regions by a line gives a *Cantor set*. The stable manifold of the saddle q_1 is a line of accumulations of the above oscillations. For this situation it is worth noting that the parameter point is in a Newhouse region. Therefore a numerically obtained image as Fig. 2b cannot make appear other eventual stable states having large periods, and very narrow "oscillating" tangled basins. This situation increases the complexity of the true "mathematical" structure of the basin of q_2, with its consequences on the structure of the singular set of the solution of the Schröder equation (6). Nevertheless the total basin being simply connected, an extension of results of Sec. 2 might present no difficulty in principle. In such a case the only "practical" difficulty lies in the fact that the domain of existence of the solution of (6) has a very complex structure. We shall say that this domain permits to define the "global" solution of Schröder's equation (6).

3.3. Non-invertible maps without canceling denominator

3.3.1. Difficulties generated by "global" solution of Schröder's equation in the simplest case

This section essentially concerns a family of two-dimensional smooth non-invertible maps, $X \to T(X)$, $X = [x, y]$, such that the critical curve LC is made up of only one branch separating the plane $\mathbb{R}^2$ in two open regions Z_0 and Z_2, the points of which have respectively 0 and 2 preimages (or antecedents or backward iterates) of rank one. The two real preimages of a point X belonging to Z_2 are given by the two inverses $T_1^{-1}(X)$, $T_2^{-1}(X)$ of T. Such noninvertible maps (which are the simplest ones) are called of $(Z_0\text{-}Z_2)$ *type* (cf. [7]). Their study is indispensable before considering more complex types, which locally may have the $(Z_0\text{-}Z_2)$ properties, plus others induced by more than two first rank preimages in certain regions of $\mathbb{R}^2$. The curve LC is the locus of points having two coincident rank-one preimages, located on a curve LC_{-1}, with $LC = T[LC_{-1}]$. If the map T is smooth, LC_{-1} is contained in the set on which the Jacobian J of T vanishes.

Denote by R_1, R_2 the two open regions such that $LC_{-1} = \overline{R_1} \cap \overline{R_2}$, and for every $X \in Z_2$, let $T_1^{-1}(X) \in R_1$, $T_2^{-1}(X) \in R_2$ be the two first rank preimages of X. If $X \in LC$ then $T_1^{-1}(X) = T_2^{-1}(X) \in LC_{-1}$.

It is recalled that a closed and invariant set A is called an *attracting set* if some neighborhood U of A exists such that $T(U) \subset U$, and $T^n(X) \to A$ as $n \to \infty$, $\forall X \in U$. An attracting set A may contain one or several *attractors* coexisting with sets of repulsive points (*strange repulsors*) giving rise to either *chaotic transients* towards these attractors, or *fuzzy boundaries* of their basin, cf. [6], [7]. The set $D = \bigcup_{n \geqslant 0} T^{-n}(U)$ is the *total basin* (or simply: basin of attraction, or influence domain) of A. That is D is the open set of points X whose forward trajectories (set of images of X with increasing rank) converge towards A. D is invariant under backward iteration T^{-1} of T, but not necessarily invariant by T:

$$T^{-1}(D) = D, \qquad T(D) \subseteq D. \tag{8}$$

In (8), the strict inclusion holds iff D contains points of Z_0, i.e., points without preimages. The relations in (8) hold also for the closure of D. The boundary (or frontier) of D is denoted by ∂D. The boundary ∂D is defined by the geometrical equality $\partial D = \overline{D} \cap \overline{C'(D)}$ where $C'(D)$ denotes the complementary set of D. This boundary satisfies:

$$T^{-1}(\partial D) = \partial D, \qquad T(\partial D) \subseteq \partial D \tag{9}$$

We remark that $T^{-1}(D) = D$ implies that D must contain the set of preimages of any of its cycles, that is ∂D must contain the stable set W^S of any cycle of T belonging to ∂D, while $T(\partial D) \subseteq \partial D$ means that the images of any of its points belongs to $\partial D \cap Z_2$. It is worth noting that, for unstable node and focus cycles, the stable set W^S is made up of the set of increasing rank preimages of cycle points (such a set does not exist in the case of an invertible map). For a saddle cycle W^S is made up of the local stable set W_l^S, associated with the determination of the inverse map which let invariant this cycle, and its increasing rank preimages.

Properties (8) and (9) with the strict inclusion are illustrated by the following example, cf. [7]:

$$x' = y, \qquad y' = 0.8x + 0.02y + x^2 + y^2, \tag{10}$$

leading to a simply connected basin of the stable fixed point $O(0;0)$. The curve of coincident first rank preimages LC_{-1} is $x = -0.4$, and the critical curve LC is the parabola $y = x^2 + 0.02x - 0.16$. The region R_1 is defined by $x > -0.4$, and R_2 by $x < -0.4$. The fixed point $O(0;0)$, $O \in R_1$, is a stable node. A second fixed point $P(0.09;0.09)$, $P \in R_1$, is a saddle with multipliers of opposite signs. Figure 3 (see p. 80) represents the boundary ∂D of the simply connected basin D of O. This boundary consists of the stable invariant set W^S of P. The determination of T^{-1} which let O and P invariant leads to T_1^{-1}. The set W_l^S is the open segment $]B, B_{-1}[$, where B_{-1} belongs to LC_{-1}, $B = T(B_{-1})$, and $W^S = W_l^S \cup T^{-1}(W_l^S) \cup T^{-2}(W_l^S)$. P has only one first rank preimage P_{-1} different from P. The two first rank preimages of B_{-1} are noted B^1_{-2} and B^2_{-2} in Fig. 3. Finally, $T^{-1}(C) = C_{-1} \in LC_{-1}$. The basin of O is simply connected, and satisfies: $T(D) = D \cap \overline{Z_2} \subset D$.

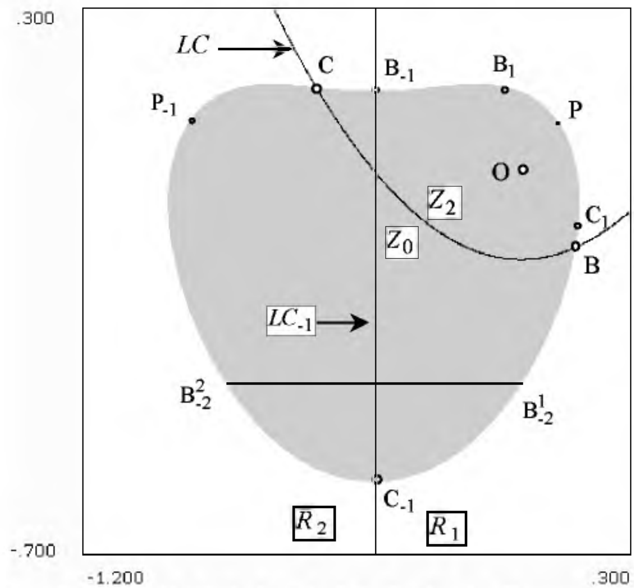

Fig. 3. Map (10). Basin (grey) of the fixed point O.

More generally a basin D may be *connected*, or *disconnected*. A connected basin may be *simply connected*, or *multiply connected* (which means connected with holes). A disconnected basin consists of a finite or infinite number of connected components (which may be simply or multiply connected). The properties and bifurcations related to these different situations will be considered in the next section. If A is a connected attractor (particular example: A is a fixed point), the *immediate basin* D_0 of A, is defined as the widest connected component of D containing A.

Let us return to the example of the map (10), where the two fixed points O and P are located in the region R_1. Consider the Schröder equation related to the stable fixed point O and note that the following properties:

$$T(D \cap Z_0) = D \cap Z_2, \quad T(D \cap R_2) = D \cap R_1 \cap Z_2$$

are satisfied. Using the Fatou's arguments, the extension of Sec. 2 results might present no essential difficulty in the region $D \cap R_1 \cap Z_2$, where the inverse of T which let O invariant is T_1^{-1}. Then one can conjecture that this region is at least a sufficient domain of existence of the solution of (6). It is not the case in the complementary region inside the basin D. Indeed a process of analytic continuation is not evident in this last region.

3.3.2. Problems generated by "global" solution of Schröder's equation in more general cases

The basin boundaries, belonging to the singular set of the solution of Schröder's equation, can have very complex structure with fractal properties described in [7]. From parameter variations they can undergo qualitative changes, related to *bifurcations* resulting from the contact of the basin boundary with a critical curve, or one of its image of a certain rank. The general case induces more complex situations with respect to the (10) one. This is due in particular to a large variety of qualitative modifications, with different types of fractalization, undergone by an immediate basin, and so by the domain of existence of the solution of the corresponding Schröder's equation.

The following example dealing with the map T:

$$x' = y, \quad y' = \left(\frac{y}{5} - \frac{x}{2}\right)\left(x^2 + y^2 - 6\frac{x}{5} - \lambda y + \frac{1}{2}\right) + x, \tag{11}$$

illustrates such modifications, when λ varies in the interval $1.05 < \lambda < 4.8$. For the case $\lambda = 4.1$ see Fig. 4 below.

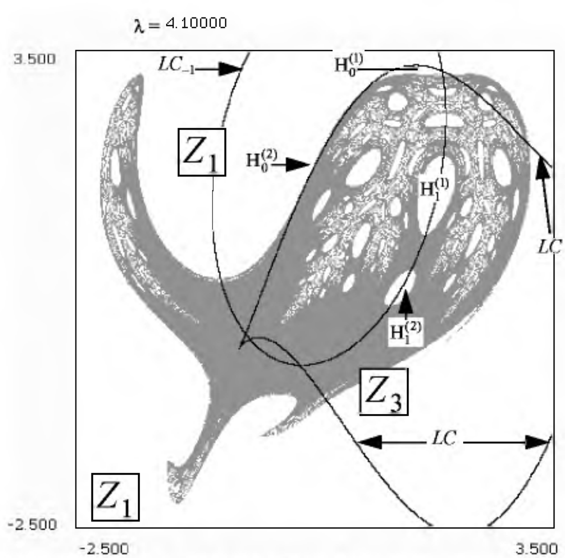

Fig. 4. Map (11), $\lambda = 4.1$. Basin (grey) of the fixed point O.

With the map T the fixed point $O(0;0)$ is always a stable node whatever be the parameter λ. The "global" solution of Schröder's equation, that is the domain of existence of the solution of (6), is considered for this fixed point O. It can be deduced from the detailed study of the basin modifications related to O described in [7] (cf. pages 439-446). Figure 4 gives an idea of the complexity of the O basin (grey marked) for $\lambda = 4.1$. This basin is multiply connected with a fractal structure.

3.4. Maps defined by functions with canceling denominator

Such maps T, invertible or non-invertible, introduce new types of singular sets, which has consequences on the determination of the domain of existence of the solution of the Schröder's equation (6). The first singularity concerns the set of nondefinition δ_s, locus of points in which at least one denominator vanishes, and the set of its successive preimages. The map is well defined provided that the initial condition belongs to the set $\hat{E}$ given by:

$$\hat{E} = \mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty} T^{-n}(\delta_s).$$

Indeed the points of the singular set δ_s, as well as all their preimages of any rank, which constitute a set of zero Lebesgue measure, must be excluded from the set of initial conditions in order to generate well defined sequences by iteration of T, so that $T : \hat{E} \to \hat{E}$.

Many other types of basin bifurcations, generated by two-dimensional non-invertible maps, and so many other qualitative changes of the existence domain of the solution of (6) are possible. Some of them are described in [7].

The presence of δ_s is followed by two other singular sets: *focal points* and *prefocal curve*. Roughly speaking a prefocal curve is a set of points for which at least one inverse exists, which maps (or "focalizes") the whole set into a single point, called focal point, which belongs to δ_s. More details on the consequences of such singularities on the structure of a basin, and on its bifurcations are given in [8]. It is evident that the domain of existence of the solutions of Schröder equation (6) must not contain a prefocal curve. Then when a prefocal curve cuts the immediate basin of a stable fixed point, separating this basin into two regions, the one which does not contain the fixed point must be excluded from the domain of existence of the solution of the Schröder equation (6). Indeed a process of analytic continuation might fail on the prefocal curve.

4. Conclusion

This text has the limited purpose to identify the difficulties generated by the determination of the domain of existence of the solution of a Schröder's equation, related to a two-dimensional real iteration (map) process, defined by functions which do not satisfy the Cauchy-Riemann conditions. The qualitative theory of dynamical systems permits to outline an answer from the notion of immediate basin of the stable fixed point of the considered iteration, and the singular sets generated by this iteration. It is worth noting that the dynamical approach permits to deal with another application of Schröder's equation. It concerns a method of construction of a class of iterations (recurrences, maps) giving rise to chaotic behaviors, which can be described from elementary functions. This process, taken on the pages 33-45 of [7] from results published in 1982, also leads to the definition of multi-dimensional function having properties similar to those of the *Chebyshev's polynomials*.

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Sur les généralisations du wronskien

Résumé. En généralisant le wronskien on montre que l'évanouissement de ce wronskien est équivalent à la dépendance linéaire du système des fonctions. Le théorème analogue est donné pour la décomposition d'une fonction $h(x, y)$ à la forme $a_1(x)b_1(y) + \cdots + a_n(x)b_n(y)$ par la généralisation du wronskien de h. On formule aussi un théorème lié à un problème ouvert concernant cette décomposition.

1. On sait bien que l'évanouissement du wronskien

$$W(f_1, \ldots, f_n) := \begin{vmatrix} f_1(x), & \ldots, & f_n(x) \\ f_1'(x), & \ldots, & f_n'(x) \\ \multicolumn{3}{c}{\dotfill} \\ f_1^{(n-1)}(x), & \ldots, & f_n^{(n-1)}(x) \end{vmatrix}$$

étant une condition nécéssaire pour la dépendence linéaire des fonctions $f_1, \ldots, f_n$, n'est pas la condition suffisante (le contre-exemple est donné déjà par G. Peano [8] en 1889).

Il y a beaucoup des théorèmes qui donnent une liaison entre l'évanouissement du wronskien et la dépendence linéaire des fonctions (voir la bibliographie dans [5] et [10]).

On sait aussi [9] que l'évanouissement du déterminant de Casorati

$$\begin{vmatrix} f_1(x_1), & \ldots, & f_n(x_1) \\ f_1(x_2), & \ldots, & f_n(x_2) \\ \multicolumn{3}{c}{\dotfill} \\ f_1(x_n), & \ldots, & f_n(x_n) \end{vmatrix}$$

est une condition équivalente à la dépendence linéaire des fonctions $f_1, \ldots, f_n$.

On peut considérer le wronskien généralisé du type de Casorati sous la forme suivante

$$\begin{vmatrix} f_1(x_1), & \ldots, & f_n(x_1) \\ f_1'(x_2), & \ldots, & f_n'(x_2) \\ \multicolumn{3}{c}{\dotfill} \\ f_1^{(n-1)}(x_n), & \ldots, & f_n^{(n-1)}(x_n) \end{vmatrix}.$$

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Il est vrai le

THÉORÈME 1

Le fonctions $f_1, \ldots, f_n$ complexes d'une variable complexe différentiables sur un ensemble E connexe sont linéairement dépendantes si et seulement si le wronskien du type de Casorati de ces fonctions reste nul pour tous $x_1, \ldots, x_n$ dans E.

Démonstration de "seulement si". Supposons que $f_1, \ldots, f_n$ soient linéairement dépendantes, c. à d. qu'ils existent des nombres complexes $\alpha_1, \ldots, \alpha_n$ pas tous nul et tels que

$$\alpha_1 f_1(x) + \cdots + \alpha_n f_n(x) = 0$$

pour chaque x dans E. Il en résulte que

$$\alpha_1 f_1^{(i)}(x_i) + \cdots + \alpha_n f_n^{(i)}(x_i) = 0 \quad \text{pour } i = 0, \ldots, n-1$$

et pour tous $x_1, \ldots, x_n$ dans E. Puisque $\alpha_1, \ldots, \alpha_n$ ne sont pas tous nul le wronskien du type de Casorati reste nul.

Démonstration de "si". Nous allons faire la démonstration par l'induction par rapport à n.

Pour $n = 1$ la supposition nous donne $f_1(x) = 0$ pour chaque x dans E, donc f_1 est linéairement dépendante.

Supposons que notre implication a lieu pour n et que le wronskien du type de Casorati de degré $n+1$ pour les fonctions $f_1, \ldots, f_n, f_{n+1}$ reste nul pour tous $x_1, \ldots, x_n, x_{n+1}$ dans E.

Considérons les deux cas suivants

a) ils existent les points $x_2, \ldots, x_{n+1}$ dans E pour lesquels la matrice

$$\left(f_j^{(i-1)}(x_i)\right)_{\substack{i = 2, \ldots, n+1 \\ j = 1, \ldots, n+1}}$$

a le rang égal à n ou

b) le rang de cette matrice est plus petit que n pour chaque $x_2, \ldots, x_{n+1}$ dans E.

Dans le cas a) en développant le wronskien du type de Casorati des fonctions $f_1, \ldots, f_{n+1}$, pour x_1 arbitraire dans E et $x_2, \ldots, x_{n+1}$ comme dans a), par rapport à la ligne première nous recevons la thèse.

Dans le cas b) d'après la supposition inductive chaques n des fonctions $f_1', \ldots, f_{n+1}'$ sont linéairement dépendantes, donc ils existent des nombres $\alpha_1, \ldots, \alpha_n$ pas tous nul tels que

$$\alpha_1 f_1'(x) + \cdots + \alpha_n f_n'(x) = 0$$

pour chaque x de E, d'où il existe c tel que

$$\alpha_1 f_1(x) + \cdots + \alpha_n f_n(x) = c.$$

Si $c = 0$ les fonctions $f_1, \ldots, f_n$ sont linéairement dépendantes, donc aussi les fonctions $f_1, \ldots, f_n, f_{n+1}$ sont les mêmes. Si $c \neq 0$ nous pouvons supposer $\alpha_1 \neq 0$ sans la restriction de la généralité, alors

$$\left(\frac{\alpha_1}{c}\right) f_1(x) + \cdots + \left(\frac{\alpha_n}{c}\right) f_n(x) = 1. \tag{1}$$

Aussi les fonctions $f_2', \ldots, f_{n+1}'$ sont linéairement dépendantes, donc ils existent des nombres $\beta_2, \ldots, \beta_{n+1}$ pas tous nul pour lesquels

$$\beta_2 f_2'(x) + \cdots + \beta_{n+1} f_{n+1}'(x) = 0,$$

d'où

$$\beta_2 f_2(x) + \cdots + \beta_{n+1} f_{n+1}(x) = d$$

pour un nombre complexe d. Si $d = 0$ la démonstration est terminée, si $d \neq 0$ nous avons

$$\left(\frac{\beta_2}{d}\right) f_2(x) + \cdots + \left(\frac{\beta_{n+1}}{d}\right) f_{n+1}(x) = 1. \tag{2}$$

En comparant les membres gauches dans (1) et (2) nous voyons que les fonctions $f_1, \ldots, f_{n+1}$ sont linéairement dépendantes puisque $\dfrac{\alpha_1}{c} \neq 0$.

La démonstration du Théorème 1 est donc terminée.

2. Pour la fonction $h(x,y)$ de deux variables l'évanouissement du wronskien de la forme

$$\begin{vmatrix} h(x,y), & h_y(x,y), & \ldots, & h_{y^{n-1}}(x,y) \\ h_x(x,y), & h_{yx}(x,y), & \ldots, & h_{y^{n-1}x}(x,y) \\ \ldots & \ldots & \ldots & \ldots \\ h_{x^{n-1}}(x,y), & h_{yx^{n-1}}(x,y), & \ldots, & h_{y^{n-1}x^{n-1}}(x,y) \end{vmatrix}$$

est une condition nécéssaire pour cette fonction soit de la forme

$$h(x,y) = a_1(x)b_1(y) + \cdots + a_{n-1}(x)b_{n-1}(y), \tag{3}$$

n'étant pas en même temps de la condition suffisante (le contre-exemple premier est donné par Th.M. Rassias en 1986, voir [9] ou [10] p. 32).

Si nous remplaçons le wronskien plus haut par le déterminant du type de Casorati de la fonction h:

$$\begin{vmatrix} h(x_1,y_1), & h(x_1,y_2), & \ldots, & h(x_1,y_n) \\ h(x_2,y_1), & h(x_2,y_2), & \ldots, & h(x_2,y_n) \\ \ldots & \ldots & \ldots & \ldots \\ h(x_n,y_1), & h(x_n,y_2), & \ldots, & h(x_n,y_n) \end{vmatrix},$$

son évanouissement est équivalent à la forme (3) de la fonction h (le résultat de F. Neuman, voir [7] ou le Théorème 22.2.1 à la page 36 et la Remarque 2.2.2 (ii) à la page 38 dans [10]).

On peut considérer le wronskien de la fonction h du type de Casorati

$$\begin{vmatrix} h(x_1,y_1), & h_y(x_1,y_2), & \dots, & h_{y^{n-1}}(x_1,y_n) \\ h_x(x_2,y_1), & h_{yx}(x_2,y_2), & \dots, & h_{y^{n-1}x}(x_2,y_n) \\ \dots & \dots & \dots & \dots \\ h_{x^{n-1}}(x_n,y_1), & h_{yx^{n-1}}(x_n,y_2), & \dots, & h_{y^{n-1}x^{n-1}}(x_n,y_n) \end{vmatrix}. \tag{4}$$

Nous avons le

THÉORÈME 2

Pour une fonction h complexe de deux variables complexes, ayant les dérivées jusqu'à

$$h_{y^{n-1}x^{n-1}}$$

sur $X\times Y$, où X et Y sont des ensembles connexes dans $\mathbb{C}$, le wronskien du type de Casorati reste nul pour tous $x_1,\dots,x_n$ dans X et $y_1,\dots,y_n$ dans Y si et seulement si la fonction h est de le forme (3), *où $a_k : X \longrightarrow \mathbb{C}$ et $b_k : Y \longrightarrow \mathbb{C}$ pour $k = 1,\dots,n-1$.*

Démonstration de "si". Le wronskien du type de Casorati de h c'est le même que le wronskien du type de Casorati des fonctions

$$h(x_1,y_1), h_y(x_1,y_2),\dots,h_{y^{n-1}}(x_1,y_n)$$

comme les fonction de x_1 avec $y_1,\dots,y_n$ fixés arbitrairement dans Y. D'après le Théorème 1 il suffit donc montrer que ces fonctions sont linéairement dépendantes dans X. Nous avons

$$h(x_1,y_1) = a_1(x_1)b_1(y_1) + \dots + a_{n-1}(x_1)b_{n-1}(y_1),$$

$$h_{y^i}(x_1,y_{i+1}) = a_1(x_1)b_1^{(i)}(y_{i+1}) + \dots + a_{n-1}(x_1)b_{n-1}^{(i)}(y_{i+1})$$

pour $i = 1,\dots,n-1$. Il en résulte que les n fonctions

$$h(x_1,y_1), h_y(x_1,y_2),\dots,h_{y^{n-1}}(x_1,y_n)$$

sont des combinaisons linéaires des $n-1$ fonctions $a_1(x_1),\dots,a_{n-1}(x_1)$ ($y_1,\dots,$ y_n sont fixés!), donc elles doivent être linéairement dépendantes.

Démonstration de "seulement si". Par l'induction par rapport à n.

Pour $n = 1$ la thèse est évidente. Supposons l'implication à prouvée pour $n-1$ et que le wronskien (4) reste nul et considérons les deux cas siuvants:

a) ils existent $x_2,\dots,x_n$ dans X et $y_2,\dots,y_n$ dans Y tels que le déterminant

$$\begin{vmatrix} h_{yx}(x_2,y_2), & \dots, & h_{y^{n-1}x}(x_2,y_n) \\ \dots & \dots & \dots \\ h_{yx^{n-1}}(x_n,y_2), & \dots, & h_{y^{n-1}x^{n-1}}(x_n,y_n) \end{vmatrix} \tag{5}$$

est différent de zéro ou

b) ce déterminant reste nul pour tous $x_2, \ldots, x_n$ dans X et pour tous $y_2, \ldots, y_n$ dans Y.

Dans de la cas a) en posant dans (4) pour $x_2, \ldots, x_n$ et $y_2, \ldots, y_n$ les nombres qui existent d'après a) et en développant le wronskien (4) par rapport à la ligne première nous pouvons compter $h(x_1, y_1)$ sous la forme (3).

Dans le cas b) puisque le déterminant (5) c'est le wronskien du type de Casorati de la fonction h_{yx} de degré $n-1$, d'après la supposition inductive nous avons

$$h_{yx}(x,y) = a_1(x)b_1(y) + \cdots + a_{n-2}(x)b_{n-2}(y),$$

d'où

$$h(x,y) = a_1(x)b_1(y) + \cdots + a_{n-2}(x)b_{n-2}(y) + c(x) + d(y) \tag{6}$$

pour certaines fonctions $c : X \longrightarrow \mathbb{C}$ et $d : Y \longrightarrow \mathbb{C}$.

Le wronskien (4) c'est le wronskien du type de Casorati des fonctions $h(x,y_1), h_y(x,y_2), \ldots, h_{y^{n-1}}(x,y_n)$ comme les fonctions de x avec $y_1, \ldots, y_n$ fixés. Puisque ce wronskien reste nul, alors ces fonctions sont linéairement dépendantes, donc ils existent des fonctions $\alpha_k : Y^n \longrightarrow \mathbb{C}$ pour $k = 1, \ldots, n$, telles que $\alpha_k(y_1, \ldots, y_n)$ ne sont pas tous nul pour chaque $(y_1, \ldots, y_n)$ dans Y^n et

$$\begin{aligned}
& \alpha_1(y_1,\ldots,y_n)\left[a_1(x)b_1(y_1) + \cdots + a_{n-2}(x)b_{n-2}(y_1) + c(x) + d(y_1)\right] \\
+ \; & \alpha_2(y_1,\ldots,y_n)\left[a_1(x)b_1'(y_2) + \cdots + a_{n-2}(x)b_{n-2}'(y_2) + d'(y_2)\right] \\
+ \; & \cdots \\
+ \; & \alpha_n(y_1,\ldots,y_n)\left[a_1(x)b_1^{(n-1)}(y_n) + \cdots + a_{n-2}(x)b_{n-2}^{(n-1)}(y_n) + d^{(n-1)}(y_n)\right] \\
= \; & 0.
\end{aligned}$$

Considérons quelques cas. S'ils existent $y_1, \ldots, y_n$ tels que $\alpha_1(y_1, \ldots, y_n) \neq 0$ nous pouvons calculer $c(x)$ de l'égalité plus haut comme la combinaison linéaire de $a_1(x), \ldots, a_{n-2}(x)$ et d'après (6) nous avons la thèse. Si α_1 reste nul toujours nous avons

$$\begin{aligned}
& \left\{b_1'(y_2)\alpha_2 + \cdots + b_1^{(n-1)}(y_n)\alpha_n\right\} a_1(x) \\
& + \ldots \\
& + \left\{b_{n-2}'(y_2)\alpha_2 + \cdots + b_{n-2}^{(n-1)}(y_n)\alpha_n\right\} a_{n-2}(x) \\
& + \alpha_2 d'(y_2) + \cdots + \alpha_n d^{(n-1)}(y_n) \\
& = 0
\end{aligned}$$

S'ils existent $y_2, \ldots, y_n$ tels que un des parenthèses { } est différent de zéro nous calculons la fonctions "a" qui se trouve à côté de cette parenthèse par les fonctions "a" restantes et la formule (6) nous donne la thèse.

Si toutes les parenthèses { } restent nul pour tous $y_2, \dots, y_n$ dans Y, dans ce cas aussi $\alpha_2 d'(y_2) + \dots + \alpha_n d^{(n-1)}(y_n) = 0$ et puisque $\alpha_2, \dots, \alpha_n$ ne sont pas tous nul le déterminant

$$\begin{vmatrix} d'(y_2), & d''(y_3), & \dots, & d^{(n-1)}(y_n) \\ b_1'(y_2), & b_1''(y_3), & \dots, & b_1^{(n-1)}(y_n) \\ \dots & \dots & \dots & \dots \\ b_{n-2}'(y_2), & b_{n-2}''(y_3), & \dots, & b_{n-2}^{(n-1)}(y_n) \end{vmatrix}$$

doit être égal à zéro pour tous $y_2, \dots, y_n$ dans Y. Il en résulte de Théorème 1 que les fonctions $b_1'(y), \dots, b_{n-1}'(y), d'(y)$ sont linéairement dépendantes. Ils existent donc des nombres $\beta_1, \dots, \beta_{n-1}$ pas tous nul et tels que

$$\beta_1 b_1'(y) + \dots + \beta_{n-2} b_{n-2}'(y) + \beta_{n-1} d'(y) = 0.$$

En calculant d'ici une des fonctions $b_1, \dots, b_{n-2}, d$ et en posant cette fonction comptée dans (6) nous recevons la thèse. Ça finit la démonstration du Théorème 2.

Remarquons qu'on peut recevoir le Théorème 1 d'après le Théorème 2. En effect il suffit considérer dans le Théorème 2

$$h(x, y) = \sum_{i=1}^{n} f_i(x) \frac{y^{i-1}}{(i-1)!} \quad \text{pour } (x, y) \in E \times \mathbb{C},$$

où $f_1(x), \dots, f_n(x)$ sont comme dans le Théorème 1. Dans ce cas le wronskien (4) de h du type de Casorati est égal au wronskien du type de Casorati du système $f_1, \dots, f_n$, donc si ce wronskien dernier est égal à zéro, d'après le Théorème 2 la fonction h est de la forme (3), d'où

$$h_{y^k}(x, y) = a_1(x) b_1^{(k)}(y) + \dots + a_{n-1}(x) b_{n-1}^{(k)}(y)$$

pour $k = 1, \dots, n-1$. Il en résulte que

$$f_j(x) = h_{y^{j-1}}(x, 0) = b_1^{(j-1)}(0) a_1(x) + \dots + b_{n-1}^{(j-1)}(0) a_{n-1}(x)$$

pour $j = 1, \dots, n$, alors les fonctions $f_1, \dots, f_n$, étantes les combinaisons linéaires des fonctions $a_1, \dots, a_{n-1}$, sont linéairement dépéndantes.

La raisonnement plus haut n'est pas la démonstration du Théorème 1, puisque dans la démonstration du Théorème 2 nous profitons le Théorème 1. Il se pose donc le

Problème

Donner la démonstration du Théorème 2 ne profitant pas du Théorème 1.

3. Nous allons considérer le problème suivant. Est-ce que si le rang de la matrice de Wronski de la fonction $h(x, y)$:

$$\begin{pmatrix} h(x,y), & h_y(x,y), & \dots, & h_{y^{n-1}}(x,y) \\ h_x(x,y), & h_{yx}(x,y), & \dots, & h_{y^{n-1}x}(x,y) \\ \dots & \dots & \dots & \dots \\ h_{x^{n-1}}(x,y), & h_{yx^{n-1}}(x,y), & \dots, & h_{y^{n-1}x^{n-1}}(x,y) \end{pmatrix} \tag{7}$$

est égal à $p < n$ pour chaque (x,y), alors la fonction h doit être de la forme $a_1(x)b_1(y) + \cdots + a_p(x)b_p(y)$?

Le problème est suggéré par les deux théorèmes suivants:

THÉORÈME (A)

Si le rang de la matrice de Wronski des fonctions $f_1, \dots, f_n$ réelles d'une variable réelle de classe C^{n-1} est stable et plus petit que n, alors ces fonctions sont linéairement dépendantes ([4]).

THÉORÈME (B) (F. Neuman — [7] ou [10] p. 30)

Si pour la fonction h ayante le dérivée $h_{y^{n-1}x^{n-1}}$ continue, le déterminant de la matrice (7) est égal à zéro pour chaque (x,y) et le déterminant de la matrice analogue à la matrice (7) mais de degré $n-1$ est différent de zéro pour chaque (x,y), alors $h(x,y) = a_1(x)b_1(y) + \cdots + a_{n-1}(x)b_{n-1}(y)$.

Le Théorème (A) sera applique chez nous pour $n = 2$ et $n = 3$.

La réponse au problème plus haut est positive pour $p < 2$ pour chaque $n > p$ et pour $p = 2$ et $n = 3$, si la fonction h est réelle des variables réelles, ayante les dérivés continues jusqu'à $h_{y^{n-1}x^{n-1}}$.

Je donnerai les démonstrations de ces faits.

Pour $p = 0$ la thèse est évidente.

Passons au cas $p = 1$. D'après le théorème de M. Čadek et J. Šimša ([1] ou [10] p. 99) il suffit montrer que pour chaque point (x_0, y_0) dans $I \times J$, où I et J forment des intervalles dans $\mathbb{R}$, il existe un entourage de ce point tel que la fonction h est de la forme $f(x)g(y)$ dans cet entourage avec les fonctions f et g telles que chaque est localement linéairement indépendante (c. à d. elle est linéairement indépendante, donc pas identiquement zéro, dans chaque sous-ensemble ouvert d'un entourage de x_0 ou y_0). Remarquons qu'il suffit montrer que la fonction h est de la forme $f(x)g(y)$, puisque si $f \equiv 0$ dans un sous-ensemble ouvert d'un entourage de x_0 ou $g \equiv 0$ dans un sous-ensemble ouvert d'un entourage de y_0, alors la matrice (7) ne peut pas avoir toujours le rang égal à 1. Nous allons montrer ça par l'induction par rapport à n. Pour $n = 2$ notre théorème est démontré dans [6], mais plus bas je donne une autre démonstration, en outre pour la commodité du lecteur (plus facile, je crois). (Rappelons que l'équation $hh_{yx} - h_x h_y = 0$ c'est bien connue l'équation différentielle de d'Alembert).

Si $h(x_0, y_0) \neq 0$, alors $h(x,y) \neq 0$ sur un entourage de (x_0, y_0). Nous pouvous supposer que $h(x,y) > 0$ sur cet entourage puisque dans le cas $h(x,y) < 0$ nous pouvons replacer h par $-h$. L'équation de d'Alembert a dans ce cas la forme

$$\frac{\partial^2 \ln h(x,y)}{\partial x \partial y} = 0,$$

d'où $h(x,y) = a(x)b(y)$.

Soit $h_y^2(x_0,y_0) + h_{yx}^2(x_0,y_0) \neq 0$, donc $h_y(x,y) \neq 0$ sur un entourage E de (x_0,y_0) ou $h_{yx}(x,y) \neq 0$ sur un entourage E de (x_0,y_0). Nous allons démontré que

$$\begin{vmatrix} h(x,y_1), & h(x,y_2) \\ h_x(x,y_1), & h_x(x,y_2) \end{vmatrix} = 0 \quad \text{pour chaque } (x,y_1),\ (x,y_1) \text{ de } E.$$

D'après de Théorème 1 de [2] ils existent $\lambda(x)$ et $\eta(x)$ tels que

$$\begin{aligned} \begin{vmatrix} h(x,y_1), & h(x,y_2) \\ h_x(x,y_1), & h_x(x,y_2) \end{vmatrix} &= \begin{vmatrix} h(x,y_1), & h(x,y_2) - h(x,y_1) \\ h_x(x,y_1), & h_x(x,y_2) - h_x(x,y_1) \end{vmatrix} \\ &= \lambda(x) \begin{vmatrix} h(x,y_1), & h_y(x,\eta(x)) \\ h_x(x,y_1), & h_{yx}(x,\eta(x)) \end{vmatrix} \\ &= 0 \end{aligned}$$

d'après le Théorème 1 de cette note, puisque les fonctions $y \longrightarrow h(x,y)$ et $y \longrightarrow h_x(x,y)$ sont linéairément dépendantes d'après le Théorème (A) plus haut. De plus le rang de la matrice

$$\begin{pmatrix} h(x,y_1), & h(x,y_2) \\ h_x(x,y_1), & h_x(x,y_2) \end{pmatrix}$$

est égal à 1, puisque dans le cas contraire il existerait un x_0 tel que $h(x_0,y_1) = h(x_0,y_2) = 0 = h_x(x_0,y_1) = h_x(x_0,y_2)$, alors ils existeraient les χ et ε tels que $h_y(x_0,\chi) = 0 = h_{xy}(x_0,\varepsilon)$, en contradiction au choix de l'entourage E. Les fonctions $x \longrightarrow h(x,y_1)$ et $x \longrightarrow h(x,y_2)$ sont donc linéairément dépendantes. Ils existent donc $\alpha(y_1,y_2)$ et $\beta(y_1,y_2)$ tels que

$$\alpha(y_1,y_2)h(x,y_1) + \beta(y_1,y_2)h(x,y_2) = 0 \quad \text{et} \quad \alpha^2(y_1,y_2) + \beta^2(y_1,y_2) \neq 0.$$

Soit $\overline{y}$ tel que $h(x,\overline{y}) \not\equiv 0$. Dans ce cas

$$\alpha(\overline{y},y)h(x,\overline{y}) + \beta(\overline{y},y)h(x,y) = 0$$

nous donne $\beta(\overline{y},y) \neq 0$, puisque dans le cas contraire $h(x,\overline{y}) \equiv 0$ ($\alpha(\overline{y},y) \neq 0$ dans ce cas) et ça est impossible. Nous avons donc

$$h(x,y) = \frac{\alpha(\overline{y},y)}{\beta(\overline{y},y)}\, h(x,\overline{y}),$$

alors la forme exigée de la fonction h.

Dans le cas $h_x(x_0,y_0) \neq 0$ il suffit raisonner comme plus haut, en changeant x et y. Notre théorème est donc démontré pour $n = 2$.

Supposons maintenant que notre théorème est vrai pour la matrice de Wronski de la dimension $n \times n$ et que la matrice de Wronski de la fonction h de la dimension $(n+1) \times (n+1)$ a le rang égal à 1. Ils existent donc i et j tels que $0 \leqslant i,\, j \leqslant n$ et $h_{y^i x^j}(x_0, y_0) \neq 0$. Il existe alors un entourage de (x_0, y_0) où $h_{y^i x^j}(x, y) \neq 0$. Considérons tous les cas possibles.

a) Les indices i et j sont tels que $0 \leqslant i,\, j \leqslant n-1$. La matrice de Wronski de h de la dimension $n \times n$ a donc le rang 1 sur un entourage de (x_0, y_0) et nous avons la thése d'après la supposition inductive.

b) Soit $i = 0$ ou $i = 1$ et $j = n$. La matrice de Wronski de la dimension $n \times n$ de la fonction h_x a donc le rang égal à 1 dans un entourage de (x_0, y_0), alors d'après la supposition inductive nous avons $h_x(x, y) = f(x)g(y)$, d'où $h(x, y) = k(x)g(y) + a(y)$, où $k'(x) = f(x)$. Puisque $hh_{yx} - h_x h_y = 0$, on a $f(ag' - a'g) = 0$. Si $f \equiv 0$ nous avons la thèse. Si $f \not\equiv 0$ on a $ag' - a'g = 0$ et puisque $g \neq 0$ dans le cas $i = 0$ $(0 \neq h_{x^n} = f^{(n)}g)$ et $g' \neq 0$ pour $i = 1$ $(0 \neq h_{yx^n} = f^{(n)}g')$, alors $a(y) = \alpha g(y)$ pour un α constant. Il en résulte la thèse.

c) Dans le cas $i = n$ et $j = 0$ ou $j = 1$ il suffit changer x et y dans le raisonnement plus haut.

d) Dans les cas $2 \leqslant i \leqslant n$ et $j = n$ la matrice de Wronski de la fonction h_{yx} de la dimension $n \times n$ a le rang égal à 1 sur un entourage de (x_0, y_0), d'où $h_{yx}(x, y) = f(x)g(y)$ et de là $h_y(x, y) = k(x)g(y) + a(y)$, où $k' = f$. Puisque

$$h_{y^{i-1}} h_{y^i x^n} - h_{y^i} h_{y^{i-1} x^n} = 0$$

nous recevons

$$a^{(i-2)}(y) k^{(n)}(x) g^{(i-1)}(y) - a^{(i-1)}(y) k^{(n)}(x) g^{(i-2)}(y) = 0,$$

d'où, puisque

$$0 \neq h_{y^i x^n} = f^{(n-1)} g^{(i-1)} = k^{(n)} g^{(i-1)}, \quad a^{(i-2)}(y) = \alpha g^{(i-2)}(y)$$

pour un α constant. Il en résulte que $a(y) = \alpha g(y) + w(y)$, où $w(y)$ est un polynôme de degré $i-3$ (le degré du polynôme “0” est égal à -1), d'où $h_y(x, y) = [k(x) + \alpha]\, g(y) + w(y)$. Puisque $h_y h_{y^i x^n} - h_{y^i} h_{yx^n} = 0$ on a $w(y) k^{(n)}(x) g^{(i-1)}(y) = 0$, d'où $w(y) = 0$, donc $h_y(x, y) = [k(x) + \alpha] g(y) = F(x)G(y)$.

Nous avons alors $h(x, y) = F(x)K(y) + b(x)$, où $K' = G$. Puisque

$$h_{x^{n-1}} h_{y^i x^n} - h_{y^i x^{n-1}} h_{x^n} = 0,$$

nous avons

$$b^{(n-1)}(x) F^{(n)}(x) K^{(i)}(y) - b^{(n)}(x) F^{(n-1)}(x) K^{(i)}(y) = 0,$$

d'où $b^{(n-1)}(x) = \gamma F^{(n-1)}(x)$ pour un γ constant, alors $b(x) = \gamma F(x) + p(x)$, où $p(x)$ est un polynôme de degré $n-2$. Il en résulte que $h(x, y) = F(x)[K(y) + \gamma] + p(x)$. Puisque

$$hh_{y^i x^n} - h_{y^i} h_{x^n} = 0,$$

nous avons

$$p(x)F^{(n)}(x)K^{(i)}(y) = 0,$$

alors $p(x) = 0$, donc $h(x,y) = [K(y) + \gamma]F(x)$.

e) Le cas $i = n$ et $0 \leqslant j \leqslant n$ est analogue.

La démonstration est donc terminée dans tous cas.

Nous allons donner la démonstration pour $p = 2$ et $n = 3$.

D'après le Théorème 5.2.1 dans [10] p. 99 il suffit montrer que pour chaque point (x_0, y_0) de $I \times J$ il existe un entourage de ce point dans lequel la fonction h est de la forme

$$h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y), \tag{8}$$

où le système des fonctions f_1, f_2 et le système des fonctions g_1, g_2 sont linéairement localement indépendantes. Il suffit montrer seulement, comme plus haut, que h est de la forme (8), puisque si f_1, f_2 sont linéairement dépendantes dans un sous-ensemble d'un entourage de x_0 ou si g_1, g_2 sont linéairement dépendantes dans un sous-ensemble d'un entourage de y_0, alors h est de la forme $f(x)g(y)$ dans un sous-ensemble ouvert d'un entourage de (x_0, y_0), donc la matrice (7) ne peut pas avoir toujours le rang égal à 2.

D'après la supposition faite et d'après le théorème dans [1] ils existent des fonctions $\alpha_1, \alpha_2, \alpha_3 : J \longrightarrow \mathbb{R}$ telles que

$$\alpha_1(y)h_{y^2}(x,y) + \alpha_2(y)h_y(x,y) + \alpha_3(y)h(x,y) = 0 \tag{9}$$

et

$$\alpha_1^2(y) + \alpha_2^2(y) + \alpha_3^2(y) \neq 0. \tag{10}$$

En différentiant (9) une fois et encore une fois par rapport à x nous recevons les deux égalités qui avec (9) nous permettent montrer la continuité de $\alpha_1, \alpha_2, \alpha_3$ par la méthode analogue à telle qui est dans la démonstration du lemme dans [3].

Soit (x_0, y_0) un point arbitraire de $I \times J$.

1. Si $\alpha_1(y_0) \neq 0$, alors il existe un entourage du point y_0 dans lequel $\alpha_1(y) \neq 0$. Nous voyons d'après (9) que la fonction $h(x, \cdot)$ comme la fonction de la variable deuxième avec x fixé, est une solution de l'équation différentielle linéaire de l'ordre 2 de la forme

   $$Y''(y) + \frac{\alpha_2(y)}{\alpha_1(y)}Y'(y) + \frac{\alpha_3(y)}{\alpha_1(y)}Y(y) = 0,$$

   alors la fonction h doit avoir la forme (8). Supposons donc dans la suite que $\alpha_1(y_0) = 0$ et désignons par M_{ji} pour $i, j = 1, 2, 3$ le mineur de la matrice (7) avec $n = 3$ pour l'élément $h_{y^{i-1}x^{j-1}}$. Puisque $\alpha_2^2(y_0) + \alpha_3^2(y_0) \neq 0$ et

$$\begin{aligned} \alpha_2(y_0)h_y(x,y_0) + \alpha_3(y_0)h(x,y_0) &= 0, \\ \alpha_2(y_0)h_{yx}(x,y_0) + \alpha_3(y_0)h_x(x,y_0) &= 0, \\ \alpha_2(y_0)h_{yx^2}(x,y_0) + \alpha_3(y_0)h_{x^2}(x,y_0) &= 0, \end{aligned} \tag{11}$$

on a

$$M_{ji}(x,y_0) = 0 \quad \text{pour } i = 3,\ j = 1,2,3. \tag{12}$$

Ils existent d'après le même théorème dans [4] des fonctions $\beta_1, \beta_2, \beta_3 : I \longrightarrow \mathbb{R}$ continues telles que

$$\beta_1(x)h_{x^2}(x,y) + \beta_2(x)h_x(x,y) + \beta_3(x)h(x,y) = 0 \tag{13}$$

et

$$\beta_1^2(x) + \beta_2^2(x) + \beta_3^2(x) \neq 0. \tag{14}$$

2. Si $\beta_1(x_0) \neq 0$ en raisonnant comme plus haut nous constatons que la fonction h est de la forme (8). Supposons donc que $\beta_1(x_0) = 0$, d'où comme plus haut

$$M_{ji}(x_0,y) = 0 \quad \text{pour } i = 1,2,3 \text{ et } j = 3. \tag{15}$$

3. Soit à présent $M_{21}(x_0,y_0) \neq 0$. Il existe donc un entourage du point (x_0,y_0) dans lequel $M_{21}(x,y) \neq 0$. Puisque la dérivée $\partial_x M_{31}$ du mineur M_{31} par rapport à x est égale à M_{21} nous avons d'après le théorème de Lagrange et d'après (15) qu'il existe ξ tel que

$$\begin{aligned} M_{31}(x,y) &= M_{31}(x,y) - M_{31}(x_0,y) \\ &= \partial_x M_{31}(\xi,y)(x-x_0) \\ &= M_{21}(\xi,y)(x-x_0), \end{aligned}$$

d'où $M_{31}(x,y) \neq 0$ pour $x \neq x_0$. S'il existerait $\overline{x} \neq x_0$ tel que $\beta_1(\overline{x}) = 0$, donc en posant $x = \overline{x}$ dans

$$\beta_1(x)h_{x^2y}(x,y) + \beta_2(x)h_{xy}(x,y) + \beta_3(x)h_y(x,y) = 0$$
$$\beta_1(x)h_{x^2y^2}(x,y) + \beta_2(x)h_{xy^2}(x,y) + \beta_3(x)h_{y^2}(x,y) = 0$$

(la différentiation de l'égalité (13) par rapport à y une fois et deux fois), nous aurions d'après $M_{31}(\overline{x},y) \neq 0$ que $\beta_2(\overline{x}) = \beta_3(\overline{x}) = 0$ et ça est impossible d'après (14). Il doit être alors $\beta_1(x) \neq 0$ pour $x \neq x_0$. Il en résulte qu'ils existent les fonctions $f_1(x)$, $f_2(x)$, $g_1(y)$, $g_2(y)$, $h_1(y)$, $h_2(y)$ telles que

$$h(x,y) = \begin{cases} g_1(y)f_1(x) + g_2(y)f_2(x) & \text{pour } x < x_0, \\ h_1(y)f_1(x) + h_2(y)f_2(x) & \text{pour } x > x_0. \end{cases} \tag{16}$$

Puisque f_1, f_2 forment le système fondamental d'une équation différentielle linéaire d'ordre 2 elles sont de classe C^2 et le wronskien $W(f_1,f_2) \neq 0$, donc les fonctions g_1, g_2, h_1, h_2 sont de classe C^3.

De plus

$$\begin{aligned} 0 &\neq M_{21}(x_0,y) \\ &= \lim_{x\to x_0} \frac{M_{31}(x,y)-M_{31}(x_0,y)}{x-x_0} = \lim_{x\to x_0} \frac{M_{31}(x,y)}{x-x_0} \\ &= \begin{cases} \left(\lim\limits_{x\to x_0^-} \dfrac{W(f_1,f_2)}{x-x_0}\right) W(g_1',g_2') \\ \left(\lim\limits_{x\to x_0^+} \dfrac{W(f_1,f_2)}{x-x_0}\right) W(h_1',h_2') \end{cases} \end{aligned}$$

et de là $W(g_1',g_2') \neq 0 \neq W(h_1',h_2')$.

Nous avons pour $x < x_0$ d'après (16)

$$h_y = g_1' f_1 + g_2' f_2, \quad h_{y^2} = g_1'' f_1 + g_2'' f_2, \quad h_{y^3} = g_1''' f_1 + g_2''' f_2.$$

En calculant f_1, f_2 de deux premières équations et en posant les dans l'équation troisième nous recevons pour $x < x_0$

$$W(g_1',g_2') h_{y^3} = F_1(y) h_{y^2} + F_2(y) h_y \tag{17}$$

pour les fonctions F_1 et F_2 correspondantes continues.

Nous recevons de même façon pour $x > x_0$

$$W(h_1',h_2') h_{y^3} = G_1(y) h_{y^2} + G_2(y) h_y \tag{18}$$

pour les fonctions G_1 et G_2 correspondantes continues.

En fixant dans (17) et (18) la variable y nous recevons d'après le Lemme 1 dans [4] p. 178 que (11) a lieu pour chaque x. La fonction $h_y(x,-)$, comme la fonction de la variable y avec x fixé, est une solution d'une équation différentielle linéaire d'ordre 2, alors

$$h_y(x,y) = \overline{a}_1(x)\overline{b}_1(y) + \overline{a}_2(x)\overline{b}_2(y)$$

pour les fonctions $\overline{a}_1$, $\overline{a}_2$, $\overline{b}_1$, $\overline{b}_2$ correspondantes. Il en résulte que

$$h(x,y) = a_1(x)b_1(y) + a_2(x)b_2(y) + c(x) \tag{19}$$

pour les fonctions a_1, a_2, b_1, b_2 correspondantes. En posant cette forme de la fonction h dans (11) avec $y = y_0$ nous recevons

$$\begin{aligned} &\alpha_2(y_0)\left[\overline{a}_1(x)\overline{b}_1(y_0) + \overline{a}_2(x)\overline{b}_2(y_0)\right] \\ &\quad + \alpha_3(y_0)\left[a_1(x)b_1(y_0) + a_2(x)b_2(y_0) + c(x)\right] = 0. \end{aligned} \tag{20}$$

Remarquons que $\alpha_3(y_0) \neq 0$, puisqe dans le cas contraire nous avons d'après (11) que $h_y(x,y_0) = 0$, d'où $M_{21}(x,y_0) = 0$, contrairement à 3. Ils existent donc d'après (20) des nombres k, l tels que $c(x) = ka_1(x)+la_2(x)$, alors (19) nous donne la forme (8) de la fonction h.

4. Dans le cas $M_{12}(x_0, y_0) \neq 0$ il suffit changer x et y dans le raisonnement plus haut.
5. Puisque $\partial_x M_{32}(x, y) = M_{22}(x, y)$, si $M_{22}(x_0, y_0) \neq 0$ nous pouvons raisonner comme dans le cas 3.
6. Nous reste seulement le cas $M_{11}(x_0, y_0) \neq 0$, donc $M_{11}(x, y) \neq 0$ dans un entourage de (x_0, y_0).

   Nous recevons en différentiant (13) une fois et deux fois par rapport à y et en posant $x = x_0$

$$\beta_1(x_0)h_{x^2y}(x_0, y) + \beta_2(x_0)h_{xy}(x_0, y) + \beta_3(x_0)h_y(x_0, y) = 0$$

$$\beta_1(x_0)h_{x^2y^2}(x_0, y) + \beta_2(x_0)h_{xy^2}(x_0, y) + \beta_3(x_0)h_{y^2}(x_0, y) = 0.$$

   En calculant $\beta_2(x_0)$ de ce système avec $y = y_0$ comme les système des inconnues $\beta_1(x_0)$ et $\beta_2(x_0)$ nous recevons que $\beta_2(x_0) = 0$ ($M_{11}(x_0, y_0) \neq 0$ et $M_{21}(x_0, y_0) = 0$). Puisque aussi $\beta_1(x_0) = 0$ on a $\beta_3(x_0) \neq 0$ (voir (14)), d'où d'après (13) avec $x = x_0$ nous avons $h(x_0, y) = 0$ et de là $h_y(x_0, y) = h_{y^2}(x, y) = 0$. Puisque $M_{11}(x_0, y_0) \neq 0$, alors $h_{yx}(x_0, y_0) \neq 0$ ou $h_{xy^2}(x_0, y_0) \neq 0$, d'où il existe un entourage de (x_0, y_0) tel que $h_{yx}(x, y) \neq 0$ dans cet entourage ou $h_{xy^2}(x, y) \neq 0$ dans un entourage de (x_0, y_0). Nous avons dans le cas premier qu'il existe ξ tel que

$$h_y(x, y) = h_y(x, y) - h_y(x_0, y) = h_{yx}(\xi, y)(x - x_0) \neq 0 \quad \text{pour } x \neq x_0$$

   et dans le cas deuxième qu'il existe ξ tel que

$$h_{y^2}(x, y) = h_{y^2}(x, y) - h_{y^2}(x_0, y) = h_{y^2x}(\xi, y)(x - x_0) \neq 0 \quad \text{pour } x \neq x_0.$$

   Il en résulte que pour $\overline{x}$ fixé et différent de x_0 on ne peut pas être $h(\overline{x}, y) \equiv 0$, donc dans chaque point (x, y) pour lequel $x \neq x_0$ nous devons avoir un des cas 1-5. Il existe alors pour chaque tel point un entourage dans lequel la fonction h est de la forme (8). De là d'après le Théorème 5.2.1 dans [10] p. 99 nous avons (16). Nous avons de plus

$$\partial_x \left(\begin{vmatrix} h_y(x, y), & h_{y^2}(x, y) \\ h_{yx^2}(x_0, y), & h_{y^2x^2}(x_0, y) \end{vmatrix} \right) = \begin{vmatrix} h_{yx}(x, y), & h_{y^2x}(x, y) \\ h_{yx^2}(x_0, y), & h_{y^2x^2}(x_0, y) \end{vmatrix},$$

   donc

$$\begin{aligned} 0 &\neq M_{11}(x_0, y) \\ &= \lim_{x \to x_0} \frac{\begin{vmatrix} h_y(x, y), & h_{y^2}(x, y) \\ h_{yx^2}(x_0, y), & h_{y^2x^2}(x_0, y) \end{vmatrix} - \begin{vmatrix} h_y(x_0, y), & h_{y^2}(x_0, y) \\ h_{yx^2}(x_0, y), & h_{y^2x^2}(x_0, y) \end{vmatrix}}{x - x_0} \end{aligned}$$

$$= \begin{cases} \lim\limits_{x \to x_0^-} \dfrac{\begin{vmatrix} f_1(x), & f_2(x) \\ f_1''(x), & f_2''(x) \end{vmatrix}}{x - x_0} W(g_1', g_2'), \\ \lim\limits_{x \to x_0^+} \dfrac{\begin{vmatrix} f_1(x), & f_2(x) \\ f_1''(x), & f_2''(x) \end{vmatrix}}{x - x_0} W(h_1', h_2'), \end{cases}$$

alors $W(g_1', g_2') \neq 0 \neq W(h_1', h_2')$. La suite est analogue que dans le cas 3.

Nous avons donc démontré le théorème suivant:

THÉORÈME 3

Soit $h : I \times J \longrightarrow \mathbb{R}$, où I et J sont des intervales réels, une fonction ayante les dérivées continues jusqu'à $h_{y^{n-1}x^{n-1}}$, pour laquelle la matrice (7) *a le rang égal à p en chaque point de $I \times J$.*

a) Si $p = 1$ et $n > p$, alors $h(x,y) = a_1(x)b_1(y)$ pour $a_1 : I \longrightarrow \mathbb{R}$ et $b_1 : J \longrightarrow \mathbb{R}$.

b) Si $p = 2$ et $n = 3$, alors $h(x,y) = a_1(x)b_1(y) + a_2(x)b_2(y)$ pour $a_1, a_2 : I \longrightarrow \mathbb{R}$ et $b_1, b_2 : J \longrightarrow \mathbb{R}$.

Remarquons enfin que la condition que la matrice (7) pour la fonction $h(x,y) = a_1(x)b_1(y) + \cdots + a_p(x)b_p(y)$ a le rang toujours égal à p est équivalente à la condition que les matrices

$$\left(a_i^{(j)}(x)\right)_{\substack{i=1,\ldots,p \\ j=0,\ldots,n-1}} \quad \text{et} \quad \left(b_i^{(j)}(y)\right)_{\substack{i=1,\ldots,p \\ j=0,\ldots,n-1}}$$

ont les rangs toujours égaux à p.

Pour les fonctions $a_i(x)$ et $b_i(y)$ de classe C^n cette condition dernière est équivalente à la suivante: le système des fonctions $a_1(x), \ldots, a_p(x)$ $(b_1(y), \ldots, b_p(y))$ forme des intégrales d'une équation différentielle ordinaire, linéaire et homogène de la forme

$$y^{(n)}(t) = p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) \tag{21}$$

avec les coefficient $p_i(t)$ continus ([3] et aussi [5]). Cela désigne que notre problème de la décomposition de la fonction h à la forme $a_1(x)b_1(y) + \cdots + a_p(x)b_p(y)$ c'est en réalité pour la fonction h de classe C^n le problème de la décomposition de cette fonction à cette forme avec les fonctions $a_i(x)$ et $b_i(y)$ étantes des solutions de l'équation de la forme (21).

L'exemple de la fonction

$$h(x,y) = x|x| + 1 \quad \text{pour } (x,y) \in \mathbb{R} \times \mathbb{R}$$

montre que la supposition de classe C^n de la fonction h est essentielle dans nos considérations dernières, même si la matrice (7) pour $n = 2$ a le rang égal à 1. On ne peut pas décomposer la fonction h à la forme $a(x)b(y)$, où $a(x)$

et $b(y)$ sont des intégrales de l'équation de la forme (21) pour $n = 2$, à cause de la régularité de $a(x)$. Remarquons que notre fonction h n'est pas aussi une solution de l'équation de la forme (21) pour $n = 1$ puisque

$$(t|t| + 1)'_{t=-1} = 2 \neq 0 = p_0(-1) \cdot 0 = p_0(-1)(t|t| + 1)_{t=-1},$$

formant en même temps cette solution dans chaque intervalle qui ne contient pas -1 (il suffit prendre $p_0(t) = \frac{(t|t|+1)'}{(t|t|+1)}$. La fonction $h(x, y) = xy$ pour $(x, y) \in \mathbb{R} \times \mathbb{R}$ est de la forme $a(x)b(y)$, sa matrice (7) pour $n = 2$ a le rang égal à 1, $a(x)$ et $b(y)$ sont des solutions de l'équation (21) d'ordre 2, n'étant pas des intégrales de l'équation de cette forme d'ordre 1 ($(t)'_{t=0} = 1 \neq 0 = p_0(0) \cdot 0$).

Travaux cités

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Zenon Moszner

Sur quelques problèmes ouverts

Résumé. On pose les problèmes au sujet de la raréfaction des ensembles, des prolongements de la mesure de Jordan et des homomorphismes, de la stabilité de l'équation de translation, de la décomposition des fonctions et sur les opérateurs déterminés par les équations fonctionnelles.

Problème 1 de la raréfaction d'un ensemble de mesure nulle [1]

E. Borel [1] a introduit une notion de la raréfaction d'un sous-ensemble de $\mathbb{R}$ de mesure lebesguienne nulle, légèrement modifiée par M. Fréchet [2]. On dit que la série $\sum_{\nu=1}^{\infty} u_\nu$ converge plus rapidement que la série $\sum_{\nu=1}^{\infty} v_\nu$ si

$$\liminf_{n\to\infty} \frac{\sum\limits_{\nu=n}^{\infty} v_\nu}{\sum\limits_{\nu=n}^{\infty} u_\nu} > 1.$$

Nous comprenons par suite majorante d'un ensemble $E \subset \mathbb{R}$ une suite d'intervalles ouvertes I_ν dont la série des longueurs $\sum_{\nu=1}^{\infty} |I_\nu|$ est convergente et qui recouvre l'ensemble E de manières que chaque point de E appartienne à une infinité d'intervalles de I_ν. E. Borel a démontré qu'il existe la suite majorante de E si et seulement si E est de mesure de Lebesgue nulle. Enfin E et F étant de mesure nulle, l'ensemble E est dit plus raréfié que l'ensemble F (en symbole $\operatorname{Rar} E > \operatorname{Rar} F$) s'il existe une suite majorante de E dont la série des longueurs converge plus rapidement que la série des longueurs de chaque suite majorante de F. Si ni $\operatorname{Rar} E > \operatorname{Rar} F$ ni $\operatorname{Rar} F > \operatorname{Rar} E$, nous disons que les ensembles E et F ont le même ordre de raréfaction.

Le problème ouvert est suivant: existe-il des ensemble E et F pour lesquels $\operatorname{Rar} E > \operatorname{Rar} F$, en particulier existe-il un ensemble E tel que l'ensemble réduit à un point est plus raréfié que E?

On peut démontré [3] que chaque sous-ensemble d'un ensemble du type F_σ et de mesure nulle a le même ordre de raréfaction que l'ensemble réduit à un point. Cette situation montre que la notion de raréfaction n'est pas bonne pour distinguer entre les ensembles de mesure nulle (il y a trop des ensembles les

[1] Mathematics Subject Classification (2000): 28E15.

plus raréfiés). Mais remarquons qu'il existe des ensembles qui ne sont pas des sous-ensembles des ensembles du type F_σ et de mesure nulle, par exemple des ensembles du type G_δ, de mesure nulle et recouvreant l'ensemble des nombres rationnels.

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Problème 2 du prolongement de la mesure de Peano-Jordan [2]

Considérons la mesure comme une fonction définie sur un corps des sous-ensembles de $\mathbb{R}^n$, non-négative, invariante par rapport à l'isométrie, additive pour les deux ensembles sans le point intérieure commun et positive pour un cube n-dimensionnel. On sait que la mesure de Peano-Jordan est de cette sorte sur la familiie $\mathbf{J}$ des ensembles mesurables au sens de Peano-Jordan et que cette mesure est unique sur cette famille. Il se pose la question est-ce qu'on peut prolonger cette mesure? La reponse est positive. On peut prolonger la mesure de Peano-Jordan sur le plus petite corps $\mathbf{C}$ recouvrant la famille $\mathbf{J}$ et la famille des ensembles non-denses. On peut aussi démontrer que si nous nous restreignons aux ensembles bornés, alors le plus grand corps sur lequel ce prolongement pourrait être possible c'est le corps $\mathbb{N}$ des ensembles bornés ayants la frontière non-dense (on a $\mathbf{J} \subsetneq \mathbf{C} \subsetneq \mathbb{N}$). Nous savons (le résultat de E. Szpilrajn-Marczewski, voir [2] p. 234) que ce prolongement existe pour $n = 1, 2$. Les problèmes suivants sont ouverts [1]:

a) est-ce que ce prolongement sur $\mathbb{N}$ est unique pour $n = 1, 2$,

b) existe-il ce prolongement pour $n > 2$ (unique)?

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Problème 3 du prolongement d'un homomorphisme [3]

On considère dans beaucoup des domaines de mathématique le groupe L_s^1 comme l'ensemble des suites $(a_1, \ldots, a_s)$, où $a_1 \neq 0$, avec l'opération définie comme il suit:

$$(a_1, \ldots, a_s) \star (b_1, \ldots, b_s) = (c_1, \ldots, c_s),$$

où c_ν pour $\nu = 1, \ldots, s$ est la dérivée d'ordre ν d'une fonction composée $f(x) = g(h(x))$ si $a_1, \ldots, a_\nu$ sont les dérivées des ordres $1, \ldots, \nu$ de la fonction extérieure $g(y)$ et $b_1, \ldots, b_\nu$ sont les dérivées des ordres $1, \ldots, \nu$ de la fonction intérieure $h(x)$.

L. Reich ([4] p. 309 — il y a là une méprise: il doit être L_3^r et L_4^r au lieu de L_r^3 et L_r^4) a posé la question suivante: quand l'homomorphisme $h_s = (f_1, \ldots, f_s)$ de $\mathbb{R}_+$ à L_s^1 est prolongeable à l'homomorphisme $h_{s+1} = (f_1, \ldots, f_s, f_{s+1})$ de $(\mathbb{R}, +)$ à L_{s+1}^1 (le problème de l'existence de la fonction f_{s+1})?

On démontre dans [3] que pour $s = 1, 2$ chaque homomorphisme peut être prolongé, mais pour $s = 3, 4$ il existe des homomorphismes qui ne sont pas prolongeables. J'ai formulé ([4] p. 309) la conjecture que ce prolongement est possible si $f_1 \neq 1$ et dans le cas si $h_s = (1, 0, \ldots, 0, f_{p+2}, \ldots, f_s)$, où $f_{p+2} \neq 0$, ce prolongement est possible si et seulement si f_{s-p} est un polynôme de f_{p+2}.

On démontre dans [2] la partie "seulement si" (la nécéssité) de la deuxième partie de cette conjecture, la partie première et la condition "si" (la suffisance) sont ouvertes.

On peut considérer le même problème pour le groupe L_s^n qui est défini analogiquement que L_s^1, seulement dans la définition de l'opération dans L_s^n on remplace la composition de deux fontions d'une variable par la superposition des deux systèmes de n fonctions des n variables (voir [1] p. 7-12). Ici s désigne l'ordre des dérivées partielles de cette superposition, donc L_s^n est la suite de $n^2 + n^3 + \cdots + n^{s+1}$ éléments, alors dans ce cas

$$\begin{aligned} &h_s(x) \\ &= ((f_{j_1}^i(x))_{i,j_1=1,\ldots,n}, (f_{j_1 j_2}^i(x))_{i,j_1,j_2=1,\ldots,n}, \ldots, (f_{j_1\ldots j_s}^i(x))_{i,j_1,\ldots,j_s=1,\ldots,n}). \end{aligned}$$

Le problème du prolongement n'est pas banal puisque pour n arbitraire l'homomorphisme de $(\mathbb{R}, +)$ à L_3^n de la forme:

$$\begin{aligned} &f_{j_1}^i(x) = 1 \quad \text{pour } i = j_1; \qquad f_{j_1}^i(x) = 0 \quad \text{pour } i \neq j_1; \\ &f_{11}^1(x) = f(x) \text{ additive}; \qquad f_{j_1 j_2}^i(x) = 0 \quad \text{pour } (i, j_1, j_2) \neq (1,1,1); \\ &f_{111}^1(x) = \tfrac{3}{2} f^2(x) + g(x), \text{ où } g(x) \text{ additive}; \\ &f_{j_1 j_2 j_3}^i(x) = \tfrac{3}{2} f(x) \quad \text{pour } (i, j_1, j_2, j_3) \neq (1,1,1,1); \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{où toujours } i, j_1, j_2, j_3 = 1, \ldots, n \end{aligned}$$

[3]Mathematics Subject Classification (2000): 20F99, 39B52.

n'est pas prolongeable à l'homomorphisme de $(\mathbb{R},+)$ à L_4^n si $f(x)$ n'est pas idéntiquement zéro et s'il n'existe pas une constante a telle que $g(x) = af(x)$.

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Problème 4 de la stabilité de l'équation de translation [4]

Nous entendons par l'équation de translation l'équation fonctionnelle de la forme

$$F(F(\alpha,x),y) = F(\alpha, x\cdot y),$$

où $F : \Gamma \times G \longrightarrow \Gamma$ est une fonction cherchée, Γ étant un ensemble arbitraire et $(G,\cdot)$ forme un groupoïde donné. Soit dans Γ une métrique ρ. On dit que cette équation est stable si pour chaque $\varepsilon > 0$ il existe un $\delta > 0$ let que pour chaque fonction $H : \Gamma \times G \longrightarrow \Gamma$ si

$$\forall \alpha \in \Gamma,\ x,y \in G:\ \rho(H(H(\alpha,x),y), H(\alpha,x\cdot y)) \leqslant \delta$$

alors il existe une solution F de l'équation de translation pour laquelle

$$\forall \alpha \in \Gamma,\ x \in G:\ \rho(H(\alpha,x),F(\alpha,x)) \leqslant \varepsilon.$$

Si par exemple G se réduit à un point $\{e\}$, l'équation de translation qui dans ce cas peut être écrite comme $f(f(\alpha)) = f(\alpha)$, où $f(\alpha) = F(\alpha,e)$, est stable pour chaque métrique dans Γ arbitraire [2].

Il se pose le problème [1]: existe-il l'espace métrique (Γ,ρ) et le groupoïde $(G,\cdot)$ pour lesquels l'équation de translation n'est pas stable dans ce sens?

On définit aussi la stabilité de l'équation de translation comme il suit: si pour une fonction $H : \Gamma \times G \longrightarrow \Gamma$ l'ensemble

$$\{\rho(H(H(\alpha,x),y), H(\alpha,x\cdot y)) :\ \alpha \in \Gamma,\ x,y \in G\}$$

est borné, alors il existe une solution F de l'équation de translation telle que l'ensemble $\{\rho(H(\alpha,x),F(\alpha,x)) :\ \alpha \in \Gamma,\ x \in G\}$ est aussi borné. L'équation

[4] Mathematics Subject Classification (2000): 39B82.

$f(f(\alpha)) = f(\alpha)$ est stable d'après cette définition pour chaque l'espace métrique (Γ, ρ), mais il existe un espace métrique (Γ, ρ) et un groupe $(G, \cdot)$ tels que l'équation de translation n'est pas stable dans ce sens [2].

Références

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Problème 5 de la décomposition de la fonction de deux variables [5]

Supposons que pour une fonction $h : I \times J \longrightarrow \mathbb{R}$, où I et J sont des intervalles réels, tous les éléments (les dérivées partielles de h) de la matrice

$$M(x,y) = \left(h_{x^i y^j}(x,y)\right),$$

où $i, j = 0, \ldots, n$, sont continues. Est-il vrai le théorème suivant (la conjecture **C**):

- la condition rang $M(x,y) = p \leqslant n$ pour chaque $(x,y) \in I \times J$ est équivalente à l'existence des fonctions $f_k : I \longrightarrow \mathbb{R}$ et $g_k : J \longrightarrow \mathbb{R}$ pour $k = 1, \ldots, p$ telles que

  $$h(x,y) = f_1(x)g_1(y) + \cdots + f_p(x)g_p(y)$$

  et

  $$\text{rang}\left(f_k^{(i)}(x)\right)_{\substack{i=0,\ldots,n \\ k=1,\ldots,p}} = p = \text{rang}\left(g_k^{(i)}(y)\right)_{\substack{i=0,\ldots,n \\ k=1,\ldots,p}} \tag{1}$$

  pour chaque $x \in I$ et $y \in J$.

On sait ([1] et [2]) que cette conjecture est vraie pour $p = 1$ et $n \geqslant 1$ arbitraire et pour $p = 2 = n$.

Si cette conjecture serait vraie en général et si nous supposons que la fonction $x \longrightarrow h(x,y)$ est de classe C^{n+1} sur I pour chaque y dans J et la fonction $y \longrightarrow h(x,y)$ est aussi de classe C^{n+1} sur J pour chaque x de I, alors dans la conjecture en considération on pourrait remplacer (1) par la condition: $f_1, \ldots, f_p$, et de même les fonctions $g_1, \ldots, g_1$, forment un systeme linéairement indépendantes solutions d'une équation différentielle de la forme

$$y^{(n+1)} = a_n(x)y^{(n)} + \cdots + a_1(x)y \tag{2}$$

avec les coefficients $a_\nu(x)$ $(\nu = 1, \ldots, n)$ continues sur I. De plus dans ce cas les fonctions $x \longrightarrow h(x,y)$, $x \longrightarrow h_y(x,y)$, ..., $x \longrightarrow h_{y^n}(x,y)$ sont des integrales

[5] Mathematics Subject Classification (2000): 35L70, 35L75, 26B40.

de la même équation de la forme (2) pour chaque y de J et les fonctions $y \longrightarrow h(x,y)$, $y \longrightarrow h_x(x,y)$, $\ldots$, $y \longrightarrow h_{x^n}(x,y)$ sont les mêmes.

Le cas $n = 1$ est en liaison à l'équation différentielle de d'Alembert

$$hh_{xy} - h_x h_y = 0$$

qui a déjà sa théorie géométrique [3]. Les fonctions $h(x,y) = f(x)g(y)$ sont des solutions de cette équation, mais il y a aussi les autres solutions ([3], p. 32). Il résulte de nos considérations dans le cas $p \leqslant n = 1$ que le rang de la matrice

$$\begin{bmatrix} h & h_y \\ h_x & h_{yx} \end{bmatrix}$$

plus petite que 2 et la stabilité de ce rang pour $(x,y) \in I \times J$ entrainent la forme $f(x)g(y)$ de la fonction $h(x,y)$.

On ne peut pas remplacer dans notre conjecture **C** la supposition $p \leqslant n$ par la condition plus faible $p \leqslant n+1$. En effect pour la fonction $h(x,y) = \frac{1}{x-y}$ sur $[0,1] \times [2,3]$ on a $hh_{xy} - h_x h_y \neq 0$ pour chaque $(x,y) \in [0,1] \times [2,3]$, mais $h(x,y)$ n'est pas de la forme $f_1(x)g_1(y) + f_2(x)g_2(y)$ puisque elle ne remplit pas de l'équation (voir [3], p. 30)

$$\begin{vmatrix} h & h_y & h_{y^2} \\ h_x & h_{yx} & h_{y^2x} \\ h_{x^2} & h_{yx^2} & h_{y^2x^2} \end{vmatrix} = 0.$$

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Problème 6 des propriétés des opérateurs déterminés par les équations fonctionnelles uniquement stables [6]

On considère dans la théorie de la stabilité des équations fonctionnelles les équations de la forme $G(f) = D(f)$ d'une fonction cherchée f qui sont uniquement stables dans ce sens (l'explication intuitive, on peut cela préciser des manières différentes) que pour chaque fonction g pour laquelle $G(g)$ n'est pas "loin" de $D(g)$ il existe exactement une solution f de cette équation qui

[6] Mathematics Subject Classification (2000): 39B82, 47B38, 47H99.

n'est pas "loin" de g. On peut dans ce cas examiner les propriétés de l'opérateur $O(g) := f$. Je donne un exemple.

En résolvant le problème de S.M. Ulam, D.H. Hyers a démontré le théorème suivant [2]:

Soient $(X, \cdot)$ et $(Y, \cdot)$ des espaces de Banach. Si pour $\varepsilon > 0$ la fonction $f : X \longrightarrow Y$ remplit la condition

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon \quad \text{pour } x, y \in X,$$

alors il existe exactement une fonction additive $a : X \longrightarrow Y$ telle que

$$|f(x) - a(x)| \leqslant \varepsilon \quad \text{pour } x \in X.$$

J'examine dans [3] pour l'opérateur $A(f(x)) = a(x)$ en outre son continuité par rapport à f et la continuité de $a(x)$ par rapport à x.

Il me semble intéressant le problème analogue pour les autres équations fonctionnelles uniquement stables.

Remarquons que

a) les opérateurs analogues sont considérés aussi dans [1] dans le cas plus générale et pour l'équation des fonctions exponentielles, mais sans des recherches des propriétés de ces opérateurs,

b) on considère dans [4] les équations fonctionnelles stables pas uniquement, c. à d. telles que pour chaque g comme plus haut il existe plus qu'une solution f comme ci-dessus et on examine les réstrictions au sujet de g et de f (par la notion de la meilleure approximation) sous lesquelles cet application de g à f est unique. Ces résultats permettent prolonger le problème formulé plus haut aussi aux équations stables pas uniquement.

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František Neuman

Functional and differential equations

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. Functional equations play an important role in the theory of differential equations. Euler functional equation for homogeneous functions, Abel and Schröder functional equations and their systems, iteration groups of functions are essential tools for studying transformations and asymptotic properties of their solutions. And conversely, differential equations give answer to some problems in the theory of functional equations, e.g., decomposition of functions.

I. Introduction

Let us start with a historical remark. Floquet theory deals with linear differential systems

$$Y' = P(x)Y, \tag{1}$$

where $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuos periodic matrix,

$$P \in C^0(\mathbb{R}), \quad P(x+1) = P(x),$$

and $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a matrix solution of the system (1).

It is known, e.g., R. Bellman [2], that the solution Y is of the form

$$Y(x) = Q(x) \cdot e^{Bx},$$

where $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $Q \in C^1(\mathbb{R})$, is a periodic matrix, $Q(x+1) = Q(x)$, and B is a constant $n \times n$ matrix with generally complex elements.

The proof of this result is essentially based on the fact that together with a solution $x \mapsto Y(x)$ of the system (1) the function $x \mapsto Y(x+1)$ is also a solution. Since $Y(x)\cdot C$, C being a regular constant matrix, is a general solution of equation (1), there exists a constant regular matrix C_0 such that

$$Y(x+1) = Y(x) \cdot C_0, \quad \det C_0 \neq 0, \quad x \in \mathbb{R}. \tag{2}$$

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However, this is a matrix functional equation. If we suppose its solution Y of the form

$$Y(x) = Q(x) \cdot e^{Bx}, \tag{3}$$

where $Q(x+1) = Q(x)$ is a periodic matrix, then

$$Y(x+1) = Q(x+1) \cdot e^{Bx} \cdot e^{B} = Q(x) \cdot e^{Bx} \cdot e^{B} = Y(x) \cdot e^{B}.$$

If now $e^B = C_0$ ($B = \ln C_0$, $\det C_0 \neq 0$) then we see that any solution Y of the functional equation (2) must have the form (3). And this is the essence of the Floquet theory.

After this historical remembrance of an application of functional equations, let us continue with more recent results when functional equations play an important role in the study of differential equations. In general we may observe that functional equations occur when solutions of differential equations are considered in different points, e.g., in consecutive zeros, with delayed or advanced arguments, or when transformations of differential equations are considered.

II. Abel functional equation and linear differential equations

Consider the second order equation in the Jacobi form

$$y'' = p(x)y, \quad p \in C_0(I),\ I = (a,b) \subset \mathbb{R}, \tag{p}$$

$-\infty \leqslant a < b \leqslant \infty$. Suppose that this equation (p) is oscillatory for $x \to b$, i.e. each solution y of (p) has infinitely many zeros when x approaches the right end of the interval of definition.

In accordance with O. Borůvka [3], introduce the following notions.

DEFINITION 1

A *phase* α of equation (p) having two linearly independent solutions y_1, y_2 is defined as a continuous function on I satisfying the relation

$$\tan \alpha(x) = \frac{y_1(x)}{y_2(x)}$$

for all x where $y_2(x) \neq 0$.

PROPERTY 1

Phase α being continuous on the whole interval I, is also in $C^3(I)$ and $\alpha'(x) \neq 0$ on I.

PROPERTY 2

If α is a phase of equation (p) then its general solution is

$$y(x) = y(x; c_1, c_2) = c_1\, |\alpha'(x)|^{-\frac{1}{2}}\, \sin(\alpha(x) + c_2).$$

DEFINITION 2

Let $x_0 \in I$ be arbitrary, and y be a nontrivial solution of equation (p) vanishing at x_0, $y(x_0) = 0$. Denote by x_1 the first zero to the right of x_0 of this solution y. Define the *dispersion* of the equation (p) as the function

$$\varphi : I \to I, \quad \varphi(x_0) = x_1 \quad \text{for each } x_0 \in I.$$

The dispersion φ is well-defined since all solutions of equation (p) having a zero in x_0 have x_1 as its first zero to the right of x_0. Moreover, all such x_1 exist because equation (p) oscillates when $x \to b$.

O. Borůvka has proved

PROPOSITION 1

The dispersion φ and the phase α of an equation (p) satisfy the Abel equation

$$\alpha(\varphi(x)) = \alpha(x) + \pi \operatorname{sign} \alpha'. \tag{4}$$

Hence

$$\varphi : I \to I, \quad \varphi(x) > x, \quad \varphi \in C^3(I) \quad \text{and} \quad \varphi'(x) > 0.$$

Using these properties we proved [9] for differential equations (p) the following result ($\varphi^{[i]}$ denoting the i-th iterate of φ).

PROPERTY 3

If the dispersion φ satisfies one of the conditions

a) $\varphi - \mathrm{id}_I$ *is a nondecreasing function, or*

b) $\varphi - \mathrm{id}_I$ *is a nonincreasing function, or*

c) $\varphi - \mathrm{id}_I = \delta = \text{const.} > 0$,

then one of the three cases hold, respectively:

a′) *the maxima of absolute values of each solution of (p) on consecutive intervals $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, $i = 0, 1, 2, \ldots$, form a nondecreasing sequence,*

b′) *those maxima form a nonincreasing sequence,*

c′) *each solution of (p) is periodic or half-periodic with the period δ.*

Roughly speaking, if the distances between consecutive zeros of solutions are increasing, or decreasing or are equal, then their maxima are increasing, or decreasing, or equal (solutions are half-periodic).

By using this Abel equation (4) and results of B. Choczewski [5], M. Kuczma [7] and E. Barvínek [1], the second order equations with prescribed properties were constructed [12].

Recently the notion of dispersion was extended to some linear differential equations of an arbitrary order. The same effect concerning relations between distances of consecutive zeros of solutions and their asymptotic behaviour was proved in [14]. Also a construction of all n-th order linear differential equations with prescribed asymptotic properties was presented there.

III. Systems of Abel and Schröder functional equations, iteration groups of functions

Consider a generall nonlinear functional differential equation,

$$F(x, y(x), \dots, y^{(n)}(x), y(\xi_1(x)), \dots, y^{(n)}(\xi_1(x)), \\ \dots, y(\xi_k(x)), \dots, y^{(n)}(\xi_k(x))) = 0,$$

and the substitution $x = h(t)$, $z(t) = y(h(t))$ converting the above equation into

$$G(t, z(t), \dots, z^{(n)}(t), z(\eta_1(t)), \dots, z^{(n)}(\eta_1(t)), \\ \dots, z(\eta_k(t)), \dots, z^{(n)}(\eta_k(t))) = 0.$$

Then $y \circ \xi_i(x) = y \circ \xi_i \circ h(t) = (y \circ h) \circ (h^{-1} \circ \xi \circ h(t)) = z(\eta_i(t))$, i.e., $h^{-1}(\xi_i(h(t)) = \eta_i(t)$, or

$$h \circ \eta_i(t) = \xi_i \circ h(t), \quad i = 1, \dots, k,$$

expressing the fact that deviating arguments ξ_i and η_i are conjugate functions.

If we consider a possibility of a special choice of *canonical* deviations $\xi_i(x) = x + c_i$, $c_i =$ const., see [10], then we come to a problem of a common solution h of a *system of Abel functional equations* for prescribed η_i:

$$h(\eta_i(t)) = h(t) + c_i, \quad i = 1, \dots, k.$$

If $k = 1$, i.e. when we have a single Abel equation, there were lot of results in the literature, see e.g., [7]. For $k > 1$ there has recently been investigated these problems in Brno, Katowice and Kraków. We discovered several sufficient conditions for the existence of a solution of a system of Abel equations [10]. Then a systematic research was done by M.C. Zdun [16].

For linear functional differential equations we may take even more general transformations of Kummer's type $z(t) = f(t)y(h(t))$ which enable us to impose one more condition on coefficients because of a rather arbitrary function f in the transformation.

In the simplest case of linear functional differential equations of the first order with one delay

$$y'(x) + a(x)y(x) + b(x)y(\xi(x)) = 0$$

we may consider their *canonical form* as

$$z'(t) + c(t)z(t-1) = 0.$$

For another choice of special deviations, e.g., of the form $\xi_i(x) = c_i x$ we get a system of Schröder functional equations,

$$h(\eta_i(t)) = c_i h(t), \quad i = 1, \dots, k.$$

In general, zeros of solutions are preserved and they may be studied on canonical forms only. Since the factor f in the transformation can be explicitly evaluated from coefficients, asymptotic properties of solutions of equations, their boundedness, classes L^p, convergency to zero, or the rate of growth, can be obtained from these properties of canonical equations.

For some cases we have also a *criterion of equivalence,* see [13].

Iteration groups of continuous functions were studied by many authors in connection with flows, dynamical systems, fractional iterates, etc. At the beginning of the eighties the study of solutions of a system of Abel equations, or equivalently, embedding of a finite number of functions into an iteration group as its elements, was initiated by investigating functional differential equations.

IV. Euler functional equation for homogeneous functions

Consider a linear differential equation of the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \quad \text{on } I, \qquad (P)$$

I being an open interval of the reals, p_i are real-valued continuous functions defined on I for $i = 0, 1, \dots, n-1$, i.e. $p_i \in C^0(I)$, $p_i : I \to \mathbb{R}$.

Take functions $f : J \to \mathbb{R}$ and $h : J \to I$ such that

$$f \in C^n(J), \quad f(t) \neq 0 \quad \text{for each } t \in J,$$

and

$$h \in C^n(J), \quad h'(t) \neq 0 \quad \text{for each } t \in J, \text{ and } h(J) = I.$$

For each solution y of equation (P) the function z defined as

$$z : J \to \mathbb{R}, \quad z(t) := f(t)\, y(h(t)), \quad t \in J, \qquad (f, h)$$

satisfies again a differential equation of the same form

$$z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \quad \text{on } J. \qquad (Q)$$

Since h is a C^n-diffeomorphism of J onto I, solutions y are transformed into solutions z on their whole intervals of definition. This is why we also speak about a *global* transformation of equation (P) into equation (Q).

Let $\mathbf{y}(x) = (y_1(x), \dots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation (P) considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \dots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \dots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation (Q), then the global transformation (f, h) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x))$$

or, for an arbitrary regular constant $n \times n$ matrix A,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x)),$$

expressing only the fact that another n-tuple of linearly independent solutions of the *same* equation (Q) is taken.

To emphasize this situation, let us denote by $(P_{\mathbf{y}})$ and $(Q_{\mathbf{z}})$ the equations (P) and (Q), respectively. Capital P refers to the coefficients p_i of the equation $(P_{\mathbf{y}})$, subscript $\mathbf{y}$ expresses a particular choice of an n-tuple of linearly independent solutions. Similarly for $(Q_{\mathbf{z}})$.

Denote by $W[\mathbf{y}](x)$ the Wronski determinant of $\mathbf{y}$, i.e.

$$\det(\mathbf{y}(x), \mathbf{y}'(x), \ldots, \mathbf{y}^{(n-1)}(x)).$$

The coefficient p_{n-1} in $(P_{\mathbf{y}})$ is given by

$$p_{n-1}(x) = -(\ln |W[\mathbf{y}](x)|)'.$$

We have $p_{n-1} \equiv 0$ exactly when $W[\mathbf{y}](x) = const. \neq 0$. Since

$$W[f \cdot \mathbf{y}(h)](t) = (f(t))^n \, (h'(t))^{\frac{n(n-1)}{2}} \, W[\mathbf{y}](h(t)),$$

for the coefficient q_{n-1} in $(Q_{\mathbf{z}})$ we have

$$q_{n-1}(t) = -n\frac{f'(t)}{f(t)} - \frac{n(n-1)}{2}\frac{h''(t)}{h'(t)} + p_{n-1}(h(t))\, h'(t). \tag{5}$$

Namely, if $p_{n-1} \equiv 0$ then $q_{n-1} \equiv 0$ occurs exactly when

$$f(t) = c\,|h'(t)|^{\frac{1-n}{2}}, \quad c = \text{const.} \neq 0. \tag{6}$$

Since the factor f belongs to $C^n(J)$, we have $h \in C^{n+1}(J)$.

For the criterion of equivalence of linear differential equations it was essential to find covariant functors from the second order equations (p) to the n-th order equations with the vanishing coefficient of $y^{(n-1)}$. The condition on the commutativity of the diagram of transformations leads to the relation

$$F\left(|h(t)|^{-\frac{1}{2}} u_1(h(t)), |h(t)|^{-\frac{1}{2}} u_2(h(t))\right) = |h(t)|^{\frac{(1-n)}{2}} \, F(u_1(h(t)), u_2(h(t))) \tag{7}$$

for linearly independent solutions u_1, u_2 of equation (p). Set $a = h(t)^{-\frac{1}{2}}$, $r = u_1(h(t))$ and $s = u_2(h(t))$, then from (7) we get

$$F(ar, as) = a^{n-1}\, F(s, r),$$

the Euler functional equation. Under the additional condition that each second order equation with analytic coefficients should be mapped on its whole interval of definition into an n-th order equation again with analytic coefficients, the only possible solutions are linear combinations with constant coefficients of

$$r^{n-1}, r^{n-2}s, \ldots, s^{n-1}.$$

It means that the n-th order linear differential equation to which the equation (p) with a couple u_1, u_2 is covariantly mapped is uniquely determined by its n-tuple of linearly independent solutions

$$u_1^{n-1}, u_1^{n-2}u_2, \ldots, u_2^{n-1}.$$

These special n-th order linear differential equations are called iterative equations and serve for effective criterion of equivalence of linear differential equations of an arbitrary order in general case, see [12].

V. Decomposition of functions

Here is a brief comment to results connected with decompositions of functions h into finite sums of the form

$$h(x,y) = \sum_{k=1}^{n} f_k(x) \cdot g_k(y). \tag{8}$$

For sufficiently smooth h, determinants of the form

$$\det \begin{pmatrix} h & h_y & \ldots & h_{y^n} \\ h_x & h_{xy} & \ldots & h_{xy^n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^n} & h_{x^n y} & \ldots & h_{x^n y^n} \end{pmatrix}$$

are involved in expressing a sufficient and necessary condition for such a decomposition. The correct formulation of the condition was first given in [11]. Functions f_k, g_k in the decomposition (8) and the number n as the minimal number possible for such a decomposition was determined there by using solutions of certain linear ordinary differential equations.

A sufficient and necessary condition for not sufficiently smooth functions h defined on arbitrary (even discrete) sets without any regularity conditions was also formulated in [11] by introducing there a new, special matrices

$$\begin{pmatrix} h(x_1,y_1) & h(x_1,y_2) & \ldots & h(x_1,y_n) \\ h(x_2,y_1) & h(x_2,y_2) & \ldots & h(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ h(x_n,y_1) & h(x_n,y_2) & \ldots & h(x_n,y_n) \end{pmatrix}.$$

Several authors, M. Čadek, H. Gauchman, Z. Moszner, Th.M. Rassias, L.A. Rubel, J. Šimša, in [4], [6], [8], [15] and others dealt with problems concerning decompositions of functions of several variables and similar questions.

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Results on the J. d'Alembert equation

Dedicated to Professor Zenon Moszner
on his 70-th birthday

Abstract. In this paper the authors provide an account of some of their recent results concerning the J. D'Alembert equation especially in a suitable category of noncommutative manifolds.

Introduction

Questions of representation of functions in several variables by means of functions of a smaller number of variables have captured the interest of mathematicians for centuries (see [14]). One of these questions is closely connected with the thirteenth problem of D. Hilbert (1862-1943) and concerns the solvability of algebraic equations (see [5]). Let us mention the surprising result of A.N. Kolmogorov here (see [6]):

> Each continuous function h on the unit n-dimensional cube can be represented in the form
> $$h(x^1, x^2, \dots, x^n) = \sum_{1\leq i\leq 2n+1} \phi_i\Big(\sum_{1\leq j\leq n} \alpha_{ij}(x^j)\Big)$$
> with some continuous functions ϕ_i and α_{ij}. Moreover, the inner functions α_{ij} can be chosen in advance, i.e., independently of the function h.

Functions of certain special forms have been investigated by several authors, including J. d'Alembert (1717-1783), who as early as 1747 proved that each sufficiently smooth scalar function h of the form $h(x,y) = f(x).g(y)$ has to satisfy the following partial differential equation

$$\frac{\partial^2 \log h}{\partial x \partial y} = 0 \tag{A}$$

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(see [2]). This equation can be also expressed in the following "Wronskian form":

$$\det W_{2+1}(h) \equiv \begin{vmatrix} h & h_y \\ h_x & h_{xy} \end{vmatrix} = 0.$$

A generalization to a finite sum of products of functions in single variables of the form

$$h(x,y) = \sum_{1\le i\le n} f_i(x).g_i(y) \tag{P}$$

has been considered since the beginning of the twentieth century. This forms *the fundamental problem* in the subject. The functions of the above tensor product play a significant role in many areas of both pure and applied mathematics. In the year 1904 in the section *Arithmetics and Algebra* at the Third International Congress of Mathematicians in Heidelberg, Cyparissos Stéphanos announced the following result ([16]):

> Functions of the type (P) form the space of all solutions of the partial differential equation with the "Wronskian" of order $(n+1)$: $\det W_{n+1}(h) = 0$.

However, no proof of the above result was given and no smoothness condition on the given function h was mentioned. In fact, Th.M. Rassias gave in [13] a counterexample to Stéphanos statement. It was F. Neuman ([7]) who proved the basic theorem involving the equation $\det W_{n+1}(h) = 0$ for functions of class C^n.

The problem of representing a function f in several (more than two) variables by:

$$h(x^1, x^2, \dots, x^k) = \sum_{1\le i\le n} f_{i1}(x^1)\cdot f_{i2}(x^2)\cdot \cdots \cdot f_{ik}(x^k), \tag{Q}$$

was proposed by Th.M. Rassias in [13]. H. Gauchman and L.A. Rubel [4] obtained some new results and extensions on finite sums expansions of the form (P), especially for real analytic functions. The first existence theorem on the decomposition (Q) was proved by F. Neuman [7]. Later M. Čadek and J. Šimša [1] found an effective criterion for the existence of the decomposition (Q) by making use of a system of functional equations, which does not require any assumption on the function h. Furthermore, they outlined a way to find systems of partial differential equations whose solutions space form the family of all sufficiently smooth functions h of type (Q). J. Šimsa [15], among other things, has introduced some new functional equations for functions of the form (P) using the so called Casorati determinant.

By using a geometric framework for partial differential equations A. Prástaro and Th.M. Rassias [11] proved that the set of solutions of the J. d'Alembert equation (A) is larger than the set of smooth functions h of two variables x, y of the form (P). This agrees with the above mentioned counterexample by Th.M.

Rassias. The book by Th.M. Rassias and J. Šimša [14] discusses the work of both past and mainly current research in the subject. Then, A. Prástaro and Th.M. Rassias [10] extended their results on the d'Alembert equation to functions of more than two variables by considering the generalized d'Alembert equation

$$\frac{\partial^n \log h}{\partial x_1 \partial x_2 \cdots \partial x_n} = 0,$$

in which $h = h(x^1, x^2, \dots, x^n)$ is a scalar unknown function, smoothly depending on the variables $x^1, \dots, x^n$. Recently A. Prástaro has given a general method to calculate integral and quantum (co)bordism groups in PDEs [8]. This method has proved to be very useful in order to show existence of global solutions, their topological structure and tunneling effects in PDE's, i.e., existence of solutions that change their sectional topology. Furthermore, A. Prástaro and Th.M. Rassias in [12] have extended such results also to generalized d'Alembert equations built in the category of quantum manifolds. These objects are noncommutative manifolds introduced by A. Prástaro who has also formulated a general geometric theory of quantum PDEs [8,9]. By utilizing such a theory we proved the existence of quantum tunneling effects for solutions of noncommutative d'Alembert equations.

In this paper we provide an account of some recent results in the subject. (For more details see the original papers [8-12].)

1. The commutative generalized d'Alembert equation

The **n-d'Alembert equation**:

$$\frac{\partial^n logf}{\partial x_1 \cdots \partial x_n} = 0, \qquad (d'A)_n$$

is an n-th order partial differential relation on the fiber bundle $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, i.e., it defines a subset $Z_n \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$. Let $\{x^\alpha, u, u_\alpha, u_{\alpha\beta}, \dots, u_{\alpha_1 \cdots \alpha_n}\}$ be a coordinate system on the jet space $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ adapted to the fiber structures $\pi_n : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n, \overline{\pi}_{n,0} : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$. Then Z_n can be defined as the following subset:

$$Z_n \equiv \{D^n f(x^1, \dots, x^n) \in J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \mid \\ f(x^1, \dots, x^n) = f_1(x^2, \dots, x^n) \cdots f_n(x^1, \dots, x^{n-1})\}.$$

Furthermore, Z_n can be locally characterized as

$$Z_n = F^{-1}(0), \quad F : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R},$$

where the value of F is a sum of terms of the type

$$F[s; r|\alpha, \beta_1\beta_2, \cdots, \gamma_1 \dots \gamma_q] \equiv s u^r u_\alpha u_{\beta_1\beta_2} \cdots u_{\gamma_1 \dots \gamma_q},$$

with $\alpha \neq \beta_1 \neq \beta_2 \neq \cdots \neq \gamma_1 \neq \ldots \neq \gamma_q \leq n, s \in \mathbb{Z}, r \in \mathbb{N} \cup \{0\}$. Furthermore, the term in F containing $u_{1\ldots n}$ is just $u_{1\ldots n}u^{n-1}$. For example,

$$F = u_{xy}u - u_x u_y \quad \text{for } n = 2;$$
$$F = u_{xyz}u^2 - u_{xy}u_z u - u_{xz}u_y u + u_x u_y u_z \quad \text{for } n = 3.$$

Note that F has not locally constant rank on all Z_n, so Z_n is not a submanifold of $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$. Furthermore, on the open subset $C_n \equiv u^{-1}(\mathbb{R} \setminus 0) \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$, one recognizes that F has locally constant rank 1. Hence $Z_n \cap C_n$ is a subbundle of $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n$, of dimension $n + \frac{(2n)!}{(n!)^2} - 1$. In the following, for abuse of notation, we shall denote by $(d'A)_n$ either Z_n or $Z_n \cap C_n$. The fundamental geometric structure of $(d'A)_n$ is given by the following:

THEOREM 1.1

1) *The n-d'Alembert equation $(d'A)_n \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ is an n-th order PDE, formally integrable on the trivial fiber bundle $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$.*

2) *The characteristic distribution of $(d'A)_n$ is zero.*

REMARK 1.1

Note that, even if the characteristic distribution of $(d'A)_n$ is zero, we can built regular solutions by means of characteristic method if one considers the infinitesimal symmetry of $(d'A)_n$ (for $n = 2$ it is generated by the following vector fields ζ on $\pi : W(\equiv)\mathbb{R}^3 \to \mathbb{R}^2$:

$$\zeta = f(u)\partial x + g(y, u)\partial y + [s(y) + r(x)]\, u\partial u, \qquad (\bullet)$$

where f, s and r are arbitrary functions of a single variable and g is an arbitrary function of two variables).

In fact we have the following:

THEOREM 1.2

Let $\psi : P \to (d'A)$ be the mapping that characterizes a 1-dimensional regular integral manifold $N \subset (d'A)$ such that the second holonomic prolongation $\zeta^{(2)}$ of a vector field ζ, as given in $(\bullet)$, for suitable functions f, g, r and s, satisfies the following conditions:

(i) transversality condition*: $\psi^*(\zeta^{(2)} \rfloor \eta) \neq 0$;*

(ii) initial conditions*: $\psi^*\mathcal{I} = 0$, $\psi^*(\zeta^{(2)} \rfloor \mathcal{I} = 0$,*

where $\mathcal{I}$ is the Pfaffian ideal defining the contact structure of $(d'A)$ (see equation (1.3) below), and η is a differential 2-form defining the horizontalization for N. Then, if ϕ is the flow associated to $\zeta^{(2)}$: $\partial\phi = \zeta^{(2)}$, one has that $V \equiv \bigcup_{s \in J} \phi_s(N)$ is a regular 2-dimensional integral manifold of $(d'A)$, where J is a suitable neighborhood of $0 \in \mathbb{R}$.

Proof. The conditions for $\zeta^{(2)}$ to be a symmetry for $\mathcal{I}$ and transversal to N imply that $\phi_s(N) \equiv N_s$ are 1-dimensional regular integral manifolds of $(d'A)$, for s in a suitable neighborhood J of $0 \in \mathbb{R}$. Furthermore, the conditions (i) and (ii) assure that the 2-dimensional manifold $V \equiv \bigcup_{s\in J} N_s$ is integral also for $(d'A)$.

REMARK 1.2

Another way to built solutions by means of the characteristic method is just to recognize characteristic strips in $(d'A)_n$, cf. [8]. In the following lemma we explicitly give the characteristic strips for the case $n = 2$.

LEMMA 1.1

The equation

$$uu_{xy} - u_x u_y = 0 \qquad (d'A)$$

admits the following two 1*-dimensional characteristic strips:*

$$\begin{aligned} v_1 &\equiv X^x\left(\partial x + u_x\partial u + u_{xx}\partial u^x + u_{xy}\partial u^y + u_{xxx}\partial u^{xx} + u_{xxy}\partial u^{xy} + u_{xyy}\partial u^{yy}\right)\\ v_2 &\equiv X^y\left(\partial y + u_y\partial u + u_{yx}\partial u^x + u_{yy}\partial u^y + u_{yxx}\partial u^{xx} + u_{xyy}\partial u^{xy} + u_{yyy}\partial u^{yy}\right) \end{aligned} \qquad (1.1)$$

where X^x and X^y are arbitrary numerical functions on $J\mathcal{D}^2(\mathbb{R}^2, \mathbb{R})$.

Now, we are ready to prove the first main theorem.

THEOREM 1.3

The set $\mathcal{S}ol(d'A)_n$ of all solutions of the n-d'Alembert equation: $(d'A)_n$, considered in domains contained in $\mathbb{R}^n$, is larger than the set of all functions f that can be represented in the form

$$\begin{aligned} &f(x^1, \dots, x^n)\\ &\quad = f_1(x^2, x^3, \dots, x^n) f_2(x^1, x^3, \dots, x^n) \cdots f_n(x^1, x^2, \dots, x^{n-1}). \end{aligned} \qquad (1.2)$$

Proof. The Cartan distribution $\mathbb{E}_n \subset T(d'A)_n$ of $(d'A)_n$ that characterizes the solutions of $(d'A)_n$ is the annihilator of the Pfaffian ideal $\mathcal{I}_n$ generated by the following differential 1-forms on $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$:

$$\omega_\alpha \equiv \begin{cases} \omega_0 & \equiv dF = (\partial x_\alpha.F)dx^\alpha + (\partial u.F)du + (\partial u^\alpha.F)du_\alpha \\ & \quad + \cdots + (\partial u^{\alpha_1\cdots\alpha_n}.F)du_{\alpha_1\cdots\alpha_n} \\ \omega_1 & \equiv du - u_\alpha dx^\alpha \\ \omega_{2\alpha} & \equiv du_\alpha - u_{\alpha\beta}dx^\beta \\ \cdots & \\ \omega_{k\alpha_1\cdots\alpha_{n-1}} & \equiv du_{\alpha_1\cdots\alpha_{n-1}} - u_{\alpha_1\cdots\alpha_{n-1}\beta}dx^\beta \end{cases} \qquad (1.3)$$

with the function F that defines $(d'A)_n$. One has a canonical embedding $((d'A)_{n-1})_{+1} \to (d'A)_n$. Let us consider, now, a vector field $\zeta : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to TJ\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ of the following type:

$$\zeta \equiv \partial x_n + u_n \partial u + u_{n\alpha}\partial^\alpha + \cdots + u_{n\alpha_1 \ldots \alpha_n}\partial u^{\alpha_1 \ldots \alpha_n} \tag{1.4}$$

such that $u_{n\alpha_1 \ldots \alpha_n}$ are functions on $J\mathcal{D}^{n+1}(\mathbb{R}^n, \mathbb{R})$ satisfying the equations which define the first prolongations of $(d'A)_n$, $\{F = 0\}$:

$$\begin{cases} F_\alpha \equiv (\partial x_\alpha.F) + (\partial u.F)u_\alpha \\ \qquad + \cdots + (\partial u^{\alpha_1 \ldots \alpha_n}.F)u_{\alpha\alpha_1 \ldots \alpha_n} \\ \quad = 0, \quad 1 \leq \alpha \leq n \\ F = 0 \end{cases} . \qquad ((d'A)_n)_{+1}$$

Then ζ is necessarily transversal to

$$((d'A)_{n-1})_{+1} = J\mathcal{D}((d'A)_{n-1}) \bigcap J\mathcal{D}^n(\mathbb{R}^{n-1}, \mathbb{R})$$

and it generates a characteristic strip for $(d'A)_n$. Therefore, if N is an $(n-1)$-dimensional integral manifold contained in $((d'A)_{n-1})_{+1}$, a vector field ζ, as defined in (1.4), generates from N an n-dimensional integral manifold V contained in $(d'A)_n$. As N is not, in general, a regular solution of the equation $(d'A)_{n-1}$, then the so generated integral manifold V, solution of $(d'A)_n$, cannot be represented as the graph of some n-derivative of function $f : \mathbb{R}^n \to \mathbb{R}$. Hence, in particular, V cannot be represented as the image of the n-derivative of a function $f(x^1, \ldots, x^n)$, of the type (1.2).

We shall prove, now, that in $\mathcal{S}ol(d'A)$ there are solutions that change their sectional topologies. We shall use some recent results obtained by A. Prástaro about tunneling effects and quantum and integral (co)bordism in PDE's [8]. In the following we shall consider the n-d'Alembert equation given as a submanifold $(d'A)_n$ of the jet space $J^n_n(\mathbb{R}^{n+1})$ by means of the embedding $(d'A)_n \hookrightarrow J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \hookrightarrow J^n_n(\mathbb{R}^{n+1})$, where $J^n_n(\mathbb{R}^{n+1}) \equiv \{[N]^n_a\}$ with $[N]^n_a$ the set of n-dimensional submanifolds of $\mathbb{R}^{n+1}$ that have with the n-dimensional submanifold $N \subset \mathbb{R}^{n+1}$ a contact of order n at an $a \in N$. In the following table we report the explicitly calculated expressions of the integral bordism groups $\Omega^{(d'A)_n}_{n-1}$ of $(d'A)_n$, for $n \in \{2, 3, 4, 5\}$.

$\Omega^{(d'A)_2}_1 = 0$	$\Omega^{(d'A)_3}_2 = \mathbf{Z}_2$	$\Omega^{(d'A)_4}_3 = 0$	$\Omega^{(d'A)_5}_4 = \mathbf{Z}_2 \oplus \mathbf{Z}_2$

Tab. 1.1 Integral bordism groups of (d'A)$_n$ for $2 \leq n \leq 5$

Now, by means of these integral bordism groups, we see that there are solutions of $(d'A)_n$ that change their sectional topology. In fact, for example, if $n = 2$ or $n = 4$ one has: $\Omega^{(d'A)_2}_1 = \Omega^{(d'A)_4}_3 = 0$. Thus, in the case $n = 2$, any compact closed admissible integral 1-dimensional manifold N of $(d'A)$ is a disjoint union of copies of S^1: $N = S^1\dot{\cup}\ldots_p\ldots\dot{\cup}S^1$. Hence, we can always find a connected 2-dimensional integral manifold V, contained into $(d'A)$, such that $\partial V = N$. In other words, if $N_0 = S^1\dot{\cup}\ldots_r\ldots\dot{\cup}S^1$ and $N_1 = S^1\dot{\cup}\ldots_s\ldots\dot{\cup}S^1$ are two compact closed admissible integral 1-dimensional manifolds of $(d'A)$, we can always find a 2-dimensional integral manifold $V \subset (d'A)$ such that $\partial V = N_0\dot{\cup}N_1$. Of course, if $r \neq s$ one has a tunnel effect, i.e., a change in the sectional topology of V, passing from N_0 to N_1. Similar considerations hold for $n = 4$. Furthermore, if $n = 3$ one has: $\Omega_2^{(d'A)_3} = \mathbb{Z}_2$. In this case we have two types of compact closed admissible integral 2-dimensional manifolds. But the above considerations can be extended to each of these types of integral manifolds.

We now state our second main theorem.

THEOREM 1.4

In the set of solutions $Sol(d'A)_n$ of the n-d'Alembert equation, $(d'A)_n \subset J\mathcal{D}^n(\mathbb{R}^n,\mathbb{R}) \subset J^n_n(\mathbb{R}^{n+1})$, there are also some manifolds enjoying a change of sectional topology (tunneling effect).

2. The quantum generalized d'Alembert equation

In order to give a geometrical model for quantum physics, A. Prástaro has introduced in [8,9] a new category of noncommutative manifolds (*quantum manifolds*) built by means of a suitable structured noncommutative Frèchet algebra, (*quantum algebra*). An example for such an algebra can be the C^*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ of continuous linear operators on a Hilbert space $\mathcal{H}$ corresponding to the canonically quantized observables of a classical system.

The aim of this section is to consider the extension of the generalized D'Alembert equation $(d'A)_m$ to this new noncommutative framework given by A. Prástaro and Th.M. Rassias in [11]. Let us recall some fundamental definitions and results on quantum manifolds as given by A. Prástaro.

A *quantum algebra* is a triplet (A, ϵ, c), where:

(i) A is a metrizable, complete, Hausdorff, locally convex topological $\mathbf{K}$-vector space, that is also an associative $\mathbf{K}$-algebra with unit;

(ii) $\epsilon : \mathbf{K} \to A_0 \subset A$ is a $\mathbf{K}$-algebra homomorphism, where A_0 is the centre of A;

(iii) $c : A \to \mathbf{K}$ is a $\mathbf{K}$-linear morphism, with $c(e) = 1$, $e =$unit of A.

A *quantum vector space* of dimension $(m_1, \ldots, m_s) \in \mathbf{N}^s$, built on the quantum algebra $A \equiv A_1 \times \ldots \times A_s$, is a locally convex topological $\mathbf{K}$-vector space E isomorphic to $A_1^{m_1} \times \ldots A_s^{m_s}$.

A *quantum manifold* of dimension $(m_1, \dots, m_s)$ over a quantum algebra $A \equiv A_1 \times \dots \times A_s$ of class Q^k_w, $0 \leq k \leq \infty, \omega$, is a locally convex manifold M modelled on E and with a Q^k_w-atlas of local coordinate mappings, where Q^k_w means class C^k_w (weak differentiability, e.g., H.H. Keller [3]), and derivatives A_0-linear. So for each open coordinate set $U \subset M$ we have a set of $m_1 + \dots + m_s$ coordinate functions $x^A : U \to A$, (*quantum coordinates*). The *tangent space* T_pM at $p \in M$, is the vector space built in the usual way (cf. [9]). Then, derived tangent spaces associated to a quantum manifold M can be naturale defined.

A *quantum PDE* (QPDE) of order k on the fibre bundle $\pi : W \to M$, defined in the category of quantum manifolds, is a subfibrebundle $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ of the jet-quantum derivative space $J\hat{\mathcal{D}}^k(W)$ over M. $J\hat{\mathcal{D}}^k(W)$ is, in the category of quantum manifolds, the counterpart of the jet-derivative space for usual manifolds.

For more details see [9, 10], where there is also formulated a geometric theory for quantum PDEs that generalizes the theory of PDEs for usual manifolds.

In order to state existence of local solutions of QPDEs the following two theorems are very useful. (For the terminology used see [9, 10].)

THEOREM 2.1 (A. PRÁSTARO [9])

1) (δ-POINCARÉ LEMMA FOR QUANTUM PDEs). *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular QPDE. If A_0 is a Noetherian $\mathbf{K}$-algebra, then $\hat{E}_k$ is a δ-regular QPDE.*

2) (CRITERION OF FORMAL QUANTUM INTEGRABILITY). *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular, δ-regular QPDE. Then if $\hat{g}_{k+r+1}$ is a bundle of A_0-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq n$, then $\hat{E}_k$ is formally quantumintegrable.*

A *solution* of $\hat{E}_k$ that satisfies the initial condition $q \in \hat{E}_k$ is an m-dimensional quantum manifold $N \subset \hat{E}_k$ such that $q \in N$ and N can be represented in a neighborhood of any of its points $q' \in N$, except for a nowhere dense subset $\Sigma(N) \subset N$ of dimension $\leq m-1$, as image of the k-derivative $D^k s$ of some Q^k_w-section s of $\pi : W \to M$. We call $\Sigma(N)$ the set of *singular points* (of Thom-Bordman type) of N. If $\Sigma(N) \neq \emptyset$ we say that N is a *regular solution* of $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$.

Let us denote by $\hat{J}^k_m(W)$ the k-jet of m-dimensional quantum manifolds (over A) contained into W. One has the natural embeddings $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W) \subset \hat{J}^k_m(W)$. Then, with respect to the embedding $\hat{E}_k \subset \hat{J}^k_m(W)$ we can consider solutions of $\hat{E}_k$ as m-dimensional (over A) quantum manifolds $V \subset \hat{E}_k$ such that V can be representable in the neighborhood of any of its points $q' \in V$; except for a nowhere dense subset $\Sigma(V) \subset V$, of dimension $\leq m-1$; as $N^{(k)}$ — the k-quantum prolongation of an m-dimensional (over A) quantum manifold $N \subset W$.

In the case that $\Sigma(V) = \emptyset$, we say that V is a *regular solution* of $\hat{E}_k \subset \hat{J}^k_m(W)$. Of course, solutions V of $\hat{E}_k \subset \hat{J}^k_m(W)$, even regular ones in general are not diffeomorphic to their projections $\pi_k(V) \subset M$, hence they are not representable by means of sections of $\pi : W \to M$. Therefore, the above two theorems allow us to obtain existence theorems of local solutions.

Now, in order to study the structure of global solutions it is necessary to consider the integral bordism groups of QPDEs. In [9] A. Prástaro has extended to QPDEs his previous results on the determination of integral bordism groups of PDEs [8]. Let us denote by $\Omega_p^{\hat{E}_k}$, $0 \leq p \leq m-1$, the integral bordism groups of a QPDE $\hat{E}_k \subset \hat{J}^k_m(W)$ for closed integral quantum submanifolds of dimension p and class Q^∞_w, over a quantum algebra A of $\hat{E}_k$. The structure of smooth global solutions of $\hat{E}_k$ are described by the integral bordism group $\Omega_{m-1}^{\hat{E}_\infty}$ corresponding to the quantum prolongation $\hat{E}_\infty$ of $\hat{E}_k$.

Let us, pass to the study of a noncommutative case. For, now, set $\mathbf{K} = \mathbb{R}$ and let A be a quantum algebra such that A_0 is Noetherian. Let us consider the following trivial fiber bundle: $\pi : W \equiv A^{m+1} \to A^m$, with quantum coordinates $(x^{A_1}, \dots, x^{A_m}, u) \mapsto (x^{A_1}, \dots, x^{A_m})$. Then the **noncommutative generalized m-d'Alembert equation**, $m \in \mathbb{N}$, $2 \leq m < \infty$, is the QPDE $\widehat{(d'A)}_m \subset J\hat{\mathcal{D}}^m(W) \subset \hat{J}^m_m(W)$ defined by means of the following Q^∞_w-function:

$$\begin{aligned} F : J\hat{\mathcal{D}}^m(A^m; A) &\to \overset{m}{\hat{A}} \\ &\equiv Hom_{A_0}(A \otimes_{A_0} \cdots_m \cdots \otimes_{A_0} A; A) \equiv (\dot{T}^m_0 A) O A \equiv (\dot{T}^m_0 A)^+, \end{aligned}$$

where F is the sum of formally the same terms with the commutative case. Of course more care must be taken on their meaning. For details see [9] and [11]. The quantum jet-derivative space $J\hat{\mathcal{D}}^m(W) \subset \hat{J}^m_m(W)$ is a quantum manifold of dimension $(m+1, m, m^2, \dots, m^m)$ over the quantum algebra $C \equiv A \times \overset{1}{\hat{A}} \times \dots \times \overset{m}{\hat{A}}$, i.e., $J\hat{\mathcal{D}}^m(W)$ is modelled on $A^{m+1} \times (\overset{1}{\hat{A}})^m \times \dots \times (\overset{m}{\hat{A}})^{m^m}$. Moreover $J\hat{\mathcal{D}}^m(W)$ is an open quantum submanifold of $\hat{J}^m_m(W)$, and $\widehat{(d'A)}_m$ is a quantum regular QPDE as the mappings $(\widehat{(d'A)}_m)_{+r} \to (\widehat{(d'A)}_m)_{+(r-1)}$, $r \geq 1$, are surjective. Hence, taking into account that $\widehat{(d'A)}_m$ is also δ-regular, it follows that $\widehat{(d'A)}_m$ is formally quantum integrable. Then, since in the open subset $Z_m \equiv u^{-1}(0) \subset J\hat{\mathcal{D}}^m(W)$ the QPDE $\widehat{(d'A)}_m$ is quantum analytic, in a suitable neighborhood U of any point $q \in Z_m \cap \widehat{(d'A)}_m$ one is able to build a quantum analytic solution that is diffeomorphic to $\pi_m(U) \subset M \equiv A^m$. Therefore we have the following:

THEOREM 2.2

The noncommutative generalized m-d'Alembert equation $\widehat{(d'A)}_m$ is a formally quantum integrable QPDE. For any point $q \in Z_m \cap \widehat{(d'A)}_m$ passes a quantum analytic solution V that is diffeomorphic to $\pi_m(V) \subset M \equiv A^m$.

In order to state existence theorems of global solutions for $\widehat{(d'A)}_m$ it is necessary to calculate the integral bordism groups $\Omega_p^{\widehat{(d'A)}_m}$, $0 \leq p \leq m-1$. From the above theorem, and since W is p-connected, $p \in \{0, \dots, m-1\}$, one has the following isomorphism: $\Omega_p^{\widehat{(d'A)}_m} \cong A \otimes_{\mathbf{K}} H_p(W; \mathbf{K})$, $\quad 0 \leq p \leq m-1$. For a proof see [9]. On the other hand $H_0(W; \mathbf{K}) \cong \mathbf{K}$, and $H_p(W; \mathbf{K}) = 0$, for $1 \leq p \leq m-1$. Therefore, one obtains: $\Omega_0^{\widehat{(d'A)}_m} \cong A$, $\Omega_p^{\widehat{(d'A)}_m} \cong 0$, for $1 \leq p \leq m-1$. Hence, in particular, the following result holds:

THEOREM 2.3

Any admissible integral closed quantum manifold $N \subset \widehat{(d'A)}_m$, of dimension $m-1$ over A, bounds an integral quantum manifold of dimension m over A that is a solution of $\widehat{(d'A)}_m$. Moreover, for two admissible integral closed quantum manifolds $N_0, N_1 \subset \widehat{(d'A)}_m$, of dimension $m-1$ over A, there exists a solution V of $\widehat{(d'A)}_m$ such that $\partial V = N_0 \dot{\cup} N_1$.

In particular if N_0 and N_1 are homotopically different and V is connected, then V is a solution with change of sectional topology. Thus, we get the following:

COROLLARY 2.1

*In the set $Sol(\widehat{(d'A)}_m)$ of solutions of the quantum generalized m-d'Alembert equation, there are solutions with change of sectional topology (*quantum tunnel effect*). Such solutions, in general, cannot be represented as mappings $f : A^m \to A$.*

Conclusions

The geometric theory of PDEs introduced by A. Prástaro in [8, 9] is a handable framework where problems in the theory of partial differential equations find their natural solutions. In fact, the J. d'Alembert equation is one such application.

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On a Pexider type equation on Δ^+

Abstract. The Pexider equation in question is defined with the use of t-norm and is considered in the space Δ^+ of all non decreasing functions F from $\mathbb{R}^+$ into I that satisfy $F(0) = 0$, $F(\infty) = 1$ and are left-continuous on $(0, \infty)$. The general solution of the equation is found as well as, under certain regularity conditions solutions of more explicit from are exhibited.

1. Introduction

Cauchy's equation on the space of distance distribution functions, $\Delta^+$was investigated in two papers [7, 8], by one of the authors. In this paper we consider the the Pexider equation on Δ^+. Work on Pexider equations on general algebraic structures was done by M.A. Taylor [12] and A. Krapež and M.A. Taylor [4]. In their work, they assumed very little structure on the domain and no regularity on the functions and showed that if solutions exist, then some have to be in terms of homomorphisms of the underlying spaces. This paper makes certain regularity assumptions about the functions and, with the results of [7], obtains rather explicit solutions of the Pexider equation. For further references on the Pexider equation as well as functional equations in general, we refer to the classic works of Aczél [1], and Aczél and Dhombres [2].

This paper is divided into four sections, Section 1 being this introduction. In Section 2, we introduce the necessary notation and known results to keep this paper reasonably selfcontained. Section 3 introduces the functional equation, some general properties of the solutions and our main result. We conclude in Section 4 with by considering certain special cases, which yield explicit formulas for the solutions.

2. Preliminaries from Δ^+

We will denote by Δ^+ the space of all nondecreasing functions F from $\mathbb{R}^+$ into I that satisfy $F(0) = 0$, $F(\infty) = 1$, that are left-continuous on $(0, \infty)$.

The following elements of Δ^+ are of particular importance and therefore merit special symbols:

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(i) For any a in $\mathbb{R}^+$, ε_a is the function in Δ^+ defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{for } 0 \leqslant x \leqslant a, \\ 1, & \text{for } a < x \leqslant \infty, \end{cases} \quad \text{if } 0 \leqslant a < \infty,$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < \infty, \\ 1, & \text{for } x = \infty. \end{cases}$$

(ii) For any a in $\mathbb{R}^+$ and b in I, $\delta_{a,b}$ is the function in Δ^+ defined by

$$\delta_{a,b}(x) = \begin{cases} 0, & \text{for } \ 0 \leqslant x \leqslant a, \\ b, & \text{for } \ a < x < \infty, \\ 1, & \text{for } \ x = \infty, \end{cases} \quad \text{if } 0 \leqslant a < \infty, \text{ and } \delta_{\infty,b} = \varepsilon_\infty.$$

We will denote the set of all the $\delta_{a,b}$ by Δ_δ^+ and note that for all a in $\mathbb{R}^+$, $\delta_{a,1} = \varepsilon_a$ and $\delta_{a,0} = \varepsilon_\infty$. We also have:

LEMMA 2.1
For $0 \leqslant a, c < \infty$ *and* $0 < b, d \leqslant 1$, $\delta_{a,b} = \delta_{c,d}$ *if and only if* $a = c$ *and* $b = d$.

The set Δ^+, partially ordered by

$$F \leqslant G \ \text{ if and only if } \ F(x) \leqslant G(x) \text{ for all } x \text{ in } \mathbb{R}^+,$$

forms a *complete lattice*, i.e., a partially ordered set in which every subset has a supremum and an infimum, see [3]. Here, for any subset S of Δ^+, the supremum of S is the pointwise supremum of all functions in S and the infimum of S is the supremum of the set of all lower bounds of S. The latter refinement is necessary since the pointwise infimum of left-continuous functions need not be left-continuous.

We note that in particular we have,

$$\varepsilon_a \leqslant \varepsilon_b\,, \quad \text{whenever } a \geqslant b,$$

and

$$\delta_{a,b} \leqslant \delta_{c,d}\,, \quad \text{whenever } a \geqslant c \text{ and } b \leqslant d.$$

Moreover, ε_∞ and ε_0 are, respectively, the least and greatest elements in this partial order.

To generalize the triangle inequality to probabilistic metric spaces, one needs a binary operation on the space of distance distribution functions: A triangle function τ is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

As an immediate consequence we have that ε_∞ is a zero for τ, i.e., that

$$\tau(\varepsilon_\infty, F) = \varepsilon_\infty, \quad \text{for all } F \text{ in } \Delta^+,$$

whence (Δ^+, τ) is a semigroup with identity and zero.

We will be principally concerned with the class of triangle functions τ_T that are induced by left-continuous t-norms via:

$$\tau_T(F,G)(x) = \sup_{u+v=x} \{T(F(u),G(v))\}, \quad \text{for all } F,G \text{ in } \Delta^+ \text{ and } x \text{ in } \mathbb{R}^+.$$

A simple calculation yields that for all a,b in $\mathbb{R}^+$ and all c,d in I,

$$\tau_T(\delta_{a,c}, \delta_{b,d}) = \delta_{a+b,T(c,d)}. \tag{1}$$

This implies that $(\Delta_\delta^+, \tau_T)$ is a subsemigroup of (Δ^+, τ_T).

Moreover, it follows from (1), that for any $\delta_{a,b}$ in Δ_δ^+, we have $\delta_{a,b} = \tau_T(\varepsilon_a, \delta_{0,b})$.

We also have the following basic lemma, which is due to R.C. Powers [6]:

LEMMA 2.2

Let F be in Δ^+, then

$$F = \sup_{a\in \mathbb{R}^+} \delta_{a,F(a)}. \tag{2}$$

DEFINITION 2.3

A function φ from Δ^+ into Δ^+ is said to be *sup-continuous* if, for any index set J and any collection $\{F_j\}$ such that F_j is in Δ^+ for all j in J, we have

$$\varphi(\sup_{j\in J} F_j) = \sup_{j\in J} \varphi(F_j).$$

The next lemma is due to R.M. Tardiff [11] (see also [10, Sec. 12.9]).

LEMMA 2.3

If T is a continuous t-norm, then τ_T is sup-continuous in each variable.

Thus we have that a sup-continuous function on Δ^+ as well as any τ_T, with continuous T, is completely determined by its values on Δ_δ^+. This is the key observation for solving functional equations for sup-continuous functions on Δ^+.

3. Pexider equation

DEFINITION 3.1

We say that the triple $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of the Pexider equation if

$$\varphi_1\left(\tau_T(F,G)\right) = \tau_T\left(\varphi_2(F), \varphi_3(G)\right), \quad \text{for all } F,G \in \Delta^+. \tag{3}$$

We begin with some obvious observations.

If φ is a solution of Cauchy's equation, see [7, 8], then $(\varphi, \varphi, \varphi)$ is a solution triple of the Pexider equation. Furthermore, if φ is a solution of Cauchy's equation and we let $\varphi_2(F) = \tau_T(\varphi(F), H_0)$, $\varphi_3(F) = \tau_T(\varphi(F), H_1)$, and $\varphi_1(F) = \tau_T(\varphi(F), \tau_T(H_0, H_1))$, then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of the Pexider equation. The latter includes all constant solutions by choosing $\varphi(F) = \varepsilon_0$. Furthermore, if $H_0 = \varepsilon_\infty$, then $\varphi_1 = \varphi_2 = \varepsilon_\infty$ and φ_3 is arbitrary; similarly for $H_1 = \varepsilon_\infty$.

This yields a large class of solutions of the Pexider equation and the rest of this section will be devoted to showing that, among sup-continuous solutions satisfying a few additional conditions, these are the only solutions of the Pexider equation on Δ^+. Krapež and Taylor, [4], showed that such solutions have to occur if the Pexider equation has a solution; they did so on spaces with very little structure and without regularity assumptions. To show that all solutions of a certain class are of this form, we need, however, some regularity assumptions.

From now on we will make more stringent assumptions on the class of solutions of (3). As was pointed out, the assumption that the solutions are sup-continuous allows us to reduce the problem to solving (3) on Δ_δ^+. We will now make two further assumption: First we assume that T is a *strict t-norm* and second, that the solutions will map Δ_δ^+ into a cancellative subsemigroup of (Δ^+, τ_T). To this end we need the following results and definitions, see [10]:

THEOREM 3.2

If T is a strict t-norm then

$$T(x, y) = g^{-1}(g(x) + g(y)), \quad \text{for all } x, y \text{ in } I,$$

where g is a continuous, strictly decreasing function from I onto $\mathbb{R}^+ = [0, \infty]$, with $g(1) = 0$.

The function g is called an *inner additive generator* (briefly, a generator); and it is well-known that g and h generate the same t-norm if and only if there is a $k > 0$ such that

$$g(x) = k \cdot h(x), \quad \text{for all } x \text{ in } I. \tag{4}$$

DEFINITION 3.3

Let T be a strict t-norm and g any inner additive generator of T. Then we let

$$\Delta_T^+ = \{F \text{ in } \Delta^+ \mid g \circ F \text{ is convex on } (b_F, \infty)\},$$

where $b_F = \sup_{x \in \mathbb{R}^+}\{F(x) = 0\}$.

In view of (4), the set Δ_T^+ does not depend on the choice of generator g. Furthermore, $(\Delta_T^+ \setminus \{\varepsilon_\infty\}, \tau_T)$ is a cancellative subsemigroup of (Δ^+, τ_T), [10, Theorem 7.8.11].

We note here that the set Δ_T^+ is often referred to as the set of T-log-concave elements of Δ^+. This terminology is due to R.A. Moynihan [5] (see also [10, Sec. 7.8]) and stems from the fact that he used multiplicative generators to define this set.

Clearly $b_{\delta_{a,b}} = a$ and, since $\delta_{a,b}$ is constant on (a, ∞), it follows that $g \circ \delta_{a,b}$ is convex, whence

$$\Delta_\delta^+ \subseteq \Delta_T^+.$$

With this we have the following lemma:

LEMMA 3.4

Assume that $\varphi_i(\Delta_\delta^+ \setminus \{\varepsilon_\infty\}) \subset \Delta_T^+ \setminus \{\varepsilon_\infty\}$, *for* $i = 1, 2, 3$ *and that for* $i = 2$ *or* $i = 3$, *we have* $\varphi_i(F) \leqslant \varphi_i(\varepsilon_0)$ *and* $\varphi_i(\varepsilon_0) \in \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. *Then* $(\varphi_1, \varphi_2, \varphi_3)$ *is a solution triple of* (3) *on* Δ_δ^+, *if and only if there is a function* φ *with*

$$\varphi(\tau_T(\delta_{a,c}, \delta_{b,d})) = \tau_T(\varphi(\delta_{a,c}), \varphi(\delta_{b,d}))$$

(i.e. a solution of Cauchy's equation on Δ_δ^+ *), such that*

$$\varphi_1(F) = \tau_T(\varphi(F), \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$$

$$\varphi_2(F) = \tau_T(\varphi(F), \varphi_2(\varepsilon_0))$$

$$\varphi_3(F) = \tau_T(\varphi(F), \varphi_3(\varepsilon_0))$$

Proof. The only if part follows from the observations in the previous sections. We will follow an approach similar to that of the proof in [4, Theorem 10]. We let $G = \varepsilon_0$ in (3) to get

$$\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0)). \tag{5}$$

For $F \in \Delta_T^+$ we have that φ_1 maps into the coset $S_3 = \tau_T(\Delta_T^+, \varphi_3(\varepsilon_0))$. Similarly, letting $F = \varepsilon_0$ we have that φ_1 maps into the coset $S_2 = \tau_T(\Delta_T^+, \varphi_2(\varepsilon_0))$. Since $\varphi_2(F) \leqslant \varphi_2(\varepsilon_0)$ and $\varphi_2(\varepsilon_0) = \delta_{a,b}$, (or the same holds for φ_3), we can write $\varphi_2(F) = \tau_T(H, \delta_{a,b})$, which is $\varphi_2(F) = T(H(x-a), b)$, so that

$$H(x) = g^{-1}\left(g(\varphi_2(F)(x+a)) - g(b)\right) \in \Delta_T^+$$

and the range of φ_1 is contained in the coset $S_{2,3} = \tau_T(\Delta_T^+, \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$. Now we let $M_i(F) = \tau_T(F, \varphi_i(\varepsilon_0))$ for $i = 2, 3$. Using the fact that τ_T is cancellative on Δ_T^+, we have that M_i is invertible on the coset $\tau_T(\Delta_T^+, \varphi_i(\varepsilon_0))$ for $i = 2$ and $i = 3$, respectively. The associativity and commutativity of τ_T yield that $M_2(M_3(F)) = M_3(M_2(F))$ and $M_i(\tau_T(F, G)) = \tau_T(M_i(F), G) = \tau_T(F, M_i(G))$. Similarly, we have $M_2^{-1}(M_3^{-1}(F)) = M_3^{-1}(M_2^{-1}(F))$, for $F \in S_{2,3}$ and $M_i^{-1}(\tau_T(G, H)) = \tau_T(M_i^{-1}(G), H)$, for $G \in S_i$, $i = 2, 3$. Using these properties of the M_i, we can now proceed and let $F = \varepsilon_0$ in (3), to get

$$\varphi_1(F) = M_3(\varphi_2(F)) \quad \text{and} \quad \varphi_1(F) = M_2(\varphi_3(F)),$$

which by the invertibility of the M_i, yields:

$$\varphi_2(F) = M_3^{-1}(\varphi_1(F)) \quad \text{and} \quad \varphi_3(F) = M_2^{-1}(\varphi_1(F)). \tag{6}$$

Substituting (6) into (3) yields

$$\varphi_1(\tau_T(F,G)) = \tau_T(M_3^{-1}(\varphi_1(F)), M_2^{-1}(\varphi_1(G))).$$

Applying $M_2^{-1}M_3^{-1}$ to both sides and simplifying yields that $\varphi = M_2^{-1}M_3^{-1}\varphi_1$ satisfies Cauchy's equation on Δ_δ^+. Using this and equations (5) and (6), gives the desired result.

Lemma 3.4 together with Lemmas 2.2 and 2.3 immediately yield our main result:

THEOREM 3.5

For $i = 1,2,3$, let $\varphi_i : \Delta^+ \to \Delta^+$, be sup-continuous. Further, assume that $\varphi_i(\Delta_\delta^+ \setminus \{\varepsilon_\infty\}) \subset \Delta_T^+ \setminus \{\varepsilon_\infty\}$, for $i = 1,2,3$ and that for $i = 2$ or $i = 3$, we have $\varphi_i(F) \leqslant \varphi_i(\varepsilon_0)$ and $\varphi_i(\varepsilon_0) \in \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. Then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of (3) *on Δ^+, if and only if there is a function φ with*

$$\varphi(\tau_T(F,G)) = \tau_T(\varphi(F), \varphi(G)), \quad \textit{for all } F, G \in \Delta^+$$

(i.e. a solution of Cauchy's equation on Δ^+), such that

$$\varphi_1(F) = \tau_T(\varphi(F), \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$$
$$\varphi_2(F) = \tau_T(\varphi(F), \varphi_2(\varepsilon_0))$$
$$\varphi_3(F) = \tau_T(\varphi(F), \varphi_3(\varepsilon_0))$$

4. Explicit solutions

Using hypotheses similar to those of Theorem 3.5, we can use the following theorem (see [7, Theorems 6.6 and 6.7]), to obtain specific explicit solutions of the Pexider equation.

THEOREM 4.1

Let T be a strict t-norm with generator g and let φ be a sup-continuous solution of Cauchy's equation for τ_T such that $\varphi(\Delta_\delta^+) \subset \Delta_T^+$. Then, given any $c \in (0,1)$, we have for all F in Δ^+,

$$\varphi(F) = \sup_{t \in \mathbb{R}^+} \tau_T\left([\varphi(\varepsilon_1)]^t, [\varphi(\delta_{0,c})]^{kg(F(t))}\right), \tag{7}$$

where $k = \frac{1}{g(c)}$.

Here F^μ is the "μ-th τ_T power" of F (for F in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$), given by

$$F^\mu(x) = g^{-1}\left(\mu \cdot g\left(F\left(\frac{x}{\mu}\right)\right)\right), \quad \text{for } 0 < \mu < \infty,$$

$$F^0 = \lim_{\mu \to 0} F^\mu = \varepsilon_0, \quad F^\infty = \lim_{\mu \to \infty} F^\mu = \begin{cases} \varepsilon_\infty, & \text{for } F \neq \varepsilon_0, \\ \varepsilon_0, & \text{for } F = \varepsilon_0. \end{cases}$$

Assuming now that φ satisfies the hypotheses of Theorem 4.1 and in particular, that $\varphi(\varepsilon_1) = \varepsilon_a$ and $\varphi(\delta_{0,c}) = \delta_{0,b}$, we see that

$$\varphi(\delta_{u,v}) = \delta_{au,\theta(v)}\,; \quad \theta(v) = g^{-1}\left(\frac{g(b)}{g(c)} g(v)\right).$$

This, in turn, implies by the sup-continuity of φ that φ is an order automorphism (see [6]), that is

$$\varphi(F)(x) = \theta(F(\gamma(x))) \quad \text{for all } F \in \Delta^+ \text{ and all } x \in \mathbb{R}^+.$$

Here $\gamma(u) = \frac{u}{a}$.

Thus the question arises whether each function of the triple $(\varphi_1, \varphi_2, \varphi_3)$ (a solution of (3)) is also induced by left and right composition. This is not quite the case as the following theorem shows:

THEOREM 4.2
For $i = 1, 2, 3$, let $\varphi_i : \Delta_\delta^+ \to \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$, be sup-continuous with

$$\varphi_i(\varepsilon_0) = \delta_{u_i,v_i}\,, \quad \varphi_i(\varepsilon_1) = \delta_{a+u_i,v_i}\,, \quad \varphi_i(\delta_{0,c}) = \delta_{u_i,T(v_i,b)} \quad \textit{for } i = 2, 3.$$

Then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of (3) if and only if there is a φ, given by (7), with $\varphi(\varepsilon_1) = \varepsilon_a$ and $\varphi(\delta_{0,c}) = \delta_{0,b}$ such that

$$\varphi_i(F)(x) = \begin{cases} 0, & \textit{if } x = 0, \\ g^{-1}\left(\frac{g(b)}{g(c)} \cdot g\left(F\left(\frac{x - v_i}{a}\right)\right) + g(u_i)\right), & \textit{if } 0 < x < \infty, \\ 1, & \textit{if } x = \infty; \end{cases}$$

for $i = 2, 3$ and $\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0))$. Here we use for ease of notation that $F(x) = 0$, for $x < 0$.

Thus we see that these solutions are not induced on Δ^+.

In a similar manner, we can obtain order automorphism solutions of the second type (see [6]), where $\varphi(\varepsilon_1) = \delta_{0,b}$ and $\varphi(\delta_{0,c}) = \varepsilon_a$. In this case we have that the left and right compositions of the order automorphism can only differ by a multiplicative constant since they need to generate the same T-norm.

Thus solutions of this type are given, for $i = 2, 3$, by

$$(\varphi_i(F))(x) = \begin{cases} 0, & \text{if } x = 0, \\ (g^{-1}F^{\vee}g^{-1})(k \cdot x - b_i), & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = \infty; \end{cases}$$

and $\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0))$; with the same notation as above and $F^{\vee}$ is the right-continuous *quasi-inverse* of F which is given by

$$F^{\vee}(y) = \begin{cases} 0, & \text{for } y = 0, \\ \inf\{x \mid F(x) > y\}, & \text{for } 0 < y < 1, \\ \infty, & \text{for } y = 1. \end{cases} \tag{8}$$

In conclusion we note that the case of non-strict T-norms requires another approach; solutions of the type of those just presented are still possible, but new methods will be needed to describe all sup-continuous solutions in that case. We further note that the restriction to triangle functions of the form τ_T is not as restrictive as it may seem, since, as was noted in [9], triangle functions that are sup-continuous and map $\Delta_\delta^+ \times \Delta_\delta^+$ into Δ_δ^+ are essentially of this form.

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On the stability of derivations of higher order

Herrn Professor Zenon Moszner mit den besten Wünschen für die Zukunft zum 70. Geburtstag gewidmet

Abstract. Derivations of order n as defined by L. Reich are *additive* and *nonlinear* functions $f:\mathbb{R}\to\mathbb{R}$ with $f(1)=0$ which satisfy the functional equation $\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}f=0$ for all $a_1,a_2,\ldots,a_{n+1}\in\mathbb{R}$, where $\delta_a f(x):=f(ax)-af(x)$. Here we prove several stability results concerning this (and similar) functional equations.

1. In [K], Chap. XIV a *derivation* $f:\mathbb{R}\to\mathbb{R}$ is defined to be an *additive* mapping which additionally satisfies the Leibniz rule

$$f(xy)=xf(y)+yf(x),\quad x,y\in\mathbb{R}.$$

In [R92] the operators δ_a are introduced. For functions $f:\mathbb{R}\to\mathbb{R}$ and reals a we have

$$\delta_a(f)(x):=f(ax)-af(x).$$

In [R98] it was shown that

an additive function $f:\mathbb{R}\to\mathbb{R}$ is a derivation if and only if and

$$f(1)=0\quad \textit{and}\quad (\delta_a\circ\delta_b)f=0,\quad a,b\in\mathbb{R}.$$

Moreover in the same paper (Satz 2) and in [UR] this leads to the following generalization.

Definition 1

A function $f:\mathbb{R}\to\mathbb{R}$ is called a derivation of order n ($\in\mathbb{N}_0$) if and only if f is additive with $f(1)=0$ and if f satisfies the equation

$$\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}f(x)=0,\quad a_1,a_2,\ldots,a_{n+1},x\in\mathbb{R}. \tag{1}$$

Actually the original definition was different. But for our purpose (stability investigations) it is convenient to use the definition above.

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Before we proceed to this topic we prove the following theorem which was mentioned in [R98] (Satz 3) and proved there for $n = 2$.

THEOREM 1
For $a \in \mathbb{R}$ and $n \in \mathbb{N}$ let $\delta_a^{n+1} := \underbrace{\delta_a \circ \delta_a \circ \ldots \circ \delta_a}_{n+1 \text{ times}}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a derivation of order n if and only if f is additive with $f(1) = 0$ and if

$$\delta_a^{n+1} f(x) = 0, \quad a, x \in \mathbb{R}. \tag{2}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be additive. Obviously it is enough to show that (2) implies (1). Moreover δ_a maps additive functions to additive functions. Thus δ_a is an endomorphism of the vector space of all additive functions defined on $\mathbb{R}$ with real values. Since $\delta_a \pm \delta_b = \delta_{a \pm b}$ and $\delta_a \circ \delta_b = \delta_b \circ \delta_a$ the ring generated by the operators δ_a, $a \in \mathbb{R}$, is commutative. It is well-known (see for example [S]) that in any commutative ring we have

$$m! 2^m x_1 \cdot x_2 \cdot \ldots \cdot x_m = \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m \left(\sum_{j=1}^{m} \varepsilon_j x_j \right)^m .$$

Using this for $m = n + 1$, $x_j = \delta_{a_j}$ (and $\cdot = \circ$) we see that the operator $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}}$ is a linear combination of certain operators δ_b^{n+1}. But this gives the desired result.

REMARK 1
Since $\delta_a(\mathrm{id}_{\mathbb{R}}) = 0$ it can easily be seen that the general additive solution of (1) or (2) is given by sums of a derivation of order n and of a linear function. In the following we will call additive solutions of (1) or (2) *generalized* derivations of order n (and for convenience omit the term "generalized").

For future use we formulate the explicit form of $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x)$.

LEMMA 1
For all $a_1, a_2, \ldots, a_{n+1}, x \in \mathbb{R}$ and all $f : \mathbb{R} \to \mathbb{R}$ we have

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x) = \sum_{J \subseteq \{1, 2, \ldots, n+1\}} (-1)^{n+1-\#J} \prod_{j \notin J} a_j \cdot f\left(\prod_{j \in J} a_j \cdot x \right) \tag{3}$$

Proof. For real a the operator δ_a may be written as $\delta_a = M_a - \mu_a$, where $M_a f(x) := f(ax)$ and $\mu_a f(x) := a f(x)$. Then $M_a \circ M_b = M_{ab}$, $\mu_a \circ \mu_b = \mu_{ab}$ and $(M_a \circ \mu_b) f(x) = b f(ax) = (\mu_b \circ M_a) f(x)$. Thus all the operators M_a, M_b, μ_c, μ_d commute in pairs and we get

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} = \bigcirc_{j=1}^{n+1} (M_{a_j} - \mu_{a_j})$$
$$= \sum_{J \subseteq \{1,2,\ldots,n+1\}} (-1)^{n+1-\#J} \bigcirc_{j \notin J} \mu_{a_j} \circ \bigcirc_{j \in J} M_{a_j},$$

from which the assertion follows.

We also mention a suitably adapted version of a stability theorem (see [K], Chap. XVII and the references given there in).

THEOREM 2
Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x+y) - f(x) - f(y)| \leqslant \varepsilon$ for all $x, y \in \mathbb{R}$. Then there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $|f - g|$ is bounded (by ε.) Moreover g is given by $g(x) = \lim_{n\to\infty} \frac{f(nx)}{n}$

Surprisingly the same problem for exponential functions has a completely different answer (see e.g. [BLZ]).

THEOREM 3
Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x+y) - f(x)f(y)| \leqslant \varepsilon$ for all $x, y \in \mathbb{R}$. Then either f is bounded or an unbounded exponential function, i.e., an unbounded function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

The phenomenon described here is called "superstability" by some authors.

2. The stability results

The possibility of investigating the stability of derivations of order n depends on the choice of equations to be replaced by suitable inequalities. One result is the following.

THEOREM 4
Let $\varepsilon > 0$, let $b : \mathbb{R}^{n+1} \to \mathbb{R}$ be an arbitrary function, and let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon, \quad x, y \in \mathbb{R} \tag{4}$$

and

$$\left|\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x)\right| \leqslant b(a_1, a_2, \ldots, a_{n+1}), \quad x, a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}. \tag{5}$$

Then we have:

i) *There is one and only one derivation d of order n such that $f - d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r : \mathbb{R} \to \mathbb{R}$ the function $f := d + r$ satisfies* (4) *and* (5) *for some suitable number ε and some function $b : \mathbb{R}^{n+1} \to \mathbb{R}$.*

iii) *If b is independent of at least one of its variables and if f satisfies* (4) *and* (5), *then f is already a derivation of order n itself.*

Proof. Using (4) and Theorem 2 we get a unique additive function d such that $f - d$ is bounded. Moreover we know that $d(x) = \lim_{m\to\infty} \frac{f(mx)}{m}$. For fixed m we put $f_m(x) := \frac{f(mx)}{m}$. Then (5) together with the linearity of the operators δ_a gives

$$\left|\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f_m(x)\right| \leqslant \frac{b(a_1, a_2, \ldots, a_{n+1})}{m}$$

for all $x, a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}$ and all $m \in \mathbb{N}$.

But for $m \to \infty$ we get that $f_m(x) \to d(x)$ and by (1)

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f_m(x) \to \delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} d(x).$$

Since $\frac{1}{m} b(a_1, a_2, \ldots, a_{n+1}) \to 0$ this means that

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} d(x) = 0$$

for all $x, a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}$, thus proving the first part of the theorem.

Let r and d be as required in the second part of the theorem, and let R be an upper bound for $|r(x)|$, $x \in \mathbb{R}$. Then

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (d + r)(x) = \delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (r)(x)$$

and by (1)

$$\left|\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (r)(x)\right| \leqslant b(a_1, a_2, \ldots, a_{n+1}) := \sum_{J \subseteq \{1,2,\ldots,n+1\}} \left|\prod_{j \notin J} a_j\right| R.$$

Moreover

$$\begin{aligned}|(d + r)(x + y) - (d + r)(x) - (d + r)(y)| &= |r(x + y) - r(x) - r(y)| \\ &\leqslant \varepsilon := 3R.\end{aligned}$$

To prove the third part we may observe that by the first part there is a unique derivation d such that $F := f - d$ is bounded. We have to show that $F = 0$. Let us assume that the function b does not depend on, say, the last variable a_{n+1}. Since $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (F) = \delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (f)$ we get

$$\left|\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} (F)(x)\right| \leqslant b(a_1, a_2, \ldots, a_n, a_{n+1}) =: B(a_1, a_2, \ldots, a_n)$$

Assuming that $a_1, a_2, \ldots, a_{n+1}$ are different from zero and using (1) we get

$$\left| \sum_{J\subseteq\{1,2,\dots,n+1\}} (-1)^{n+1-\#J} \frac{F\left(\prod_{j\in J} a_j \cdot x\right)}{\prod_{j\in J} a_j} \right| \leqslant \frac{B(a_1,a_2,\dots,a_n)}{|a_1a_2\cdots a_{n+1}|}.$$

If we fix $a_1,a_2,\dots,a_n$ and if we let a_{n+1} tend to infinity we get

$$\sum_{J\subseteq\{1,2,\dots,n\}} (-1)^{n+1-\#J} \frac{F\left(\prod_{j\in J} a_j \cdot x\right)}{\prod_{j\in J} a_j} = 0$$

since $\frac{B(a_1,a_2,\dots,a_n)}{|a_1a_2\cdots a_{n+1}|} \to 0$ for $a_{n+1} \to \infty$ and since also $\frac{F(\prod_{j\in J} a_j\cdot x)}{\prod_{j\in J} a_j} \to 0$ if the subset $J \subseteq \{1,2,\dots,n+1\}$ is such that $n+1 \in J$.

But the sum above contains the term $\pm F(x)$ (for $J=\emptyset$) and all the other terms tend to zero when all the a_j tend to infinity, which means that $F(x)=0$ (for arbitrary x).

Concerning the characterization of derivations as given by Theorem 1 we have the following result.

THEOREM 5

Let $\varepsilon > 0$, let $b : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let $f : \mathbb{R} \to \mathbb{R}$ be such that (4) *is satisfied and such that*

$$\left|\delta_a^{n+1} f(x)\right| \leqslant b(a), \quad x,a \in \mathbb{R} \tag{6}$$

holds. Then we have:

i) *There is one and only one derivation d of order n such that $f-d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r : \mathbb{R} \to \mathbb{R}$ the function $f := d + r$ satisfies* (4) *and* (6) *for some suitable number ε and some function $b : \mathbb{R} \to \mathbb{R}$.*

iii) *If b is constant and if f satisfies* (4) *and* (6), *then f is already a derivation of order n itself.*

Proof. The first part and also the second one can be proved as the corresponding parts of the theorem above. Especially the desired derivation of order n is given by $d(x) := \lim_{m\to\infty} \frac{f(mx)}{m}$. As for the third part we put $F := f - d$ and observe that F is bounded and satisfies $\delta_a^{n+1} F = \delta_a^{n+1} f$. Moreover

$$\delta_a^{n+1}F(x) = \sum_{J\subseteq\{1,2,\ldots,n+1\}} (-1)^{n+1-\#J} a^{n+1-\#J} F\left(a^{\#J}x\right)$$
$$= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} a^{n+1-j} F\left(a^j x\right).$$

Thus for $a \neq 0$ (and putting $B := b(a)$)

$$\left|\frac{\delta_a^{n+1}F(x)}{a^{n+1}}\right| = \left|(-1)^{n+1}F(x) + \sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\frac{F\left(a^j x\right)}{a^j}\right| \leqslant \frac{B}{|a|^{n+1}}.$$

This for $a \to \infty$ implies $F(x) = 0$, as desired.

REMARK 2

The last parts of both theorems are remarkable since they show the phenomenon of “strong” superstability: *Every solution of the inequality is also a solution of the equation!*

3. Characterization of derivations of order n by a single equation (and its stability)

Actually the definition of a derivation of order n contains two requirements (additivity and the condition connected with the operators δ_a). For $n = 0$ formally this is nothing but the definition of *linearity* by the two requirements $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$ which are equivalent to the single condition $f(a(x+y)) = af(x) + af(y)$. Generalizing this we have the following theorem.

THEOREM 6

For $n \in \mathbb{N}_0$ *and* $f : \mathbb{R} \to \mathbb{R}$ *the conditions* (a) *and* (b) *below are equivalent.*

(a) f *is a derivation of order* n.

(b) $f\left(a^{n+1}(x+y)\right) + \sum_{j=0}^{n}(-1)^{n+1-j}\binom{n+1}{j}a^{n+1-j}\left(f(a^j x) + f(a^j y)\right) = 0$ *for all* $x, y, a \in \mathbb{R}$.

Proof. Obviously the condition given in (b) is nothing but

$$f\left(a^{n+1}(x+y)\right) + \delta_a^{n+1}f(x) + \delta_a^{n+1}f(y) - f(a^{n+1}x) - f(a^{n+1}y) = 0. \qquad (7)$$

If f is a derivation of order n the terms $\delta_a^{n+1}f(x)$ and $\delta_a^{n+1}f(y)$ vanish. Moreover by the additivity of f the three remaining terms on the left-hand side of (7) also disappear.

Conversely, if (7) is satisfied, we may put $x = y = 0$ in this relation to get

$$f(0) + 2(1-a)^{n+1}f(0) - 2f(0) = 0 \quad \text{or} \quad \left(2(1-a)^{n+1} - 1\right)f(0) = 0.$$

Using this with (for example) $a = -1$ we get $(2^{n+2} - 1)f(0) = 0$, i.e. $f(0) = 0$. This gives $\delta_a^{n+1} f(0) = 0$. Applying the equation for $y = 0$ gives

$$f(a^{n+1}x) + \delta_a^{n+1} f(x) + 0 - f(a^{n+1}x) - 0 = 0 \quad \text{or} \quad \delta_a^{n+1} f(x) = 0.$$

Using this and (7) for $a = 1$ once more also gives the additivity of f.

The corresponding stability result is contained in the following theorem.

THEOREM 7

Let $b : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and let $f : \mathbb{R} \to \mathbb{R}$ be given such that

$$\left| f\left(a^{n+1}(x+y)\right) + \delta_a^{n+1} f(x) + \delta_a^{n+1} f(y) - f(a^{n+1}x) - f(a^{n+1}y) \right| \leqslant b(a) \quad (8)$$

for all $x, y, a \in \mathbb{R}$. Then we have:

i) *There is one and only one derivation d of order n such that $f - d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r : \mathbb{R} \to \mathbb{R}$ the function $f := d + r$ satisfies (8) for some suitable function $b : \mathbb{R} \to \mathbb{R}$.*

iii) *If b is constant and if f satisfies (8), then f is already a derivation of order n itself.*

Proof. Since $\delta_1^{n+1} f(x) = (1-1)^{n+1} f(x) = 0$, equation (8) for $a = 1$ implies

$$|f(x+y) - f(x) - f(y)| \leqslant b(1) =: \varepsilon.$$

Putting $y = 0$ in (8) leads to

$$\left| f(a^{n+1}x) + \delta_a^{n+1} f(x) + \delta_a^{n+1} f(0) - f(a^{n+1}x) - f(0) \right| \leqslant b(a). \quad (9)$$

Moreover

$$\delta_a^{n+1} f(0) - f(0) = \left((1-a)^{n+1} - 1\right) f(0) =: c(a).$$

Thus with $b'(a) := b(a) + |c(a)|$ we get

$$\left|\delta_a^{n+1} f(x)\right| \leqslant b'(a), \quad x, a \in \mathbb{R}.$$

Accordingly we may apply Theorem 5 to get the first part of the theorem.

The second part may be proved in a similar way as in previous cases. (If R is an upper bound for $|r|$ we may take $b(a) = 3R + 2(1 + |a|)^{n+1}$.)

With $F = f - d$ we have (again as in previous cases) that F is bounded and that

$$|F(x+y) - F(x) - F(y)| \leqslant b(1) = \varepsilon.$$

Thus (8) with $x = y$ implies

$$\left|2\delta_a^{n+1} f(x)\right| \leqslant 2\varepsilon, \quad x, a \in \mathbb{R}.$$

Now we again apply Theorem 5.

REMARK 3

It is possible to formulate (and prove) similar results with δ_a^{n+1} replaced by the operator $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}}$. But we will not do this here.

The results of this paper have been presented earlier for example at a joint Graz-Maribor seminar in 1996 and at the 36-th ISFE in Brno in 1998.

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A Wiener Tauberian Theorem on discrete abelian torsion groups

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. One version of the classical Wiener Tauberian Theorem states that if G is a locally compact abelian group then any nonzero closed translation invariant subspace of $L^\infty(G)$ contains a character. In other words, spectral analysis holds for $L^\infty(G)$. In this paper we prove a similar theorem: if G is a discrete abelian torsion group then spectral analysis holds for $C(G)$, the space of all complex valued functions on G.

1. Introduction

A possible formulation of one version of the classical Wiener Tauberian Theorem on locally compact abelian groups is the following: if G is a locally compact abelian group then any nonzero closed translation invariant subspace of $L^\infty(G)$ contains a character. Similar problem can be formulated concerning $C(G)$ instead of $L^\infty(G)$ but in that case "characters" should be replaced by "generalized characters": does any nonzero closed translation invariant subspace of $C(G)$ contain a generalized character? The answer is positive in some special cases but the general problem is far from being solved. The problem is still open even in the case where G is discrete. In this paper we give a positive answer to the above question if G is a discrete abelian torsion group.

2. Spectral: analysis and synthesis on discrete abelian groups

In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is an abelian group then $C(G)$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the topology of pointwise convergence. The dual of $C(G)$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G.

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Homomorphisms of G into the additive group of complex numbers, resp. into the multiplicative group of nonzero complex numbers are called *additive*, resp. *exponential functions.* Bounded exponential functions are exactly the characters of G, hence exponential functions are sometimes called generalized characters. Products of additive functions are called *monomials*, products of monomials and exponential functions are called *exponential monomials.*

The basic question of spectral analysis on $C(G)$ can be formulated as follows: does any nonzero closed translation invariant subspace of $C(G)$ contain an exponential function? If so, then we say that *spectral analysis holds for* $C(G)$. For instance, if G is finite then by the Wiener Tauberian Theorem the answer is "yes". Another basic problem concerns spectral synthesis on $C(G)$: given a nonzero closed translation invariant subspace of $C(G)$, do the linear combinations of exponential monomials in this subspace form a dense subset? If so, then we say that *spectral synthesis holds for* $C(G)$. This is the case, for instance, if G is finitely generated, due to the following theorem.

THEOREM 1 (M. Lefranc [2])

If G is a finitely generated discrete abelian group then spectral synthesis holds for $C(G)$.

3. Exponential monomials on abelian torsion groups

Let G be an abelian group. We say that G is a *torsion group* if every element of G has finite order. In other words, for every x in G there exists a positive integer n with $nx = 0$. Hence G is not a torsion group if and only if there exists an element of G which generates a subgroup isomorphic to $\mathbb{Z}$, the additive group of integers.

In what follows we need the following lemma (see [1]).

LEMMA 2

Let G be an abelian group, $H \subseteq G$ a subgroup and let D be a divisible abelian group. If $\varphi : H \to D$ is a homomorphism, then there exists a homomorphism $\Phi : G \to D$ which extends φ, that is, $\Phi(x) = \varphi(x)$ for all x in H.

THEOREM 3

Let G be an abelian group. Then G is a torsion group if and only if every nonzero exponential monomial on G is a character.

Proof. Suppose that G is a torsion group and let $a : G \to C$ be an additive function, and $m : G \to \mathbb{C}$ an exponential function. For every x in G there exists a positive integer n with $nx = 0$ and hence $0 = a(nx) = na(x)$, which implies $a(x) = 0$. This means that every additive function on G is zero. Further $1 = m(nx) = m(x)^n$, which implies $|m(x)| = 1$. This means that every exponential function on G is a character. Now we conclude that if G is a torsion group then every nonzero exponential monomial on G is a character.

Assume now that G is not a torsion group, that is, there exists an x_0 in G such that the cyclic group generated by x_0 is isomorphic to $\mathbb{Z}$. Let $\alpha \neq 0$ be a complex number and we define $\varphi(nx_0) = n\alpha$ for any integer n. Then φ is a homomorphism of the subgroup generated by x_0 into the additive group of complex numbers. As this latter group is divisible, by Lemma 2. this homomorphism can be extended to a homomorphism $a : G \to \mathbb{C}$ of G into the additive group of complex numbers. By $a(x_0) = \varphi(x_0) = \alpha \neq 0$ we have that a is a nonzero additive function, that is, a nonzero exponential monomial on G, which is obviously not a character. The theorem is proved.

4. A Wiener Tauberian Theorem on abelian torsion groups

In this paragraph we show that if G is a discrete abelian torsion group, then any nonzero closed translation invariant subspace of $C(G)$ contains a character. The proof heavily depends on Theorem 1.

THEOREM 4

Let G be an abelian torsion group. Then any nonzero closed translation invariant subspace of $C(G)$ contains a character.

Proof. Let $V \subseteq C(G)$ be any nonzero closed translation invariant subspace. Then by the Hahn-Banach theorem V is equal to the annihilator of its annihilator, that is, there exists a set $\Lambda \subseteq \mathcal{M}_c(G)$ of finitely supported complex measures on G such that V is exactly the set of all functions in $C(G)$ which are annihilated by all members of Λ:

$$V = V(\Lambda) = \{f \mid f \in C(G),\ \lambda(f) = 0 \quad \text{for all } \lambda \in \Lambda\}.$$

We show that for any finite $\Gamma \subseteq \Lambda$, its annihilator, $V(\Gamma)$, contains a character. Indeed, let F_Γ denote the subgroup generated by the supports of the measures belonging to Γ. Then F_Γ is a finitely generated torsion group. The measures belonging to Γ can be considered as measures on F_Γ and the annihilator of Γ in $C(F_\Gamma)$ will be denoted by $V(\Gamma)_{F_\Gamma}$. This is a closed translation invariant subspace of $C(F_\Gamma)$. It is also nonzero. Indeed, if f belongs to V then its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$. If, in addition, we have $f(x_0) \neq 0$ and y_0 is in F_Γ, then the translate of f by $x_0 - y_0$ belongs to V, its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$ and at y_0 it takes the value $f(x_0) \neq 0$. Hence $V(\Gamma)_{F_\Gamma}$ is a nonzero closed translation invariant subspace of $C(F_\Gamma)$. As F_Γ is finitely generated, by Theorem 1. spectral synthesis holds for $C(F_\Gamma)$, and, in particular $V(\Gamma)_{F_\Gamma}$ contains nonzero exponential monomials. As F_Γ is a torsion group, any nonzero exponential monomial on F_Γ is a character. That means, $V(\Gamma)_{F_\Gamma}$

contains a character of F_Γ. By Lemma 2. any character of F_Γ can be extended to a character of G, and obviously any such extension belongs to $V(\Gamma)$.

We have proved that for any finite $\Gamma \subseteq \Lambda$ the annihilator $V(\Gamma)$ contains a character. Let $char(V)$ denote the set of all characters contained in V. Obviously $char(V)$ is a compact subset of $\hat{G}$, the dual of G, because $char(V)$ is closed and $\hat{G}$ is compact. On the other hand, the system of sets $char(V(\Gamma))$, where $\Gamma \subseteq \Lambda$ is finite, is a centered system of nonempty compact sets:

$$char(V(\Gamma_1 \cup \Gamma_2)) \subseteq char(V(\Gamma_1)) \cap char(V(\Gamma_2)).$$

We infer that the intersection of this system is nonempty, and obviously

$$\emptyset \neq \bigcap_{\Gamma \subseteq \Lambda \text{ finite}} char(V(\Gamma)) \subseteq char(V).$$

That means, $char(V)$ is nonempty, and the theorem is proved.

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On a problem of H.-H. Kairies concerning Euler's Gamma function

Abstract. The Bohr-Mollerup theorem on the Euler Γ function states: If $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the functional equation $f(x+1) = x\,f(x)$ on $\mathbb{R}_+$, $\log \circ f$ is convex on $(\gamma, +\infty)$ for some $\gamma \geqslant 0$ and $f(1) = 1$ then $f = \Gamma$. We give some partial answers to the question posed by H.-H. Kairies: By what other function can the logarithm be replaced in this statement.

Introduction

Let us introduce the family of functions

$$\mathbf{F} := \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid \forall x \in \mathbb{R}_+ : \ f(x+1) = xf(x) \text{ and } f(1) = 1\}.$$

Then for every $f \in \mathbf{F}$ and $n \in \mathbb{N}$ we have $f(n) = \Gamma(n) = (n-1)!$ where Γ is the Euler function defined by the formula

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x), \tag{Γ}$$

where

$$\Gamma_n(x) = \frac{n^x n!}{x(x+1)\dots(x+n)}. \tag{Γ_n}$$

Moreover, $f \in \mathbf{F}$ iff $f(x) = p(x)\Gamma(x)$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is a periodic function of period 1 with $p(1) = 1$.

We begin our considerations with reminding the Bohr-Mollerup Theorem, cf. [1] p. 14.

Theorem (Bohr-Mollerup)

If $f \in \mathbf{F}$ and $\log \circ f$ is convex on $(\gamma, +\infty)$ for some $\gamma \geqslant 0$, then $f = \Gamma$.

H.-H. Kairies proposed (private communication) to investigate the properties of the following set:

$$\mathbf{M} := \{g : \mathbb{R}_+ \to \mathbb{R} \mid (f \in \mathbf{F} \text{ and } g \circ f \text{ is convex on } (\gamma, +\infty) \text{ for some } \gamma \geqslant 0) \Rightarrow f = \Gamma\}.$$

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In this paper we find some elements of the set $\mathbf{M}$ and study its properties.

1. Properties of the set $\mathbf{M}$

We start with the two lemmas which follow directly from the definitions of monotonicity and concavity of the functions involved.

LEMMA 1

If $g : \mathbb{R} \to \mathbb{R}$ is increasing and concave then $g^{-1} : \mathbb{R} \to \mathbb{R}$ is increasing and convex.

LEMMA 2

Let X, Y, Z be some intervals of $\mathbb{R}$. If $f : X \to Y$ is a convex function and $g : Y \to Z$ is increasing and convex then $g \circ f : X \to Z$ is convex.

THEOREM 1

Let $g \in \mathbf{M}$ and let $h : \mathbb{R} \to \mathbb{R}$ be increasing and concave. Then $h \circ g : \mathbb{R}_+ \to \mathbb{R}$ belongs to the set $\mathbf{M}$.

Proof. Let g and h satisfy the assumptions and let $f \in \mathbf{F}$. By Lemma 1 the function h^{-1} is increasing and convex. If $h \circ g \circ f$ is convex then (by Lemma 2) $h^{-1} \circ h \circ g \circ f$ is convex, too. Thus, since $g \in \mathbf{M}$, we have $h \circ g \in \mathbf{M}$.

The above theorem implies

REMARK 1

Let $a > 0$. If $g : \mathbb{R}_+ \to (a, +\infty)$ is increasing and logarithmically convex then $g^{-1} \in \mathbf{M}$.

Proof. By our assumption $\log \circ g$ is a convex function. By Lemma 1 and Lemma 2 the function $g^{-1} \circ \exp = (\log \circ g)^{-1}$ is increasing and concave for $x \in (\log a, +\infty)$. By the Bohr-Mollerup Theorem, $\log \in \mathbf{M}$. Thus, by Theorem 1, $g^{-1} = (\log \circ g)^{-1} \circ \log \in \mathbf{M}$.

In particular, we have

REMARK 2

The function $G = \left(\Gamma\,|_{(2,+\infty)}\right)^{-1}$ is in $\mathbf{M}$.

THEOREM 2

If $g \in \mathbf{M}$, $a > 0$, $b \in \mathbb{R}$, then $a \cdot g + b \in \mathbf{M}$.

Proof. Let $g \in \mathbf{M}$, $a > 0$ and $b \in \mathbb{R}$. Take a function $f \in \mathbf{F}$. If the function $(a \cdot g + b) \circ f$ is convex then so is the function $\frac{1}{a} \cdot [(a \cdot g + b) \circ f] - \frac{b}{a} = g \circ f$. Since we have assumed that $g \in \mathbf{M}$, $f = \Gamma$ and $a \cdot g + b \in \mathbf{M}$.

Besides $f = \Gamma$ there are other convex functions belonging to $\mathbf{F}$, e.g., the functions $f_c : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f_c(x) = \Gamma(x) \exp\left[c \sin(2\pi x)\right].$$

They are convex for sufficiently small $c > 0$ on $(0, +\infty)$ (see [2]). Thus we obtain the following remarks:

Remark 3
The function $\mathrm{id}_{\mathbb{R}_+}$ does not belong to $\mathbf{M}$.

Remark 4
If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and convex function then $h \notin \mathbf{M}$.

Proof. Let h be a function satisfying the assumptions and let $f \in \mathbf{F}$. By Lemma 2 if f is convex then so is $h \circ f$. Because Γ is not the only convex element of $\mathbf{F}$, we have $h \notin \mathbf{M}$.

2. Special elements of the set $\mathbf{M}$

Theorem 3
Let us assume that $h : \mathbb{R}_+ \to \mathbb{R}$ *and*

$$\lim_{x \to +\infty} h(x) = m \in \mathbb{R}. \tag{1}$$

Then $\log + h$ *is an element of the set* $\mathbf{M}$.

Proof. We put $g = \log + h$, and take a function $f \in \mathbf{F}$. Moreover we fix an $n \in \mathbb{N}$ and $x \in (0, 1]$. If $g \circ f$ is convex then the following inequalities hold true:

$$\begin{aligned} g \circ f(n) - g \circ f(n-1) &\leqslant \frac{g \circ f(n+x) - g \circ f(n)}{x} \\ &\leqslant g \circ f(n+1) - g \circ f(n). \end{aligned} \tag{2}$$

Using $f(x+1) = x f(x)$, we have

$$\begin{aligned} x\left(g_{n-1} - g_{n-2}\right) &\leqslant g\left[x(x+1)\dots(x+n-1)f(x)\right] - g_{n-1} \\ &\leqslant x\left(g_n - g_{n-1}\right) \end{aligned} \tag{3}$$

where we have put, for short,

$$\begin{aligned} g_n &:= g(n!), \\ h_n &:= h(n!). \end{aligned}$$

Since $g = \log + h$, we get

$$\begin{aligned} x\left[\log(n-1)+h_{n-1}-h_{n-2}\right] \leqslant{} & \log\left[x(x+1)\ldots(x+n-1)f(x)\right] \\ & + h\left[x(x+1)\ldots(x+n-1)f(x)\right] \\ & - \log(n-1)! - h_{n-1} \\ \leqslant{} & x\left[\log n + h_n - h_{n-1}\right]. \end{aligned}$$

Since the exponential function is increasing, we obtain

$$\begin{aligned} &(n-1)^x \left[\frac{\exp h_{n-1}}{\exp h_{n-2}}\right]^x \\ &\quad \leqslant \frac{x(x+1)\ldots(x+n-1)f(x)}{(n-1)!} \cdot \frac{\exp\circ h\left[x(x+1)\ldots(x+n-1)f(x)\right]}{\exp h_{n-1}} \\ &\quad \leqslant n^x \left[\frac{\exp h_n}{\exp h_{n-1}}\right]^x . \end{aligned}$$

Hence

$$\begin{aligned} &(n-1)^x (n-1)!\, \frac{(\exp h_{n-1})^{x+1}}{(\exp h_{n-2})^x} \\ &\quad \leqslant x(x+1)\ldots(x+n-1)f(x) \cdot \exp\circ h\left[x(x+1)\ldots(x+n-1)f(x)\right] \\ &\quad \leqslant n^x (n-1)!\, \frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}}. \end{aligned}$$

In turn,

$$\begin{aligned} &n^x n!\, \frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} \\ &\quad \leqslant x(x+1)\ldots(x+n)f(x) \cdot \exp\circ h\left[x(x+1)\ldots(x+n)f(x)\right] \\ &\quad \leqslant n^x n!\, \frac{x+n}{n} \cdot \frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}} \cdot \frac{\exp\circ h\left[x(x+1)\ldots(x+n)f(x)\right]}{\exp\circ h\left[x(x+1)\ldots(x+n-1)f(x)\right]} \end{aligned}$$

and

$$\begin{aligned} &\Gamma_n(x) \cdot \frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} \\ &\quad \leqslant f(x) \exp\circ h\left[x(x+1)\ldots(x+n)f(x)\right] \\ &\quad \leqslant \frac{x+n}{n} \cdot \Gamma_n(x) \cdot \frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}} \cdot \frac{\exp\circ h\left[x(x+1)\ldots(x+n)f(x)\right]}{\exp\circ h\left[x(x+1)\ldots(x+n-1)f(x)\right]}. \end{aligned}$$

Notice that by the relations resulting from (1) ($\lim_{n\to\infty} h_n = m$)

$$\lim_{n\to\infty} \exp\circ h\left[x(x+1)\ldots(x+n)f(x)\right] = e^m$$

$$\lim_{n\to\infty} \frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} = e^m$$

$$\lim_{n\to\infty} \frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}} \cdot \frac{\exp \circ h\,[x(x+1)\dots(x+n)f(x)]}{\exp \circ h\,[x(x+1)\dots(x+n-1)f(x)]} = e^m$$

and by (Γ) we obtain

$$\Gamma(x)e^m \leqslant f(x)e^m \leqslant \Gamma(x)e^m.$$

Thus $f(x) = \Gamma(x)$ for $x \in (0,1]$. We shall show that $f(x) = \Gamma(x)$ for each real positive x.

Let $x \in \mathbb{R}_+$. We proceed by induction. There exists a $k \in \mathbb{N}$ such that $x \in (k-1,k]$. If $k = 1$ then we have already proved that $f(x) = \Gamma(x)$. Let us assume that $f(x) = \Gamma(x)$ for $x \in (k-1,k]$. Take $x \in (k,k+1]$ and $y = x-1$. Since $y \in (k-1,k]$, by the inductive assumption we have $f(y) = \Gamma(y)$. By the functional equation for f we have $f(y+1) = yf(y)$. Thus $f(y+1) = y\Gamma(y) = \Gamma(y+1)$ hence $f(x) = \Gamma(x)$ for $x \in (k,k+1]$. Therefore $f(x) = \Gamma(x)$ for $x \in \mathbb{R}_+$.

REMARK 5

The function $g = \log + \arctan$ belongs to $\mathbf{M}$.

REMARK 6

Let $a,b > 0$. Then $\log \circ \left(a\,\mathrm{id}_{\mathbb{R}_+} + b\right) \in \mathbf{M}$.

Proof. Take $h = \log \circ \left(a + \frac{b}{\mathrm{id}_{\mathbb{R}_+}}\right)$, so that $\log \circ \left(a\,\mathrm{id}_{\mathbb{R}_+} + b\right) = \log + h$ and $\lim_{x\to+\infty} h(x) = a$. Thus, by Theorem 3, $\log \circ \left(a\,\mathrm{id}_{\mathbb{R}_+} + b\right) \in \mathbf{M}$.

THEOREM 4

Let $m, a > 0$ and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $h(x) = m - \frac{a}{x} + R(x)$, where $R(x) = o\left(\frac{1}{x}\right)$, $x \to +\infty$. Then $h \cdot \log \in \mathbf{M}$.

Proof. We put $g = h \cdot \log$, and we take a function $f \in \mathbf{F}$. Moreover we fix an $n \in \mathbb{N}$ and $x \in (0,1)$.

If $g \circ f$ is convex then inequalities (2) and (3) (as in the proof of Theorem 3) hold true. Since $g = h \cdot \log$, we get

$$\begin{aligned} x\ &\{h_{n-1} \log[(n-1)!] - h_{n-2} \log[(n-2)!]\} \\ &\leqslant h \circ f(n+x) \log[x(x+1)\dots(x+n-1)] \\ &\quad - h_{n-1} \log[(n-1)!] \\ &\leqslant x\ \{h_n \log(n!) - h_{n-1} \log[(n-1)!]\}. \end{aligned} \tag{4}$$

By properties of the logarithmic function we have

$$\begin{aligned} \log\left[\frac{(n-1)!^{h_{n-1}}}{(n-2)!^{h_{n-2}}}\right]^x &\leqslant \log \frac{[x(x+1)\dots(x+n-1)f(x)]^{h\circ f(x+n)}}{(n-1)!^{h_{n-1}}} \\ &\leqslant \log\left[\frac{n!^{h_n}}{(n-1)!^{h_{n-1}}}\right]^x. \end{aligned}$$

Thus

$$\frac{\left[(n-1)!^{h_{n-1}}\right]^{x+1}}{\left[(n-2)!^{h_{n-2}}\right]^{x}} \leqslant [x(x+1)\dots(x+n-1)f(x)]^{h\circ f(x+n)} \leqslant \frac{\left[n!^{h_n}\right]^{x}}{\left[(n-1)!^{h_{n-1}}\right]^{x-1}}$$

and next

$$\left[\frac{(n-1)!^{(x+1)h_{n-1}}}{(n-2)!^{xh_{n-2}}}\right]^{\frac{1}{h\circ f(x+n)}} \leqslant x(x+1)\dots(x+n-1)f(x) \leqslant \left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}}.$$

It follows easily that

$$\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} \leqslant x(x+1)\dots(x+n)f(x) \leqslant (x+n)\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}}.$$

So we have

$$\frac{\Gamma_n(x)}{n!n^x}\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} \leqslant f(x) \leqslant \frac{\Gamma_n(x)(x+n)}{n!n^x}\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}} \tag{5}$$

Let us put

$$l_n = \frac{1}{n!n^x}\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} \tag{6}$$

and

$$r_n = \frac{(x+n)}{n!n^x}\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}}. \tag{7}$$

We notice that

$$l_n = [a_n\, b_n\, c_n]^{\frac{1}{h\circ f(n+1+x)}} \tag{8}$$

and

$$r_n = [a_n\, b_{n-1}\, d_n]^{\frac{1}{h\circ f(n+x)}} \cdot \frac{x+n}{n} \tag{9}$$

where

$$\log a_n = x(h_n - h_{n-1})\log[(n-1)!]\,, \tag{10}$$

$$\log b_n = [h_n - h\circ f(n+1+x)]\log(n!)\,, \tag{11}$$

$$\log c_n = x[h_n - h \circ f(n+1+x)] \log n \tag{12}$$

and

$$\log d_n = x[h_n - h \circ f(n+x)] \log n. \tag{13}$$

We shall prove that

$$\lim_{n\to\infty} l_n = 1 \quad \text{and} \quad \lim_{n\to\infty} r_n = 1.$$

By (10) it is obvious that

$$\frac{\log a_{n+1}}{\log a_n} = A_n \cdot \frac{\log(n!)}{\log[(n-1)!]}, \tag{14}$$

where

$$A_n = \frac{h_{n+1} - h_n}{h_n - h_{n-1}}.$$

By the assumptions of the theorem we have

$$A_n = \frac{m - \frac{a}{(n+1)!} + R[(n+1)!] - m + \frac{a}{n!} - R(n!)}{m - \frac{a}{n!} + R(n!) - m + \frac{a}{(n-1)!} - R[(n-1)!]}.$$

Thus

$$A_n = \frac{(n-1)!}{n!} \cdot \frac{a - \frac{a}{n+1} + n!R[(n+1)!] - n!R(n!)}{a - \frac{a}{n} + (n-1)!R(n!) - (n-1)!R[(n-1)!]}.$$

Because $R(x) = o(\frac{1}{x})$, $x \to +\infty$, we have

$$\lim_{x\to+\infty} xR(x) = 0. \tag{15}$$

Consequently $\lim_{n\to\infty} A_n = 0$ and by (14) $\lim_{n\to\infty} \log a_n = 0$, which gives

$$\lim_{n\to\infty} a_n = 1. \tag{16}$$

Similarly, by (11),

$$\frac{\log b_{n+1}}{\log b_n} = B_n \cdot \frac{\log[(n+1)!]}{\log(n!)}, \tag{17}$$

where

$$B_n = \frac{h_{n+1} - h \circ f(n+2+x)}{h_n - h \circ f(n+1+x)}$$

and by (1) we have

$$B_n = \frac{m - \frac{a}{(n+1)!} + R[(n+1)!] - m + \frac{a}{f(n+2+x)} - R[f(n+2+x)]}{m - \frac{a}{n!} + R(n!) - m + \frac{a}{f(n+1+x)} - R[f(n+1+x)]}$$

and further

$$B_n = \frac{n!}{(n+1)!} \times \frac{\frac{a(n+1)!}{f(n+2+x)} - a + (n+1)!R\left[(n+1)!\right] - (n+1)!R\left[f(n+2+x)\right]}{\frac{an!}{f(n+1+x)} - a + n!R(n!) - n!R\left[f(n+1+x)\right]}. \tag{18}$$

Because $f(n+1+x) = x(x+1)\dots(x+n)\,f(x)$, we have (by (Γ_n))

$$\frac{n!}{f(n+1+x)} = \Gamma_n(x)\cdot\frac{1}{f(x)\,n^x}$$

and

$$\lim_{n\to\infty}\frac{n!}{f(n+1+x)} = 0.$$

Note that

$$n!R\left[f(n+1+x)\right] = f(n+1+x)R\left[f(n+1+x)\right]\frac{n!}{f(n+1+x)}$$

so that by (15) we have

$$\lim_{n\to\infty} n!\,R\left[f(n+1+x)\right] = 0.$$

Thus (18) yields $\lim_{n\to\infty} B_n = 0$ whence $\lim_{n\to\infty}\log b_n = 0$ (by (17)), and finally

$$\lim_{n\to\infty} b_n = 1. \tag{19}$$

Similarly (using (12)) we can prove that

$$\lim_{n\to\infty} c_n = 1. \tag{20}$$

Finally, by (13) it follows that

$$\frac{\log d_{n+1}}{\log d_n} = D_n\frac{\log(n+1)}{\log n},$$

where

$$D_n = \frac{h_{n+1} - h\circ f(n+1+x)}{h_n - h\circ f(n+x)}.$$

We can observe that

$$D_n = \frac{f(n+x)}{f(n+1+x)}\cdot\frac{S_{n+1}}{S_n},$$

with

$$S_n := a - f(n+x)\left[\frac{a}{n!} - R[f(n+x)] + R(n!)\right]. \tag{21}$$

But

$$\frac{f(n+x)}{n!} = \frac{1}{\Gamma_n(x)}\cdot\frac{n^x f(x)}{x+n} = \frac{f(x)}{\Gamma_n(x)}\cdot\frac{1}{\left(\frac{x}{n}+1\right)n^{1-x}}$$

and because of $x \in (0,\, 1)$ we have

$$\lim_{n\to\infty} \frac{f(n+x)}{n!} = 0.$$

The identity

$$f(n+x)\, R(n!) = n!\, R(n!) \cdot \frac{f(n+x)}{n!}$$

together with (15) imply

$$\lim_{n\to\infty} f(n+x)\, R(n!) = 0.$$

So $\lim_{n\to\infty} D_n = 0$ because of (15) and (21), whence $\lim_{n\to\infty} \log d_n = 0$ and finally

$$\lim_{n\to\infty} d_n = 1. \tag{22}$$

Thus by relations (8), (9), (16), (19), (20) and (22) we obtain

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n = 1. \tag{23}$$

This implies that $f(x) = \Gamma(x)$ for $x \in (0,\, 1]$ (as $f(1) = 1 = \Gamma(1)$).

Applying the same inductive argument as in the proof of Theorem 3 we find that $f(x) = \Gamma(x)$ for $x \in \mathbb{R}_+$, and the proof is completed.

The starting point of the proofs of Theorem 3 and of Theorem 4 is analogous to that in Artin's proof [1] of the Bohr-Mollerup Theorem.

We notice that in a vicinity of $+\infty$ the function arctan is represented by

$$\arctan \mathrm{x} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots\ .$$

Thus we have the following

REMARK 7

The function $\arctan \cdot \log$ is in $\mathbf{M}$.

3. Special convex compositions with Γ

It is known that $g \circ \Gamma$ is convex on $\mathbb{R}_+$ for $g = \log$. We want to present other functions g with this property.

THEOREM 5

Let the functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $g_1 := \log + \arctan$ *and* $g_2 := \log \circ \left(\mathrm{id}_{\mathbb{R}_+} + a\right)$, *where* $a > 0$. *Then there is a* $\gamma > 0$, *such that* $g_1 \circ \Gamma$ *and* $g_2 \circ \Gamma$ *are convex on* $(\gamma, +\infty)$.

Proof. 1°. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be given by $\psi = (\log \circ \Gamma)'$, and let $g_1 = \log + h$ (where $h = \arctan$). We notice by (Γ) that the function $\psi = \frac{\Gamma'}{\Gamma}$ is represented by the formula

$$\psi(x) = \log x - \sum_{n=0}^{\infty} \left[\frac{1}{x+n} - \log\left(1 + \frac{1}{x+n}\right) \right],$$

so (by the inequality $\log(1+x) \leqslant x$) we have

$$\psi(x) \leqslant \log x. \tag{24}$$

Moreover, the derivative of ψ is given by

$$\psi'(x) = \frac{\Gamma''(x)\Gamma(x) - [\Gamma'(x)]^2}{[\Gamma(x)]^2} = \frac{1}{x} + \frac{1}{2x^2} + \int_0^{+\infty} \frac{4tx}{(t^2+x^2)^2(e^{2\pi t}-1)}\, dt \tag{25}$$

(see [4] p. 250-251). Thus $\psi' = (\log \circ \Gamma)'' \geqslant 0$ on $(0, +\infty)$. By the definition of g_1 we see that

$$(g_1 \circ \Gamma)'' = \frac{\Gamma''\Gamma - (\Gamma')^2}{\Gamma^2} + (h'' \circ \Gamma)(\Gamma')^2 + (h' \circ \Gamma)\Gamma''.$$

By properties of h we can see that

$$\forall x \in \mathbb{R}_+ : \; h' \circ \Gamma(x) \cdot \Gamma''(x) \geqslant 0,$$

and

$$h'' \circ \Gamma = \frac{-2\Gamma}{(1+\Gamma^2)^2}.$$

So we obtain:

$$\begin{aligned} h'' \circ \Gamma(x) \cdot [\Gamma'(x)]^2 &\geqslant \frac{-2\Gamma'(x)}{[\Gamma(x)]^3} \\ &= -2\left[\frac{\Gamma'(x)}{\Gamma(x)}\right]^2 \cdot \frac{1}{\Gamma(x)} = \frac{-2[\psi(x)]^2}{\Gamma(x)}. \end{aligned}$$

Thus by conditions (24) and (25) we have:

$$(g_1 \circ \Gamma)''(x) \geqslant \frac{1}{x} + \frac{1}{2x^2} - 2\frac{\log^2 x}{\Gamma(x)} \geqslant 0$$

for sufficiently large x, say for $x \geqslant \gamma$. Hence $g_1 \circ \Gamma$ is convex on $(\gamma, +\infty)$.

2°. Now let $a > 0$ and let $g_2 = \log \circ \left(\mathrm{id}_{\mathbb{R}_+} + a\right)$. The function $(g_2 \circ \Gamma)''$ is given by the formula

$$(g_2 \circ \Gamma)''(x) = \frac{\Gamma''(x)\Gamma(x) - [\Gamma'(x)]^2 + a\Gamma''(x)}{[\Gamma(x)+a]^2}.$$

Because $\log \circ \Gamma$ and Γ are convex and twice differentiable, we have

$$\forall x \in \mathbb{R}_+ : \; \Gamma''(x)\Gamma(x) - [\Gamma'(x)]^2 \geqslant 0 \text{ and } \Gamma''(x) \geqslant 0$$

Hence $g_2 \circ \log$ is convex on $\mathbb{R}_+$.

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Report of Meeting

7th International Conference on Functional Equations and Inequalities, Zlockie, September 12-18, 1999

The Seventh International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University in Kraków, was held from September 12 to September 18, 1999, in the hotel "Geovita" at Złockie. The preceding ICFEI took place at: Sielpia (1984), Szczawnica (1987), Koninki (1991), Krynica (1993) and Muszyna-Złockie (1995 and 1997). The substantial support of the Polish State Committee for Scientific Research (KBN) and of the Foundation for Advancement of Science "Kasa im. Józefa Mianowskiego" is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Eugeniusz Wachnicki, the Dean of the Faculty of Mathematics, Physics and Technics of the Pedagogical University in Kraków. He conveyed participants' best greetings and congratulations to Professor János Aczél, whom the University of Miskolc conferred on September 11, 1999 the degree of Doctor Honoris Causa. It was the fourth of Prof. Aczél's honorary doctorates, after those granted by the Universities of Karlsruhe, Graz and Katowice.

There were 73 participants who came from: Austria (1), Canada (3), Czech Republic (2), Germany (3), Hungary (8), Italy (2), Japan (1), Russia (1), Sweden (1), Ukraine (3), The U.S.A. (1), Venezuela (1), Yugoslavia (1); and from Poland: Gdańsk (1), Gliwice (1), Katowice (16), Kielce (1), Kraków (19), Rzeszów (5), Zielona Góra (2).

During 19 sessions 63 talks were delivered mainly on: functional equations in several variables, their stability, and applications; iteration theory (also for multifunctions) and dynamical systems; functional and general inequalities.

The organizing Committee was chaired by Professors Dobiesław Brydak and Bogdan Choczewski. Dr. Jacek Chmieliński acted as a scientific secretary. Miss Ewa Dudek, Mrs. Anna Grabiec, Miss Janina Wiercioch and Mr. Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference office at Złockie.

At Wednesday, September 15, there was a half-day excursion to the Pieniny region. A group of participants took part in the scenic rafting race on the Dunajec river through a gorge in the Pieniny mountains. Other group visited the castle Dunajec at Niedzica (constructed about 1310 and owned by Hungarian noble families till 1943) and the world famous wooden 15th century church at Dębno (with its polychromy which have kept its fastness for 500 years up to the present).

When closing the Conference, Professor D. Brydak first asked to commemorate two our colleagues, who attended former meetings and passed away in the last two years.

Professor György Targonski died at the age of 69, on January 10, 1998, in Munich. The European Conference on Iteration Theory held in September, 1999, also in "Geovita", was dedicated to his memory.

Mr. Martin Grinč, Slovakian citizen, who studied at the Silesian University in Katowice, died of cancer at the age of 28, on January 18, 1999, at Stará L′ubovňa, a week after submitting his Ph. D. thesis.

Expressing cordial thanks to the participants, and especially to Professor János Aczél, Prof. Brydak pointed out that the present meeting was the best also in numbers: of participants (73), talks (63) and contributions (14) presented on problems-and-remarks parts of many sessions. He extended best thanks to the members of the office staff at Złockie for their effective and dedicated work and assistance, and to the managers of the hotel "Geovita" for their hospitality and quality of services.

The 8th ICFEI was announced to be held in 2001, most probably in September, at the same place.

The abstracts of talks are printed in the alphabetical order, and the contributions to the problems-and-remark sessions — in the order of presentation. They were completed by Dr. J. Chmieliński and prepared for printing by him and Mr. W. Wilk.

Bogdan Choczewski

Abstracts of Talks

János Aczél *The strictly monotonic solutions of a functional equation arising from coordination of two ways to measure utility*

Joint work with Gy. Maksa and Zs. Páles.

Gyula Maksa, Duncan Luce and I dealt in 1996 with the pair of functional equations

$$H(x,y)z = H[xz, yP(x,z)],$$

$$G[H(x,y)] = G(x)G(y),$$

(x and y are in $[0,1[$, z in $[0,1]$) originating from a problem of utility theory described in the title of this talk. That problem makes it natural to assume that G is strictly monotonic and maps $[0,1[$ onto $]0,1]$ (this determines H from the second equation) and P maps $[0,1[\times[0,1]$ into $[0,1]$. From these conditions we proved that

$$P(x,z)=g(x)/g(xz),$$

where g is continuous, strictly decreasing and maps $]0,1[$ into the set of positive reals, while $P(0,z)=z$ and $P(x,0)=0$. This reduces the first of the above equations to

$$G[yg(x)/g(xz)]=G[H(x,y)z]/G(xz),$$

with H defined, as before, by the second equation. We in 1996 and several others since then, however, succeeded to advance further to the complete solution of the problem only under differentiability conditions. Eventually I came to the paradox idea that the limit equation (as y tends to 1; continuity has already been established)

$$G[g(x)/g(xz)]=G(z)/G(xz)$$

may be easier to solve. We succeeded to do this with help of an idea (by now method) of Páles which derives the Jensen inequality from this equation. That is what this talk is about.

Roman Badora *On approximate additive derivations*

The aim of the talk is to present a stability theorem for additive derivations.

Karol Baron *On a linear functional equation in a complex domain*

Studies of the problem how the brain works have led Thomas L. Saaty (University of Pittsburgh) to the functional equation

$$f(a_1z_1,\dots,a_Nz_N)=bf(z_1,\dots,z_N),$$

where $a_1,\dots,a_N$ and b are given complex numbers. It is the purpose of the talk to present a result on its solutions $f:(\mathbb{C}\setminus\{0\})^N\to\mathbb{C}$ which are continuous on polycircles about the origin.

Lech Bartłomiejczyk *Solutions with big graph of iterative functional equations of the first order*

We obtain a result on the existence of a solution with big graph of functional equations of the form

$$g(x,\varphi(x),\varphi(f(x)))=0$$

and we show that it is easily applicable to some particularly important equations, both linear and nonlinear, as, e.g., those of Abel, Böttcher and Schröder. The graph of such a solution has some strange properties: it is dense and connected, has full outer measure and is topologically big.

Bogdan Batko *On the stability of an alternative Cauchy equation*

The talk is based on the results obtained jointly with Jacek Tabor.

Let G be a commutative semigroup and let $f : G \to \mathbb{R}$. We deal with the stability (in the Hyers-Ulam sense) of the functional equation

$$|f(x+y)| = |f(x) + f(y)| \quad \text{for } x, y \in G$$

and its generalizations. We obtain the following results.

THEOREM 1

Let $V \subset G$ be such that for every $x \in G \setminus \{0\}$ there exists an $n \in \mathbb{N}$ with $kx \notin V$ for $k \geqslant n$. Suppose that $f : G \to \mathbb{R}$ satisfies for some $\delta > 0$ the inequality

$$\big|\, |f(x+y)| - |f(x) + f(y)| \,\big| \leqslant \delta \quad \textit{for } (x,y) \in G \times G \setminus V \times V.$$

We prove that there exists a unique additive function $\gamma : G \to \mathbb{R}$ such that

$$|f(x) - \gamma(x)| \leqslant 3\delta \quad \textit{for } x \in G.$$

The constant of the approximation is, in general, the best possible one.

We also show that an analogon of this result for functions $f : G \to \mathbb{R}^2$ does not hold.

THEOREM 2

Let L be a complete Archimedean Riesz Space. Suppose that $F : G \to L$ satisfies for some $e \in L_+$ the inequality

$$\big|\, |F(x+y)| - |F(x) + F(y)| \,\big| \leqslant e \qquad \textit{for } x, y \in G.$$

Then there exists a unique additive mapping $A : G \to L$ such that

$$|F(x) - A(x)| \leqslant e \qquad \textit{for } x \in G.$$

As the method of the proof we use the Johnson-Kist Representation Theorem.

Zoltán Boros *Stability of the Cauchy equation in ordered fields*

Let R be an ordered field and $I \subset R$ be an interval. We give sufficient conditions for R and I so that the following statement hold: if $(X, +)$ is a commutative semigroup and $f : X \to R$ such that

$$f(x) + f(y) - f(x+y) \in I \quad \text{for every } x, y \in X,$$

then there exists an additive function $g : X \to R$ such that $f(x) - g(x) \in I^*$ for every $x \in X$, where $I^* = I$ (if I denotes the set of infinitesimal or finite elements) or I^* is infinitesimally close to I (if I is of the form $[-\delta, \delta]$).

Janusz Brzdęk *On the isosceles orthogonally exponential mappings*

Let X be a real normed space with $\dim X > 1$ and K be a field. We have the following theorem.

THEOREM 1
Suppose $f : X \to K$ satisfies

$$f(x+y) = f(x)f(y) \quad \textit{whenever } \|x+y\| = \|x-y\|. \tag{1}$$

Then $f(X \setminus \{0\}) = \{0\}$ or $0 \notin f(X)$.

Theorem 1 yields the subsequent two corollaries.

COROLLARY 1
Suppose X is not an inner product space and $\dim X > 2$. Then every solution $f : X \to K$ of (1) is exponential, i.e.

$$f(x+y) = f(x)f(y) \quad \textit{for every } x, y \in X. \tag{2}$$

COROLLARY 2
Let X be as in Corollary 1, $(S, \cdot)$ be a commutative semigroup with the neutral element e, and $f : X \to S$ be a solution of (1). Suppose that $f(x_0)$ is invertible (in S) for some $x_0 \in X \setminus \{0\}$. Then (2) holds.

Jacek Chmieliński *Almost approximately inner product preserving mappings*

Motivated by previous papers dealing with mappings preserving the inner product almost everywhere well as by stability results for the orthogonality equation we investigate a combination of these two problems. We show that a mapping that preserves inner product approximately and up to a negligible set of arguments has to be almost everywhere close to an exact solution of the orthogonality equation.

Jacek Chudziak *Continuous solutions of the generalized Gołąb-Schinzel equation*

We consider the functional equation

$$f(g(x)\phi(f(y)) + h(y)\varphi(f(x))) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{1}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$ and $\phi, \varphi : f(\mathbb{R}) \to \mathbb{R}$ are unknown functions such that

(i) f and ϕ are continuous;

(ii) $f(0) = 1$;

(iii) g, h are bijections with $g(0) = h(0) = 0$.

The equation (1) is a generalization of the well known Gołąb-Schinzel equation.

Krzysztof Ciepliński *On non-singular iteration groups on the unit circle*

Let S^1 be the unit circle with positive orientation and V be a vector space over $\mathbb{Q}$ such that $\dim V \geqslant 1$.

We consider non-singular iteration groups $\{F^v,\ v \in V\}$ of homeomorphisms of S^1, that is iteration groups possesing at least one element without periodic point. We present some results on such groups with no further assumptions as well as we give the general form of some particular non-singular iteration groups.

Stefan Czerwik *Stability of the quadratic functional equation in L^p spaces*

Joint work with Krzysztof Dłutek.

Let (X, v) be an abelian complete measurable group with $v(X) = \infty$ and let E be a metric abelian group. A function $f : X \to E$ is called quadratic iff it satisfies the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x, y \in X. \tag{1}$$

We define the quadratic difference Qf by

$$Qf(x, y) := 2f(x) + 2f(y) - f(x + y) - f(x - y). \tag{2}$$

By L_p^+, $p > 0$ we denote some generalization of the space L^p. The following results can be proved.

Lemma 1

Let E be a space without elements of order two. If $f(x) = q(x) + c$, $x \in X$, where q is quadratic and $c \in E$ is fixed and $f \in L_p^+(x, E)$ for a certain $p > 0$, then

$$q = 0 \quad \text{and} \quad c = 0.$$

Lemma 2

Let G, H be abelian groups. Then for every $f : G \to H$, the quadratic difference Qf satisfies the functional equation:

$$\begin{aligned} Qf(x, u + v) + Qf(x, u - v) + 2Qf(u, v) \\ = Qf(x + u, v) + Qf(x - u, v) + 2Qf(x, u). \end{aligned} \tag{3}$$

Theorem 1

Let E be uniquely divisible by two. Let $f : X \to E$ be such that $Qf(x, y) \overset{v \times v}{=} 0$. Then there exists a quadratic function $q : X \to E$ such that

$$f(x) \overset{v}{=} q(x), \quad x \in X. \tag{4}$$

THEOREM 2

Let E be a space without elements of order two. If $f : X \to E$ is such that $Qf \in L_p^+(X \times X, E)$, then

$$Qf(x,y) \overset{v\times v}{=} 0. \tag{5}$$

Zoltán Daróczy *Characterization of Matkowski pairs*

This work is joint with Gy. Maksa and Zs. Páles.

Let $I \subset \mathbb{R}$ be an open interval and let $CM(I)$ denote the class of all continuous and strictly monotonic real-valued functions defined on I. If $\phi \in CM(I)$, then we define

$$A_\phi(x,y) = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)$$

for all $x, y \in I$. A pair of functions $(\phi, \psi) \in CM(I)^2$ is called a Matkowski pair if the functional equation

$$A_\phi(x,y) + A_\psi(x,y) = x + y$$

holds for all $x, y \in I$. We characterize Matkowski pairs in the following two cases:

(i) there exists a nonvoid open interval $K \subset I$ such that either ϕ or ψ is continuously differentiable on K;

(ii) there exists a nonvoid open interval $K \subset I$ such that A_ϕ and A_ψ are strictly comparable in K.

Thomas M.K. Davison *D'Alembert's functional equation and the Chebyshev polynomials*

The functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{d'Alembert}$$

is studied where the domain of f is the additive group of the integers, and the codomain of f is an arbitrary commutative ring R. We show there is a function

$$T : \mathbb{Z} \to \mathbb{Z}[X] \quad \text{denote} \quad n \mapsto T_n$$

such that if $f : \mathbb{Z} \to R$ satisfies d'Alembert and $f(0) = 1$ then, for all $n \in \mathbb{Z}$

$$f(n) = T_n(f(1)).$$

The sequence T_n of polynomials is identified with the sequence of Chebyshev polynomials using Kannappan's fundamental result

$$f(x) = \frac{e(x) + e(-x)}{2}$$

where $e(x+y) = e(x)e(y)$ and $e(0) = 1$.

In our case

$$e(n) := \begin{bmatrix} X & 1 \\ X^2-1 & X \end{bmatrix}^n \qquad n \in Z.$$

Certain consequences of our result are discussed.

[1] P. Kannappan, *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc. **19** (1968), 69-74.

Joachim Domsta *Regularly varying solutions of Schröder's and related linear equations*

This is a presentation of an extension of former results by B. Choczewski, M. Kuczma, E. Seneta, A. Smajdor and other authors on regularly varying at 0 solutions of the linear equations

$$\Psi(f(x)) = g(x)\cdot\Psi(x), \quad x \in \mathbb{R}_+ := (0,\infty)\,. \tag{$\mathrm{S}_{f,g}$}$$

DEFINITION

We say, that $h : \mathbb{R}_+ \to \mathbb{R}_+$ *is almost constant at 0*, if

(C_0) h is continuous and $h(0^+) \in \mathbb{R}_+$;

(C_1) at least one of the following conditions is fulfilled,

(a) h is of bounded variation locally at 0;

(b) $h(x) - h(0^+) = O(|\log x|^{-1-\nu})$, as $x \to 0^+$, where $\nu > 0$.

The solutions are analysed with the use of the following assumptions and notation,

(H_1) f is a nondecreasing continuous selfmapping of $\mathbb{R}_+$, $0 < f(x) < x$, for $x \in \mathbb{R}_+$ and $f'_+(0) := \lim_{x\to 0^+} \frac{f(x)}{x} \in (0,1)$;

(H_2) g is almost constant at 0;

$$\gamma_n(x|y) := \frac{G_n(y)}{G_n(x)}\,,$$

where $G_n(x) := \prod_{k=0}^{n-1} g_0(f^k(x))$, $g_0 := \frac{g}{g(0^+)}$, $x \in \mathbb{R}_+$, $n \in \mathbb{N}$.

LEMMA

For every $y \in \mathbb{R}_+$, the limit $\gamma(x\,|\,y) := \lim_{n\to\infty} \gamma_n(x\,|\,y)$, $x \in \mathbb{R}_+$, exists and forms a continuous slowly varying solution of (S_{f,g_0}).

COROLLARY 1

If besides (H_1), $\frac{f(x)}{x}$, $x \in \mathbb{R}_+$, *is almost constant at* 0, *then the canonical Schröder equation* $\Phi(f(x)) = f'_+(0)\cdot\Phi(x)$, $x \in \mathbb{R}_+$, *possesses exactly one (up* *to a multiplicative constant) regularly varying solution; it equals the principal function, i.e.*

$$\Phi(x) = \Phi(y) \cdot \varphi(x\,|\,y) \qquad \text{where } \varphi(x\,|\,y) := \lim_{n\to\infty} f^n(x)/f^n(y),\ x, y \in \mathbb{R}_+ .$$

Corollary 2

Under the assumptions of the above Lemma and Corollary 1, the linear equation $(S_{f,g})$ *possesses exactly one regularly varying solution; it is given by the formula*

$$\Psi(x) = \Psi(y) \cdot (\varphi(x\,|\,y))^{\rho} \cdot \gamma(x\,|\,y), \quad x, y \in \mathbb{R}_+ ,$$

where $\rho = \frac{\log g(0^+)}{\log(f'_+(0))}$.

Tibor Farkas *On the associativity of algorithms*

Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{i=1}^{\infty} \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called *interval filling* if, for any $x \in [0, L(\lambda)]$, there exists a sequence (δ_n) such that $\delta_n \in \{0, 1\}$ for all $n \in \mathbb{N}$ and $x = \sum_{i=1}^{\infty} \delta_n \lambda_n$. (This concept has been introduced and discussed in Daróczy-Járai-Kátai [1].)

An *algorithm* (with respect to an interval filling sequence λ) is defined as a sequence of functions $\alpha_n : [0, L(\lambda)] \to \{0, 1\}$ $(n \in \mathbb{N})$ for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x)\lambda_n \quad (x \in [0, L(\lambda)]).$$

The most important and frequently observed algorithms are the *regular* (or *greedy*), the *quasi-regular* and the *anti-regular* (or *lazy*) ones. In [2] Gy. Maksa introduced the following concept: an algorithm (α_n) is called *associative* if the binary operation $\circ : [0, L(\lambda)]^2 \to [0, L(\lambda)]$ defined by

$$x \circ y = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(y)\lambda_n \quad (x, y \in [0, L(\lambda)])$$

is associative. In the same paper the author characterized the associative algorithms and proved the associativity of the regular algorithm with respect to any interval filling sequence and the non-associativity of the anti-regular algorithm in the case of a special class of interval filling sequences.

The purpose of this presentation is to prove the non-associativity of the anti-regular and the quasi-regular algorithms and the existence of associative algorithms different from the regular one.

[1] Z. Daróczy, A. Járai, I. Kátai, *Intervalfüllende Folgen und volladditive Funktionen*, Acta Sci. Math. **50** (1986), 337-350.

[2] Gy. Maksa, *On associative algorithm*, Acta Acad. Paed. Agriensis, Sectio Matematicae (to appear).

Carlos Finol *Inequalities arising from Schlumprecht's construction of an arbitrarily distortable Banach space*

T. Schlumprecht (Israel J. of Math., **76** (1991), 81-95) furnished an explicit construction of an arbitrarily distortable Banach space. The construction is accomplished by using a concrete submultiplicative function,

$$\frac{\log(1+x)}{\log 2}, \quad x \geqslant 1,$$

whose Matuszewska-Orlicz index, at infinity, is zero; along with other properties. That similar spaces can be constructed with functions which share some properties of that function is stated therein.

We single out those properties which characterize the 'Schlumprecht Class' of functions which produce such a spaces and derive a general inequality which all the functions in this class satisfy.

Margherita Fochi *Conditional functional equations in normed spaces*

Let X be a real normed vector space with $\dim X \geqslant 3$ and $f : X \to \mathbb{R}$. We study the exponential Cauchy equation

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \tag{1}$$

and its following conditional forms

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \text{ with } \|x\| = \|y\| \qquad (1)_1$$

and

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \text{ with } x \perp_I y \qquad (1)_2$$

where the James isosceles orthogonality $x \perp_I y$ is defined as follows

$$x \perp_I y \iff \|x+y\| = \|x-y\|.$$

Referring to recent results of Gy. Szabó on the additive Cauchy equation conditioned in the above domains, we prove the equivalence of equations (1), $(1)_1$ and $(1)_2$.

Roman Ger *Ring homomorphisms equation revisited*

We deal with a functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{$*$}$$

considered by J. Dhombres (*Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186-212) for functions f mapping a given ring into another one. In this paper both rings were supposed to have unit elements; additionally the division by 2 had to be performable.

Without these assumptions the study of equation $(*)$ becomes considerably more sophisticated (see author's paper *On an equation of ring homomorphisms*, Publicationes Math. **52** (1998), 397-417). At present, we deal with equation $(*)$ assuming that the domain is a unitary ring with no assumptions whatsoever upon the target ring.

Attila Gilányi *Hyers-Ulam stability of monomial functional equations on a general domain*

In this talk the Hyers-Ulam stability of mononomial functional equations for functions defined on a power-associative, power-symmetric groupoid is investigated.

Throughout the talk $(S, \circ)$ denotes a groupoid, that is, a nonempty set S with binary operation $\circ : S \times S \to S$. The powers of an element $x \in S$ are defined by $x^1 = x$ and, for a positive integer m, by $x^{m+1} = x^m \circ x$. An operation $\circ$ (or the groupoid $(S, \circ)$) is called *power-associative* if $x^{k+m} = x^k \circ x^m$ for all positive integers k, m and each $x \in S$, it is said to be l^{th}*-power-symmetric* (or simply *power-symmetric*) if $l \geqslant 2$ is a given integer such that $(x \circ y)^l = x^l \circ y^l$ for all $x, y \in S$. Using this notation we call a function f mapping from $(S, \circ)$ into a linear normed space X a *monomial function of degree* n if $\Delta_y^n f(x) - n! f(y) = 0$ for all $x, y \in S$, where Δ denotes the well-known difference operator.

Our main result reads as follows. If n is a positive integer, $(S, \circ)$ is a power-associative, power-symmetric groupoid, B is a Banach space, $f : S \to B$ is a function, and, for a nonnegative real number ε, we have

$$\|\Delta_y^n f(x) - n! f(y)\| \leqslant \varepsilon \quad (x, y \in S),$$

then there exists a unique monomial function $g : S \to B$ of degree n such that

$$\|f(x) - g(x)\| \leqslant \frac{1}{n!}\varepsilon \quad (x \in S).$$

In the special case when S is an Abelian group, this result yields the Hyers-Ulam stability of monomial functional equations in a well-known form, furthermore, if $n = 1$, we get the stability of the Cauchy equation.

Roland Girgensohn *Non-affine fractal interpolation functions*

Let $b \in \mathbb{N}$ and choose $b+1$ data points (t_ν, y_ν), where $0 = t_0 < t_1 < \ldots < t_b = 1$ and $y_\nu \in \mathbb{R}$. Then the fractal interpolation functions of M.F. Barnsley, which are defined via certain iterated function systems, satisfy $f(t_\nu) = y_\nu$ and exhibit a fractal behaviour. The same functions can be defined as the solutions of systems of functional equations of the form

$$f((t_{\nu+1} - t_\nu)\,x + t_\nu) \;=\; a_\nu f(x) + g_\nu(x) \quad \text{for } \nu = 0, \ldots, b-1,$$

where $|a_\nu| < 1$ and $g_\nu : [0,1] \to \mathbb{R}$ are given, and $f : [0,1] \to \mathbb{R}$ is unknown. In the talk, we will point out a connection with certain Schauder bases on $C[0,1]$, we will give an explicit formula for the box dimension of these functions in the case of equidistant t_ν, and we will discuss certain singular solutions.

Andrzej Grząślewicz *On the functional equation $F(x,y) \bullet F(y,z) = F(x,z)$*

Let $(M, \bullet)$ be a "group with zero", $(B, \leqslant)$ a linearly ordered set. M. Fréchet in [1] and Z. Moszner in [2] characterized the solutions of equation

$$F(x,y) \bullet F(y,z) = F(x,z) \tag{1}$$

and their extensions in the case, where $M = \mathbb{R} = B$, $\bullet$ is the usual multiplication, $\leqslant$ is the usual order in $\mathbb{R}$ and F is a function defined on the set $\mathbb{R}\times\mathbb{R}\cap \leqslant$. A. Grząślewicz in [3] generalized these results assuming only, that F is defined on the set $B \times B \cap \leqslant$.

In our report we present the general solution of (1) assuming, that F is a function defined on the set $R_{A,B} := \{(x,y) \in A \times B : \ x \leqslant y\}$, where A is a subset of B. Moreover, as the common result with Angelo Grząślewicz, the extensions of considered solutions are characterized.

[1] M. Fréchet, *Solution continue la plus générale d'une équation fonctionelle de la théorie des probabilités en chaine,* Bull. Soc. Math. France **60** (1932), 232-280.

[2] Z. Moszner, *Ogólne rozwiązanie równania $F(x,y) \cdot F(y,z) = F(x,z)$ przy warunku $x \leqslant y \leqslant z$*, Rocznik Nauk.-Dydakt. WSP w Krakowie, **25**, Matematyka (1966), 123-138.

[3] A. Grząślewicz, *O pewnych homomorfizmach i homomorfizmach ciągłych grupoidu Brandta*, Rocznik Nauk.-Dydakt. WSP w Krakowie, **41**, Prace Matematyczne **6** (1970), 15-30.

Grzegorz Guzik *On embedding of a linear functional equation*

Let the iterative equation

$$\varphi(f(x)) = g(x)\varphi(x) + h(x), \tag{L}$$

where f, g, h are given continuous functions defined on an real interval X, have on X a continuous solution φ, and let f be embeddable in a continuous iteration group F defined on $\mathbb{R}\times X$. We say that (L) has embedding with respect to F if there exist functions G and H defined on $\mathbb{R}\times X$ and satisfying some functional equations such that $G(1,\cdot) = g$ and $H(1,\cdot) = h$ and each continuous solutions φ of (L) defined on X satisfies

$$\varphi(F(t,x)) = G(t,x)\varphi(x) + H(t,x). \tag{Lt}$$

We can prove that (L) has embedding with respect to F whenever it has a one parameter family of continuous solutions. We can prove moreover when embeddability is possible if the zero function is the only continuous solution of (L). Our results yield an answer (under assumptions of continuity) to the problem of L. Reich posed in 1997 on the 35-th ISFE (24. Remark).

Wojciech Jabłoński *On graph of non-affine continuous functions*

In 1970 Marek Kuczma and Roman Ger introduced a class

$$\mathcal{A}_n = \left\{ T \subset \mathbb{R}^n : \begin{array}{l} \text{every convex function } g : D \to \mathbb{R} \\ \text{where } T \subset D \subset \mathbb{R}^n,\ D \text{ is open and convex,} \\ \text{bounded from above on } T \text{ is continuous on } D \end{array} \right\}.$$

In 1973 Marek Kuczma [2] proved that for every continuous non-affine function $f : [a, b] \to \mathbb{R}$ we have $\operatorname{Gr} f \in \mathcal{A}_2$. This result has next been generalized to higher dimensions by Roman Ger. In a special case his result reads as follows

Theorem G.
Let $D \neq \emptyset$ be an open and connected subset of $\mathbb{R}^n$ and let f be a non-affine real-valued function of class C^1, defined on D. Then $\operatorname{Gr} f \in \mathcal{A}_n$.

(Even in this special case the assumption that f is of class C^1 is necessary.)

That theorem does not contain the result proved by Marek Kuczma as a particular case, because of the regularity assumptions on f. Therefore there arises a question whether these assumptions are necessary. It appears that the assumptions on f can be weakened, and we have the following

Theorem
Let $D \neq \emptyset$ be an open and connected subset of $\mathbb{R}^n$ and let f be non-affine continuous real-valued function defined on D. Then $\operatorname{Gr} f \in \mathcal{A}_n$.

[1] R. Ger, *Note on convex functions bounded on regular hypersurfaces,* Demonstratio Math. **6** (1973), 97-103.
[2] M. Kuczma, *On some set classes occurring in the theory of convex functions,* Annales Soc. Math. Pol., Comment. Math. **17** (1973), 127-135.

Witold Jarczyk *On mutual relations between Mulholland's and Tardiff's theorems*

Joint work with J. Matkowski.

We give a complete solution to the following problem posed by B. Schweizer (presented also by A. Sklar during the 37th ISFE in Huntington this year):

"Compare the assumptions imposed on $\varphi : [0, \infty) \to [0, \infty)$ in Mulholland's theorem [Proc. London Math. Soc. (2) **51** (1950), 294-307] and in Tardiff's theorem [Aequationes Math. **27** (1984), 308-316] which guarantee that φ satisfies the inequality

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all $x_1, x_2, y_1, y_2 \in [0, \infty)$."

Peter Kahlig *On the Dido functional equation*

Joint work with J. Matkowski.

By some geometrical considerations we formulate an equation which is related to the ancient isoperimetric problem of Dido. The continuous solution of this Dido functional equation depends on an arbitrary function. However, we show that in a class of functions of suitable asymptotic behavior at infinity, the Dido functional equation has a one-parameter family of "principal" solutions. Some applications are given.

Hans-Heinrich Kairies *On a Banach space automorphism and its connections to functional equation and cnd functions*

Denote by $\mathcal{H}$ the Banach space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even. It turns out that $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

is a Banach space automorphism. Important properties of F are closely related to a de Rham type functional equation for $F[\varphi]$.

The class $F[\mathcal{H}]$ contains many continuous nowhere differentiable functions $F[\varphi]$. A large part of them can be identified by simple properties of the generating function φ.

Palaniappan Kannappan *On the stability of the generalized cosine functional equations*

This is a joint work with G.H. Kim.

We study among others the stability problem of the functional equations

$$f(x+y) + f(x-y) = 2f(x)g(y) \tag{1}$$

and

$$f(x+y) + f(x-y) = 2g(x)f(y) \tag{2}$$

for complex and vector valued functions.

Mikio Kato *Clarkson-type inequalities and their relations to type and cotype*

Joint work with Lars-Erik Persson and Yasuji Takahashi.

The celebrated Clarkson inequalities (**CI**) which might be regarded as the origin of the Banach space geometry, have been proved, originally for L_p and for various concrete Banach spaces. On the other hand, as a multi-dimensional global version of **CI**'s, "generalized Clarkson's inequality" (**GCI**) was given for L_p by M. Kato in connection with behavior of the operator norms of the Littlewood matrices. This includes Boas' and Koskela's inequalities; the former, considered in the context with uniform convexity, is the first one of this type (with two elements). **GCI** was further extended in parameters by L. Maligranda and L.-E. Persson. Also related to **GCI**, A. Tonge proved random Clarkson inequality (**RCI**).

In this talk we characterize all these inequalities by means of type and cotype in the general Banach space setting. As far as we know in literature, M. Milman first observed Clarkson's and type inequalities in the same frame work. Thus our results provide a conclusion to his observation (in an extended setting). Let X be a Banach space. Let $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

CI for L_p: *Denote* $\| \cdot \| = \| \cdot \|_p$. *Then for all* $x, y \in L_p$

$$\left(\|x+y\|^{p'} + \|x-y\|^{p'}\right)^{\frac{1}{p'}} \leqslant 2^{\frac{1}{p'}} \left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}} \quad \text{if } 1 \leqslant p \leqslant 2, \tag{1}$$

$$\left(\|x+y\|^{p} + \|x-y\|^{p}\right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}} \left(\|x\|^{p'} + \|y\|^{p'}\right)^{\frac{1}{p'}} \quad \text{if } 2 \leqslant p \leqslant \infty. \tag{2}$$

These (1) and (2) are characterized in a Banach space X as follows.

THEOREM 1

(i) *X satisfies* (1) *if and only if X is of type p and 'type p constant' is 1.*

(ii) *X satisfies* (2) *if and only if X is of cotype p and 'cotype p constant' is 1.*

Let $1 \leqslant p \leqslant 2$. Then (1) for L_p and (2) for $L_{p'}$ are equivalent. Indeed, it is easily seen by duality argument that (1) holds in a Banach space X if and only if it holds in the dual space X'. Thus it is enough to treat **CI** (1) for the case $1 \leqslant p \leqslant 2$ in our discussion, which we refer to as (p, p')-*Clarkson inequality.*

THEOREM 2

Let $1 \leqslant p \leqslant 2$. Then

(i) **GCI** *of Kato form, resp. of Maligranda-Persson form, holds in X, if and only if X is of type p and 'type p constant' is 1.*
In addition to this, the same implications for the dual space X' are equivalent.

(ii) **RCI** *holds in X with an 'absolute constant' K if and only if X is of type p.*

THEOREM 3

Let $1 \leqslant p \leqslant 2$. **GCI** *holds in X if and only if* **RCI** *holds in $L_{p'}(X)$ with the absolute constant $K = 1$.*

Denis Khusainov *Stability of difference systems with rational right hand side*

The report describes the investigation of the stability of zero solution of difference equation systems with a rational right hand side with and without delay. The investigation uses the quadratic Lyapunov's function. The following system of difference equations is considered.

$$x(k+1) = [E + X(k)D]^{-1}[A + X(k)B]x(k), \quad k = 0, 1, \dots \qquad (1)$$

with $x(k) \in \mathbb{R}^n$, E – identity matrices, $X(k), B, D$ – block structure matrix with corresponding size. The following results were obtained.

THEOREM 1

Let A be asymptotically stable matrix (i.e. $\max_{i=\overline{1,n}} |\lambda_i(A)| < 1$. Then zero solution of system (1) is asymptotically stable. Stability region contains the sphere U_R with the radius

$$R = \frac{\sqrt{\gamma(H) + \Psi^2(H)} - \Psi(H)}{|A||D| + |B| + |D|\left[\sqrt{\gamma(H) + \Psi^2(H)} - \Psi(H)\right]} \cdot \frac{1}{\sqrt{\varphi(H)}},$$

where

$$\gamma(H) = \frac{\lambda_{\min}(H - A^T HA)}{\lambda_{\max}(H)}, \quad \Psi(H) = \frac{|HA|}{\lambda_{\max}(H)}, \quad \varphi(H) = \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)}.$$

The obtained results are extended to the system with several delays

$$x(k+1) = \left[E + \sum_{j=0}^{m} X(k-j)D_j\right]^{-1} \sum_{j=0}^{m} A_j x(k-j), \quad n = 0, 1, 2, \dots. \qquad (2)$$

Let us denote

$$\overline{A} = \sum_{j=0}^{m} \alpha_j A_j, \quad a(H) = \sum_{j=0}^{m} |A_j| \left(\alpha_j + \sqrt{\varphi(H)}\right),$$

$$d(H) = |D_0| + \sqrt{\varphi(H)} \sum_{j=0}^{n} |D_j|.$$

THEOREM 2

Let the constants $\alpha_j, j = \overline{0,m}$ and positive defined matrix H exist with the condition

$$\gamma(H) > a_2(H) + 2d(H)|\overline{A}|.$$

Then zero solution of system (2) is asymptotically stable. Stability region contains the sphere U_R with radius

$$R = \left[1 - \frac{a(H) + |\overline{A}|}{\sqrt{\gamma(H) + |\overline{A}|^2}}\right] \frac{1}{d(H)\sqrt{\varphi(H)}}.$$

Barbara Koclęga *On a generalized Cauchy equation*

This is a joint work with Professor Roman Ger.

A description of all continuous (resp. differentiable) solutions f mapping the real line $\mathbb{R}$ into a real normed linear space $(X, \|\cdot\|)$ (not necessarily strictly convex) of the functional equation

$$\|f(x+y)\| = \|f(x) + f(y)\|$$

has been presented by Peter Schöpf in [2]. Looking for more readable representations we have shown that any function f of that kind fulfilling merely very mild regularity assumptions has to be proportional to an odd isometry mapping $\mathbb{R}$ into X.

To gain a proper proof tool we have also established an improvement of Edgar Berz's [1] result on the form of Lebesgue measurable sublinear functionals on $\mathbb{R}$.

[1] E. Berz, *Sublinear functions on* $\mathbb{R}$, Aequationes Math. **12** (1975), 200-206.
[2] P. Schöpf, *Solution of* $\|f(\xi+\eta)\| = \|f(\xi)+f(\eta)\|$, Mathematica Pannonica **8/1** (1997), 117-127.

Zygfryd Kominek *On ε-convex functions*

Joint result with Bogdan Batko and Jacek Tabor.

We give a different proof of the known result of Hyers and Ulam on approximately convex functions getting somewhat better estimation. Moreover, we prove that in an arbitrary infinite-dimensional linear space this result is no longer true.

Aleksandar Krapež *Functional equations on almost quasigroups*

Quasigroups may be defined as groupoids in which all left and right translations are permutations. *Almost quasigroups* are groupoids in which some (but not all) of the translations may be constant mappings. If we additionally require that almost quasigroup has a unit, we get an *almost loop.* Similarly, associative (and commutative) almost quasigroup is an *almost group (almost Abelian group).*

A *quasizero* of a groupoid is a triple (p, q, r) of elements such that for all x, y $px = xq = r$. If $p = q = r$, the notion of the quasizero reduces to the familiar notion of zero.

A *quasigroup with quasizero* is a groupoid with quasizero (p, q, r), such that equation $xy = z$ is uniquely solvable in x for all $y \neq q$ and uniquely solvable in y for all $x \neq p$. Note that a quasigroup with quasizero is not a quasigroup.

A quasizero of a quasigroup with quasizero which has a unit, reduces to zero. Therefore we get notions of *loop with zero, group with zero* and *Abelian group with zero.* The last two are familiar from the semigroup theory, in particular the last one which is a multiplicative reduct of a field.

We have the following representation theorem:

Theorem 1

Any almost quasigroup is either a quasigroup or a quasigroup with quasi-zero.

This result enables us to solve the two classical functional equations in the case of almost quasigroups:

Theorem 2

If the four (six) almost quasigroup operations $A, B, C, D(, E, F)$ satisfy the generalized associativity (GA) *(the generalized bisymmetry* (GB)*) equation*

$$A(x, B(y, z)) = C(D(x, y), z) \tag{GA}$$

$$A(B(x, y), C(u, v)) = D(E(x, u), F(y, v)) \tag{GB}$$

then they are all isotopic to the same almost (Abelian) group.

The formulas of general solutions of these equations are also given.

In a similar way we can solve any generalized balanced functional equation on almost quasigroups.

Dorota Krassowska *A system of functional inequalities related to Cauchy's functional equation*

Joint work with J. Matkowski.

We consider the system of functional inequalities

$$f(a + x) \leqslant \alpha + f(x), \quad f(b + x) \leqslant \beta + f(x), \quad x \in \mathbb{R}.$$

Assuming the continuity of f at least at one point and some algebraic conditions on a, b, α, β, we show that every solution f of that system must be an affine function.

We also show that if the algebraic conditions are not satisfied, then the continuous solution of the system of functional equations

$$f(a + x) = \alpha + f(x), \quad f(b + x) = \beta + f(x), \quad x \in \mathbb{R}.$$

depends on an arbitrary function.

The relevant results for remaining three types of Cauchy's system of functional inequalities or equations are also considered.

Károly Lajkó *Further functional equations in the theory of conditionally specified distributions*

Let (X, Y) be an absolutely continuous bivariate random variable with support in the positive quadrant. Let us denote the joint, marginal, and conditional densities by $f_{X,Y}$, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$, respectively. We can write $f_{X,Y}$ in two ways and obtain the relationship

$$f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x) \qquad (x, y \in \mathbb{R}_+). \tag{1}$$

It is natural to inquire about all joint densities whose conditional densities satisfy

$$f_{X|Y}(x|y) = g_1(x(c_1 + c_2y)); \quad f_{Y|X}(y|x) = g_2(y(d_1 + d_2x)) \tag{2}$$

or

$$f_{X|Y}(x|y) = g_3\left(\frac{x - a_1 - a_2y}{1 + cy}\right); \quad f_{Y|X}(y|x) = g_4\left(\frac{y - b_1 - b_2x}{1 + dx}\right), \tag{3}$$

where $c_1, c_2, d_1, d_2, c, d \in \mathbb{R}_+$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$. In case (2) or (3) we have from (1) the functional equation

$$g_1(x(c_1 + c_2y))f_Y(y) = g_2((d_1 + d_2x)y)f_X(x) \quad (x, y \in \mathbb{R}_+) \tag{4}$$

or

$$g_3\left(\frac{x - a_1 - a_2y}{1 + cy}\right)f_Y(y) = g_4\left(\frac{y - b_1 - b_2x}{1 + dx}\right)f_X(x) \quad (x, y \in \mathbb{R}_+), \tag{5}$$

respectively, for functions $f_X, f_Y, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $g_3, g_4 : \mathbb{R} \to \mathbb{R}_+$. Solving these functional equations, it is possible to determine the nature of the joint distributions associated with (2) or (3).

Zbigniew Leśniak *Iterative roots of homeomorphisms of the plane*

We give a construction of iterative roots of a free mapping f of the plane. In particular, we deal with the case where f cannot be embedded in a flow.

Andrzej Mach *La solution générale de l'équation:*

$$\varphi(\varphi(\varphi(...\varphi(\varphi(\alpha, x_1), x_2), \ldots), x_{n-1}), x_n) = \varphi(\alpha, x_1 \cdot \mu(x_2) \cdot \mu^2(x_3) \cdot \ldots \cdot \mu^{n-2}(x_{n-1}) \cdot x_n)$$

Dans l'équation considérée nous avons: n est un nombre naturel supérieur ou égal à deux, $\varphi : \Gamma \times G \to \Gamma$, Γ est un ensemble arbitraire non-vide, $\langle G; \cdot\rangle$ est un groupe binaire, $\mu \in \mathrm{Aut}(\langle G; \cdot\rangle)$ et $\mu^{n-1}(x) = x$, $x \in G$ (μ^ν dénote ν-ième itération).

Le travail [1] donne une construction générale des solutions de l'équation considérée et en conséquence donne une généralisation de la construction de l'équation de translation classique.

[1] A. Mach, *The construction of the solutions of the generalized translation equation,* submitted.

Elena N. Makhrova *Limit sets of continuous mappings of dendrites with closed periodic points set*

This is a joint work with L.S. Efremova.

In this report we consider piecewise monotone mappings of dendrites with countable ramification points set.

Let D be the class of dendrites such that for every $X \in D$ the next properties hold: (1) the ramification points set $R(X)$ is closed; (2) for any point $x \in R(X)$ the number of components of $X \setminus \{x\}$ is finite.

A continuous mapping f is called piecewise monotone if there exists a finite nonempty set $A = \{a_1, a_2, \dots, a_n\}$ such that for any component $C \subset X \setminus A$ the restriction $f|_C$ is monotone.

Let us formulate the main results.

THEOREM A.

Let f be a piecewise monotone mapping of a dendrite $X \in D$ into itself. Then the next statements are equivalent:

(A1) *the periodic points set $Per(f)$ is closed;*

(A2) $C(f) = Per(f)$, *where $C(f)$ is the center of f;*

(A3) *ω-limit set of any trajectory is a periodic orbit;*

(A4) $\Omega(f) = \cup_{z \in X} \omega(z, f) = Per(f)$, *where $\Omega(f)$ is f-nonwandering set, $\omega(z, f)$ is ω-limit set of the point z trajectory.*

COROLLARY

If f is a piecewise monotone mapping with closed set of periodic points of a dendrite $X \in D$ into itself, then the topological entropy of f equals zero.

Note that there are a dendrite $X \notin D$ and a continuous mapping $f : X \to X$ such that f has the closed set $Per(f)$ and a recurrent nonperiodic point.

THEOREM B.

For every unbounded set M of natural numbers there exist a dendrite $X \in D$ with countable ramification points set and continuous mapping $f : X \to X$ such that

(B1) *topological entropy of f equals* 0,

(B2) *the set of the least periods of f-periodic points coincides with M.*

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[1] L.S. Efremova, E.N. Makhrova, *On dynamics of monotone mappings of dendrites* (in Russian), Algebra & Analisis, (1999) (to appear).

Gyula Maksa *On a problem of Matkowski*

This work is joint with Z. Daróczy.

Let $I \subset \mathbb{R}$ be an open interval of positive length and let $CM(I)$ denote the class of all continuous and strictly monotonic real-valued functions defined on I. A function $M : I^2 \to I$ is called quasi-arithmetic mean if there exists $\phi \in CM(I)$ such that

$$M(x,y) = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) =: A_\phi(x,y)$$

for all $x, y \in I$. J. Matkowski proposed the following problem: for which pair of functions $\phi, \psi \in CM(I)$ does the functional equation

$$A_\phi(x,y) + A_\psi(x,y) = x + y$$

hold for all $x, y \in I$.

In this talk we give a partial solution of this problem supposing comparability properties for A_ϕ and A_ψ in addition.

Lech Maligranda *The failure of the Hardy inequality and interpolation of intersections*

The main idea here is to clarify why it is sometimes incorrect to interpolate inequalities in a "formal" way. For this we consider two Hardy type inequalities, which are true for each parameter α different from 0 but they fail for the "critical" point $\alpha = 0$. This means that we cannot interpolate these inequalities between the noncritical points $\alpha = 1$ and $\alpha = -1$ and conclude that it is also true at the critical point $\alpha = 0$. Why? An accurate analysis shows that this problem is connected with the investigation of the interpolation of intersections $(N \cap L_p(w_0), N \cap L_p(w_1))$, where N is a linear space which consists of all functions with the integral equal to 0.
We calculate the K-functional for the couple $(N \cap L_p(w_0),\ N \cap L_p(w_1))$, which occurs to be essentially different from the K-functional for $(L_p(w_0), L_p(w_1))$, even for the case when $N \cap L_p(w_i)$ is dense in $L_p(w_i)$ $(i = 0, 1)$. This essential difference is the reason why the "naive" interpolation gives a wrong result.

[1] N. Krugljak, L. Maligranda, L.E. Persson, *The failure of the Hardy inequality and interpolation of intersections*, Arkiv Mat., to appear.

Janusz Matkowski *On a functional equation satisfied by pairs of exponential functions*

We prove that the functions $f, g : \mathbb{R} \to (0, \infty)$, satisfy the functional equation

$$f^{-1}[t(f(x)+f(y))] + g^{-1}(t[g(x)+g(y)]) = x + y, \quad x, y \in \mathbb{R},\ t > 0,$$

if, and only if, the function $\frac{f}{f(0)}$ is an exponential bijection, and the product fg is a constant function.

For $t = \frac{1}{2}$ this functional equation was considered recently by Z. Daróczy, Gy. Maksa, Zs. Páles, and the present author.

The solutions f, g of the above functional equation satisfy the functional equation

$$f^{-1}[tf(x) + (z-t)f(y)] + g^{-1}[(z-t)g(x) + tg(y)] = x + y$$

for all $x, y \in \mathbb{R}$, $z > 0$, $t \in (0, z)$. Specializing z and t we obtain some new functional equations. Open problems will be presented.

Janusz Morawiec *On compactly supported solutions of the two-coefficient dilation equation*

We consider the equation

$$\varphi(x) = a\varphi(2x) + b\varphi(2x-1) \tag{1}$$

and its compactly supported solutions $\varphi : \mathbb{R} \to \mathbb{R}$, where a and b are real parameters. In the present contribution we determine the sets $B_{a,b}$ and $C_{a,b}$ defined in the following way: Let $x \in [0,1]$. We say that $x \in B_{a,b}$ [resp. $x \in C_{a,b}$] if and only if the zero function is the only compactly supported solution of (1) which is bounded in a neighbourhood of x [resp. continuous at x].

Zenon Moszner *Sur les généralisations du wronskien*

Théorème

Les fonctions $f_1, \dots, f_n$ réelles (complexes) différentiables jusqu'à l'ordre $n-1$ sur un intervalle réel I sont linéairement dépendantes sur cet intervalle si et seulement si le wronskien généralisé

$$\begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ f_1'(x_2) & \dots & f_n'(x_2) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x_n) & \dots & f_n^{(n-1)}(x_n) \end{vmatrix}$$

reste nul pour tous $x_1, \dots, x_n$ dans I.

Théorème

Pour une fonction h réelle (complexe) de deux variables réelles (complexes), ayant les dérivées jusqu'à $h_{y^{n-1}x^{n-1}}$ sur $I \times J$, où I et J sont des ensembles connexes dans $\mathbb{R}$, le wronskien généralisé

$$\begin{vmatrix} h(x_1,y_1), & h_y(x_1,y_2), & \ldots, & h_{y^{n-1}}(x_1,y_n) \\ h_x(x_2,y_1), & h_{yx}(x_2,y_2), & \ldots, & h_{y^{n-1}x}(x_n,y_n) \\ \ldots & \ldots & \ldots & \ldots \\ h_{x^{n-1}}(x_n,y_1), & h_{yx^{n-1}}(x_n,y_2), & \ldots, & h_{y^{n-1}x^{n-1}}(x_n,y_n) \end{vmatrix}$$

reste nul pour tous $x_1, \ldots x_n$ *dans* I *et* $y_1, \ldots y_n$ *dans* J *si et seulement si* $h(x,y) = a_1(x)b_1(y) + \cdots + a_{n-1}(x)b_{n-1}(y)$, *où* $a_k : I \to \mathbb{R}$ *et* $b_k : J \to \mathbb{R}$ *pour* $k = 1, \ldots, n-1$.

Problème

Est il vrai le théorème suivant: Si pour une fonction h comme dans le théorème 2 la matrice de Wronski de h

$$\begin{pmatrix} h(x,y), & h_y(x,y), & \ldots, & h_{y^{n-1}}(x,y) \\ h_x(x,y), & h_{yx}(x,y), & \ldots, & h_{y^{n-1}x}(x,y) \\ \ldots & \ldots & \ldots & \ldots \\ h_{x^{n-1}}(x,y), & h_{yx^{n-1}}(x,y), & \ldots, & h_{y^{n-1}x^{n-1}}(x,y) \end{pmatrix}$$

a le rang égal à $p < n$ pour chaque point $(x,y) \in I \times J$, alors $h(x,y) = a_1(x)b_1(y) + \ldots + a_p(x)b_p(y)$ pour certaines $a_\nu : I \to \mathbb{R}$ et $b_\nu : J \to \mathbb{R}$ et $\nu = 1, \ldots, p$?

Théorème

La réponse est "oui" dans le cas reel si $p = 1$ *pour chaque* $n > 1$ *et si* $p = 2$ *et* $n = 3$ *et si* h *a les dérivées considèrées continues.*

František Neuman *Iteration groups and functional differential equations*

Iteration groups of continuous functions were studied by many authors in connection with flows, dynamical systems, fractional iterates, etc. At the beginning of the eighties the study of solutions of a system of Abel equations, or equivalently, embedding of a finite number of functions into an iteration group as its elements, was initiated by investigating functional differential equations. These questions became important when we considered transformations of functional differential equations with several deviating arguments into special, canonical equations with deviations in the form of shifts: $t + c_i$, c_i being constants (systems of Abel equations), or in the form of rays: $c_i \cdot t$ (systems of Schröder equations).

First we discovered several sufficient conditions for the existence of a solution of a system of Abel equations, then a systematic research was done by M.C. Zdun.

We explain why systems of Abel equations (or systems of Schröder equations) are important for transforming functional differential equations into their canonical forms.

Jolanta Olko *On an application of Banach–Steinhaus theorem*

Applying a set-valued version of Banach–Steinhaus theorem on the uniform boundedness, we generalize theorems concerning iteration semigroups of linear continuous set-valued functions.

Zsolt Páles *Solution and regularity theory of composite functional equations*

We deal with the functional equation

$$\begin{aligned} &f(x+y) - f(x) + \phi(g(y+z) - g(y)) \\ &\qquad = \psi(g(x+y+z) - g(y+z) - g(x+y) + g(y)) \\ &\qquad\qquad (x, y, z > 0,\ x+y+z < \alpha), \end{aligned}$$

where $0 < \alpha \leq \infty$ and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $\phi : J \to \mathbb{R}$, $\psi : H \to \mathbb{R}$ are strictly monotonic functions defined on the sets

$$I :=]0, \alpha[, \qquad J := \{g(y+z) - g(y) \,|\, y, z > 0,\ y+z < \alpha\},$$

$$H := \{g(x+y+z) - g(y+z) - g(x+y) + g(y) \,|\, x, y, z > 0,\ x+y+z < \alpha\}.$$

The solution of the above equation is done in two steps. First, using the Bernstein–Doetsch theorem and the Lebesgue theorem on the almost everywhere differentiability of monotonic functions, we show that J, H are intervals and all the functions f, g, ϕ and ψ are everywhere differentiable. Then, after differentiation with respect to the variables x, y, z, we eliminate the parts where composite functions appear. Thus, an equation containing only f' and g' is obtained, which can be solved by using standard techniques.

Tomasz Powierża *Set-valued iterative square roots of bijections*

There are different ideas how to generalize the notion of an iterative root, especially when a function does not have such a root. We consider a multifunction as a substitute for this notion.

Following an idea of S. Łojasiewicz [Ann. Soc. Polon. Math. **24** (1951), 88-91] we show how to construct a set-valued iterative square root of a bijection which is single-valued if the function has a "real" square iterative root. We show also that every square iterative root of a bijection can be obtained using our construction.

Zbigniew Powązka *Functional equation connected with Schröder's equation*

Let a, b be positive real numbers. Let $f : [0, \infty) \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ fulfill equation

$$af(x) + bf(y) = f(ax + by)g(y - x). \tag{1}$$

There are mostly studied solutions of (1) in the class of locally integrable functions in $\mathbb{R}$.

Thomas Riedel *On some functional equations on the space of distance distribution functions*

Joint work with Kelly Wallace.

We present some lattice theoretic properties of the space of distance distribution functions which are then employed to solve various functional equations on this space. Some of the known results are reviewed and we will present new, joint work with Kelly Wallace on a Pexider type equation on the space of distance distribution functions.

Maciej Sablik *On a Chini's functional equation*

This is a report on a joint work with Thomas Riedel and Prasanna Sahoo.

In connection with some problems related to actuarial mathematics, M. Chini had considered in [1] the following equation

$$f(x+y)+f(x+z)=c\,f(x+h(y,z)), \tag{E}$$

where $f:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}^2\to\mathbb{R}$ are unknown functions, and c is a non-zero fixed constant. Chini gave all differentiable solutions of the equation. We present the continuous solutions of (E) and of some more general equations.

[1] M. Chini, *Sopra un'equazione funzionale da cui discendono due notevoli formule di Matematica attuariale.* Periodico di Matematica **4** (1907), 264-270.

Alexander N. Sharkovsky *Asymptotical behavior of solutions of the simplest nonlinear q-difference equations*

Joint work with G.A. Derfel and E.Yu. Romanenko.

We consider nonlinear q-difference equations of the form

$$x(qt+1)=f(x(t)),\quad q>1,\ t\in R^+.$$

The behavior of solutions is studied as $t\to+\infty$. The investigation of asymptotical properties of solutions is based, in particular, on the comparison of these with the properties of solutions of the difference equation $x(t+1)=f(x(t))$. We show that asymptotical properties of solutions of the q-difference equations are "similar" to those of the corresponding difference equations when $q>1$ is not "very large".

Justyna Sikorska *On a functional equation related to the power means*

M.E. Kuczma in [1] has considered analytic solutions of the functional equation

$$x+g(y+f(x))=y+g(x+f(y))$$

on the real line. In [2] solutions in the class of twice differentiable functions are given.

We present solutions in other classes of functions.

[1] M.E. Kuczma, *On the mutual noncompatibility of homogeneous analytic non-power means.* Aequationes Math. **45** (1993), 300-321.
[2] J. Sikorska, *Differentiable solutions of a functional equation related to the non-power means.* Aequationes Math. **55** (1998), 146-152.

Stanisław Siudut *Cauchy difference operator in some F^*-spaces*

Some abstract stability theorems with applications are presented. In particular, a necessary and sufficient condition of stability of the Cauchy equation in certain class of F^*-spaces is proved.

Fulvia Skof *On some mutually equivalent alternative quadratic equations*

The search of connections between the classes of solutions to different alternative equations stemming from the quadratic equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0$$

for operators f with values in a real normed space, points out a variety of situations in dependence on some peculiarities of the norm of the space (pre-Hilbert, strictly convex etc.). More specially, we consider here the property of pairwise equivalence of such equations, with special regard to the case that equivalence occurs if and only if the target space is endowed with a suitable norm.

Some remarks in this context, giving rise to new characterizations for the kind of norm involved, are presented in this talk.

Andrzej Smajdor *Concave iteration semigroups of linear set-valued functions and differential problems*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the set of all nonempty compact convex subsets of K. Suppose that $\{A^t : t \geqslant 0\}$ is a concave iteration semigroup of continuous linear functions $A^t : K \to \mathrm{cc}(K)$ such that $A^0(x) = \{x\}$. Then there exists a continuous linear set-valued function G such that

$$D_tA^t(x) = A^t(G(x)),$$

where D_t denotes the Hukuhara derivative of $A^t(x)$ with respect to t.

An existence and uniqueness theorem for the differential problem

$$D_t\Psi(t,x) = \Psi(t,G(x)),$$
$$\Psi(0,x) = \Psi_0(x)$$

is given.

Wilhelmina Smajdor *Entire solutions of a functional equation*

Joint work with Andrzej Smajdor.

All entire solutions of order less than 4 of the equation

$$|f(s+it)f(s-it)| = |f(s)^2 - f(it)^2|, \quad s,t \in \mathbb{R}$$

are

$$f(z) = az \ \text{ and } \ f(z) = a \sin bz,$$

where a, b are arbitrary complex constants.

Tomasz Szostok *Equation of Jensen type and orthogonal additivity in normed spaces*

A conditional functional inequality that is often considered in papers by authors dealing with Orlicz spaces is studied. Namely, under some assumptions on the arguments, the right-hand side of the Jensen inequality is multiplied by a constant. Related equation is considered. For functions defined on $(0,\infty)$ solutions of this equation are expressed in terms of multiplicative functions. After suitable modifications the same equation can be considered in normed spaces. Close connection between the resulting equation and that of orthogonal additivity is obtained.

László Székelyhidi *On a functional equation for a two-variable function*

This is a joint work with Prasanna Sahoo.

In this work we prove that the function $f : G \times G \to \mathbb{C}$, where G is a 2-divisible abelian group, satisfies the functional equation

$$f(x-t,y)+f(x+t,y+t)+f(x,y-t) = f(x-t,y-t)+f(x,y+t)+f(x+t,y)$$

for all x, y, t in G if and only if

$$f(x,y) = B(x,y) + \varphi(x) + \psi(y) + \chi(x-y),$$

where $B : G\times G \to \mathbb{C}$ is a biadditive function and $\varphi, \psi, \chi : G \to \mathbb{C}$ are arbitrary functions.

Jaromír Šimša *Some finite decompositions of three-place functions*

In the early 80's, F. Neuman found and stated a general criterion for a given two-place function $h : X \times X \to \mathbb{R}$ (or $\mathbb{C}$) to be decomposed in the form

$$h(x,y) = \sum_{i=1}^{m} a_i(x)b_i(y) \quad (x \in X,\ y \in Y) \tag{1}$$

(see e.g. the book Th.M. Rassias and J.Š. *Finite sums decompositions in mathematical analysis,* John Wiley and Sons, 1995.) Later on, M. Čadek and J.Š. showed that a tree-place function h can be represented as

$$h(x,y,z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \alpha_{ijk} a_i(x)b_j(y)c_k(z) \quad (x \in X,\ y \in Y,\ z \in Z)$$

if and only if h possesses the following three decompositions

$$h(x,y,z)=\sum_{i=1}^{m}a_i(x)u_i(y,z)=\sum_{j=1}^{n}b_j(y)v_j(x,z)=\sum_{k=1}^{p}c_k(z)w_k(x,y),$$

which are of type (1) and hence the criterion of F. Neuman applies to them.

In the present talk, we discuss decompositions of the form

$$h(x,y,z)=\sum_{i=1}^{m}a_i(x)u_i(y,z)+\sum_{j=1}^{n}b_j(y)v_j(x,z)+\sum_{k=1}^{p}c_k(z)w_k(x,y).$$

The main interest is devoted to the crucial case $m=n=p=1$.

Józef Tabor *Stability and the Chebyshev center*

We study the existence and uniqueness of the best approximate of a given function in classes of solutions of the Cauchy type functional equations. The notion of the Chebyshev center is applied to get the results.

Maryna B. Vereykina *Dynamics of solutions of a class of nonlinear boundary value problems*

We consider the simple boundary value problem, namely, the system of two equations with one spatial variable

$$\begin{aligned}\frac{\partial u}{\partial t}&=a\frac{\partial u}{\partial x}+b_1u,\\ \frac{\partial v}{\partial t}&=-a\frac{\partial v}{\partial x}+b_2v\end{aligned}\tag{1}$$

where $x\in[0,1]$, $t\in\mathbb{R}^+$, for $a,b_1,b_2\in\mathbb{R}$, with nonlinear boundary conditions

$$\begin{aligned}u|_{x=0}&=v|_{x=0},\\ u|_{x=1}&=f(v(t))|_{x=1},\quad t\in\mathbb{R}^+\end{aligned}\tag{2}$$

and with the initial conditions

$$\begin{aligned}u|_{t=0}&=u_0(x),\\ v|_{t=0}&=v_0(x),\quad x\in[0,1]\end{aligned}\tag{3}$$

and assume that $a>0$ and f is a nonlinear function.

The solutions of the BVP (1) – (3) are represented as solutions of difference equations with continuous arguments

$$w(\tau+2)=e^{\frac{b_1+b_2}{a}}f(w(\tau))\tag{4}$$

with initial conditions

$$w|_{\tau\in[-1,1)} = \begin{cases} v_0(-\tau)\cdot e^{\frac{b_2}{a}(\tau+1)} & \text{for } \tau\in[-1,0), \\ u_0(\tau)\cdot e^{\frac{b_1\tau+b_2}{a}} & \text{for } \tau\in[0,1). \end{cases}$$

Peter Volkmann *On a Cauchy equation in norm*

Jointly with Roman Ger we investigate the equation

$$\|f(x+y)\| = \|f(x)f(y)\| \tag{C}$$

for functions $f:\mathbb{R}\to C(K)$, K being compact, $K\neq\emptyset$. We have the theorem: Let $f:\mathbb{R}\to C(K)$ solve the inequalities

$$\|f(x)\|\cdot\|f(-x)\|\leqslant 1 \tag{I$_1$}$$

and (for $n\geqslant 2$)

$$\|f(x_1+\cdots+x_n)\|\leqslant\|f(x_1)\cdots f(x_n)\|. \tag{I$_n$}$$

Then there are $\tau\in K$ and $\varphi:\mathbb{R}\to\mathbb{R}$, such that $\varphi(x+y)=\varphi(x)\varphi(y)$ and $\|f(x)\|=|f(x)(\tau)|=\varphi(x)$ (for $x,y\in\mathbb{R}$). The inequalities (I_1), (I_2) imply (C), and it is an interesting question, whether the theorem holds, when only requiring these two inequalities for f.

Anna Wach-Michalik *On special convex compositions with Euler's Gamma function*

Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a function satisfying the following properties:

$$\forall\, x\in\mathbb{R}_+ : f(x+1)=xf(x) \;\text{ and }\; f(1)=1. \tag{$*$}$$

Thus $f(x)=p(x)\Gamma(x)$, where $p:\mathbb{R}_+\to\mathbb{R}_+$ is a periodic function of period 1 and $p(1)=1$ and Γ is the Euler Γ-function defined by the formula

$$\Gamma(x)=\lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}\,. \tag{Γ}$$

Prof. H. H. Kairies proposed to investigate the following set:

$$M=\left\{ g:\mathbb{R}_+\to\mathbb{R}_+ : \begin{array}{l} \text{if } f:\mathbb{R}_+\to\mathbb{R}_+ \text{ satisfies } (*) \\ \text{and } g\circ f \text{ is convex, then } f=\Gamma \end{array}\right\}.$$

By Bohr-Mollerup's theorem we know that $\log\in M$. We find some further elements of the set M and study its properties.

Janusz Walorski *On a problem connected with convexity of derivatives*

The aim of the talk is to present an answer to the question posed by Milan Merkle in [*Conditions for convexity of the derivative and some applications to the Gamma function,* Aequationes Math. **55** (1998), 273-280.]

Problems and Remarks

1. Remark. Let I be a real interval and f be a homeomorphisms mapping I onto I.

During the 6th International Conference on Functional Equations and Inequalities (Muszyna-Złockie, 1997) I proved that

If f has no fixed points, then it can be represented as a composition of at most 2 continuous involutions.

Now I can prove essentialy more:

If f is increasing [decreasing], then it can be represented as a composition of at most 4 [at most 3] continuous involutions.

The functions $(0,1) \ni x \mapsto x^2$, $(0,\infty) \ni x \mapsto x^2$, and $(0,1) \ni x \mapsto 1-x^2$ serve as examples showing that the numbers 2, 4, and 3 are the best possible here.

Witold Jarczyk

2. Remark and Problem. Inscribe a convex n-gon ($n \geqslant 3$) in the unit circle. Now, by drawing tangents, you get a circumscribed n-gon to the circle. László Fuchs and I proved fifty years ago (Compos. Math. **8** (1950), 61-67) that the sum of the areas of these two n-gons have the minimum 6, independent of n, realized by a pair of squares. The proof was analytical, using a function which is first strictly concave then strictly convex.

No elementary (not using calculus, say geometrical) proof has been found since. It seems very difficult to find one.

Clearly no minimal pair of n-gons exists for $n > 4$ because one can always slightly distort squares by joining (several) small additional sides.

Pál Erdös asked several years ago whether among pairs of triangles ("3-gons") constructed as above the pair of regular (equilateral) triangles has the minimal area-sum. I proved, by analytic tools similar to those used for the 1950 theorem, that this is true. I believe for this an elementary proof (geometric or at least without derivation) would be relatively easy to find.

János Aczél

3. Problem. Let $f :]0,\infty[\to \mathbb{R}$. If f is Jensen-convex, i.e.

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \qquad (x,y>0), \tag{1}$$

then the inequality

$$f\left(\frac{x+y+z}{3}\right) \leqslant \frac{f(x)+f(y)+f(z)}{3} \qquad (x,y,z>0)$$

also holds. Therefore, with $z = \sqrt{xy}$,

$$f\left(\frac{x+y+\sqrt{xy}}{3}\right) \leqslant \frac{f(x)+f(y)+f(\sqrt{xy})}{3} \qquad (x,y>0). \tag{2}$$

If f is continuous, then we can prove that (2) implies (1) as well. Is it true that (1) follows from (2) without any regularity assumptions?

Zoltán Daróczy and Zsolt Páles

4. Remark. In a recent issue of the Bulletin of the London Mathematical Society Braden and Byatt-Smith [1] have considered the equation

$$\begin{vmatrix} 1, & 1, & 1 \\ f(x), & f(y), & f(z) \\ f'(x), & f'(y), & f'(z) \end{vmatrix} = 0; \qquad x+y+z=0.$$

Solutions include $f(x)=x$, $f(x)=\exp x$, and $f(x)=p(x)$ where $p(x)$ is the Weierstrass pe function.

This equation arises from an equation of Sutherland (1974)

$$F(x)F(y)+F(y)F(z)+F(z)F(x)=G(x)+G(y)+G(z)$$

subject to $x+y+z=0$.

This was solved by Calogero (1970) at least in the physical situation that gave rise to Sutherland's equation.

[1] H.W. Braden, J.G.B. Byatt-Smith, *On a functional differential equation of determinental type*, Bull. London Math. Soc. **31** (1999), 463-470.

Thomas M.K. Davison

5. Problem. (Presented by János Aczél.)
What is known about the system of equations

$$F(x,x)=x, \quad F[F(x,y),z]=F(x,z) \qquad (x,y,z\in\mathbb{R};\ F:\mathbb{R}^2\to\mathbb{R})?$$

I can prove that if $y\mapsto F(x,y)$ is differentiable, then $F(x,y)\equiv x$ and $F(x,y)\equiv y$ are the only solutions. The original question is also of interest in (general) vector spaces.

Günter Pickert (Giessen)

6. Remark. *Remarque au problème de G. Pickert.*
La théorème suivant est démontré dans la note [L. Piechowicz, S. Serafin, *Solution of the translation equation on some structures*, Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace Mat. **21** (1979), 109-114]:

Theorem

A mapping $F: M\times S\to M$ is a solution of $F(F(x,a),b)=F(x,b)$ if and only if it is constructed as follows

a) *We take a partition* $(M_i)_{i\in I}$ *of* M.

b) *We denote by* $\mathcal{F}$ *the set of all functions* $f : M \to M$ *such that*

$$\bigwedge_{i\in I} (f(M_i) \subset M_i \ \wedge \ \operatorname{card} f(M_i) = 1). \tag{$*$}$$

c) *We take an arbitrary function* $\varphi : S \to \mathcal{F}$.

d) *We define* $F(x,a) := (\varphi(a))(x)$ *for* $(x,a) \in M \times S$.

Nous avons des relations suivantes:
$(*)$ $\iff$ f est stable sur chaque M_i et sa valeur sur M_i est dans M_i $\implies$ f est l'idéntité sur l'ensemble de ses valeurs $\iff$ $\bigwedge_{x\in M} f(f(x)) = f(x)$.
Passons au cas $M = S = \mathbb{R}$. Nous avons des équivalences suivantes:

$$F(a,a) = a \iff (\varphi(a))(a) = a \iff a \in \varphi(a)(\mathbb{R}).$$

La solution générale du problème de G. Pickert est donnée par la construction dans le théorème avec $\varphi(a)$ remplissante la condition: $\bigwedge_{a\in\mathbb{R}} a \in \varphi(a)(\mathbb{R})$.

Exemples

$$\left.\begin{array}{ll} 1) & \bigwedge_{a,x\in\mathbb{R}} \varphi(a)(x) = x, \ \text{donc } F(x,a) = x \\ 2) & \bigwedge_{a,x\in\mathbb{R}} \varphi(a)(x) = a, \ \text{donc } F(x,a) = a \end{array}\right\} \text{les exemples de Pickert.}$$

3)

$$\varphi(a)(x) = \left\{\begin{array}{l} a \text{ pour } x \in [[a],[a]+1) \\ k \text{ pour } x \in [k,k+1) \text{ et } k \neq [a] \text{ et } k = 0, \pm 1, \dots \end{array}\right\} = F(x,a).$$

Zenon Moszner

7. Problem. (Presented by János Aczél).
What is the general solution of the integral equation (somewhat similar to the "integrated Cauchy equation")

$$2f(u) = 2\int_0^\infty f(x+u)f(x)\,dx \quad \text{(if the integral exists).}$$

$f(\lambda) = \lambda e^{-\lambda x}$ is a solution. There are applications in Statistics.

József Bukszár (Miskolc)

8. Problem. (Presented by János Aczél).
1. After the "fermatian" statement "I do not think it right to occupy space by a very full development of the demonstration: the following will be

enough for anyone who has an ordinary acquaintance with functional algebra and the differential calculus", A. De Morgan states in a paper that (with slightly changed notation)

$$\varphi(x+u)+\varphi(y+u)=\varphi[z(x,y)+u] \qquad (x,y,u \text{ and } \varphi \text{ nonnegative}) \tag{1}$$

implies that there exist nonnegative functions c and F such that

$$\varphi(x+u)=c(x)F(u). \tag{2}$$

Ingram Olkin and I were unable to fill in what was missing here. (There are applications to actuarial mathematics, among others.)

2. Could the injectivity assumption be weakened?

Albert W. Marschall (Lumnisland, WA, UBC, Canada)

9. Remark. *To A.W. Marschall's Problem 1.*
If φ is an injection, then we get from (1) with $u=0$: $z(x,y)=\varphi^{-1}[\varphi(x)+\varphi(y)]$. Putting this into (1) and writing

$$\varphi_u(x):=\varphi(x+u), \quad s=\varphi(x), \quad t=\varphi(y) \tag{3}$$

we get

$$\varphi_u\circ\varphi^{-1}(s+t)=\varphi_u\circ\varphi^{-1}(s)+\varphi_u\circ\varphi^{-1}(t) \qquad (s,t\in[\varphi(0),\lim_{x\to\infty}\varphi(x)[\,),$$

the Cauchy equation on a domain from which it can be extended to $[0,\infty[^2$ (or to $\mathbb{R}^2$). Since $\varphi_u\geqslant 0$ (cf. (3)): $\varphi_u\circ\varphi^{-1}(s)=c_u s$; thus $\varphi(x+u)=\varphi_u(x)=c_u\varphi(x)=c(u)\varphi(x)$. This proves (2), moreover $\varphi(x)=be^{A(x)}$ $(b\geqslant 0)$. Here A is an arbitrary injective additive function. This is not enought to guarantee $A(x)=ax$, thus $\varphi(x)=be^{ax}$ but local boundedness (on a small proper interval or on a set of positive measure) of φ is enough.

János Aczél

10. Remark. The following generalization of the regularity result presented in Zs. Páles' talk holds.
Consider the functional equation

$$\begin{aligned} f(x+y)-f(x)+\sum_{i=1}^{n}\phi_i[g_i(y+z_i)-g_i(y)] \\ =\sum_{i=1}^{n}\psi_i[g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y)] \\ (x,y,z_i>0,\ x+y+z_i<\alpha;\ i=1,\dots,n), \end{aligned} \tag{1}$$

where $0<\alpha\leqslant\infty$, $f,g_i:I\to\mathbb{R}$, $\phi_i:J_i\to\mathbb{R}$, $\psi_i:H_i\to\mathbb{R}$ and

$$I=(0,\alpha),$$

$$J_i = \{g_i(y+z_i) - g_i(y) \mid y, z_i > 0,\ y+z_i < \alpha\},$$
$$H_i = \{g_i(y+x+z_i) - g_i(y+x) - g_i(y+z_i) + g_i(y) \mid x, y, z_i > 0,$$
$$y+x+z_i < \alpha\},$$

for $i = 1, \dots, n$. Suppose, that the functions $\phi_1, \dots, \phi_n$, $\psi_1, \dots, \psi_n$, $g_1, \dots, g_n$ and f satisfy (1), furthermore, $\phi_1, \dots, \phi_n$; $\psi_1, \dots, \psi_n$ and $g_1, \dots, g_n$ are strictly monotonic in the same sense, respectively.
Then

- f is strictly convex or strictly concave and continuously differentiable on I;
- $g_1, \dots, g_n$ are strictly convex or strictly concave on I;
- $J_1, \dots, J_n$ and $H_1, \dots, H_n$ are open intervals;
- $\phi_1, \dots, \phi_n$ and $\psi_1, \dots, \psi_n$ are differentiable on $J_1, \dots, J_n$ and $H_1, \dots, H_n$, respectively.

Attila Gilányi and Zsolt Páles

11. Problem. Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing convex function such that $\varphi(0) = 0$. The complementary function in the sense of Young to φ is defined by
$$\varphi^*(u) = \sup\{uv - \varphi(v) :\ v > 0\}, \quad u \geqslant 0.$$
We can describe φ^* also by the formula $\varphi^*(u) = \int_0^u p^{-1}(t)\,dt$, where p^{-1} denotes the inverse of p from the integral representation of φ, i.e. $\varphi(u) = \int_0^u p(t)\,dt$. Consider a new function $h_\varphi : (0, \infty) \to [1, 2]$ given by

$$h_\varphi(u) = \frac{\varphi^{-1}(u)(\varphi^*)^{-1}(u)}{u}, \quad u > 0.$$

The function h_φ stems from the theory of Orlicz spaces. In fact

$$h_\varphi(u) = \frac{\|\chi_{(0,\frac{1}{u})}\|^o_{L^\varphi}}{\|\chi_{(0,\frac{1}{u})}\|_{L^\varphi}}$$

where $\|\cdot\|^o_{L^\varphi}$ denotes the Orlicz norm and $\|\cdot\|_{L^\varphi}$ the Luxemburg norm (cf. [1], [2]).
Since $\|x\|_{L^\varphi} \leqslant \|x\|^o_{L^\varphi} \leqslant 2\|x\|_{L^\varphi}$ for any x in the Orlicz space L^φ it follows that $1 \leqslant h_\varphi(u) \leqslant 2$ for all $u > 0$.
Notice also that if $\varphi(u) = u^p$, $1 \leqslant p < \infty$, then $h_\varphi(u) = p^{\frac{1}{p}}(p')^{\frac{1}{p'}}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. It is easy to show that if $h_\varphi(u) = 2$ for all $u > 0$, then $\varphi(u) = cu^2$ for some $c > 0$ (cf. [2]). I have a rather complicated proof that if

$$h_\varphi(u) = a \quad (1 < a < 2) \quad \text{for all } u > 0$$

then

$$\varphi(u) = cu^p$$

for some $c > 0$ and $p > 1$.

My question is: to find a simple proof of the last statement.

[1] M.A. Krasnoselskii, Ya.B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff Groningen 1961.
[2] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. **5**, Univ. of Campinas, Campinas SP, Brazil. 1989.

Lech Maligranda

12. Problem. Let $\varphi_n, \psi_n, \chi_n : \mathbb{Z} \to \mathbb{C}$ be functions, $B_n : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ biadditive functions, and $F_n : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ $2\mathbb{Z}$-periodic functions in both variable, that is

$$F_n(x + 2z, y) = F_n(x, y + 2z)$$

is satisfied for all x, y, z in $\mathbb{Z}$. let

$$f_n(x, y) = \varphi_n(x) + \psi_n(y) + \chi_n(x - y) + B_n(x, y) + F_n(x, y)$$

for all x, y in $\mathbb{Z}$ and for $n = 1, 2, \ldots$. Suppose that the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{Z}$ to the limit f. Is it true, that f has the form

$$f(x, y) = \varphi(x) + \psi(y) + \chi(x - y) + B(x, y) + F(x, y)$$

where $\varphi, \psi, \chi : \mathbb{Z} \to \mathbb{C}$ are arbitrary, $B : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is biadditive and F is $2\mathbb{Z}$-periodic in the above sense?

László Székelyhidi

13. Remark. *On A. W. Marschall's problem.*
The equation $\varphi(x+u)+\varphi(y+u) = \varphi[z(x, y)+u]$ is Chini's equation (cf. [Talk by M. Sablik, p. 187], [2]). Using [3], it suffices to assume that z is continuous in each variable and φ is locally bounded (either from above or below) and non constant to obtain the continuity of φ. This allows us to use [4] to reduce the equation to

$$\varphi(x + z) = M(x)\varphi(z) + P(x) \qquad (x, z \geqslant 0)$$

and using $z = 0$ and $g(x) = \varphi(x) - \varphi(0)$ we obtain

$$g(x + z) = M(x)g(z) + g(x) \qquad (x, z \geqslant 0).$$

By Corollary 2 in Chapter 15 of [1], we obtain, after the elimination of extraneous solutions, that

$$\varphi(x) = Ke^{ax} \quad \text{and} \quad z(x, y) = \frac{1}{a}\ln(e^{ax} + e^{ay}) \quad \text{for } K > 0,\ a \neq 0.$$

We further note that with the direct methods presented in M. Sablik's talk, it suffices to assume that z is continuous in one variable, but we then need to assume the continuity of φ.

[1] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.

[2] M. Chini, *Sopra un'equazione funzionale da cui discendono due notevoli formule di Matematica attuariale.* Periodico di Matematica **4** (1907), 264-270.

[3] C.T. Ng, *Local boundedness and continuity for a functional equation on topological spaces*, Proc. Amer. Math. Soc. **39.3** (1973), 525-529.

[4] C.T. Ng, *On the functional equation* $f(x) + \sum_{i=1}^{n} g_i(y_i) = h(T(x, y_1, \dots, y_n))$, Ann. Polon. Math. **27** (1973), 329-336.

Thomas Riedel and Maciej Sablik

14. Problem. Characterize

$$\mathcal{F} := \{p \in \mathbb{Z}[X] : \ \forall x \in \mathbb{R} \ \ |p(x)| \leqslant 1 \ \Leftrightarrow \ |x| \leqslant 1\}.$$

Comments. $X^n, T_n(X) \in \mathcal{F}$. If $p \in \mathcal{F}$, then $-p \in \mathcal{F}$. If $p, g \in \mathcal{F}$ then $p \circ q \in \mathcal{F}$. But there are more than these: $kX^4 - kX^2 + 1 \in \mathcal{F}$ for $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Thomas M.K. Davison

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