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Ternary semigroups of linear mappings and matrices

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. Following the classical Green's equivalences we examine, by means of some equivalence relations, the structure of the ternary semigroup of linear mappings. The suitable results for the ternary semigroup of matrices are consequences of the considerations for the linear mappings.

1. Introduction

In this paper we introduce the notion of a ternary semigroup of linear mappings of two vector spaces. The ternary semigroup of linear mappings is a counterpart of the semigroup of endomorphisms of a vector space. By means of the ternary semigroup of linear mappings we can define a ternary (linear) algebra of linear mappings. The last one is isomorphic to a ternary (linear) algebra of matrices. The purpose of the present paper is to examine the structure of the ternary semigroup of linear mappings. To this end, we shall use the relations constructed after the pattern of Green's equivalences in the theory of semigroups. We shall show that the structure of the ternary semigroup of linear mappings is similar to that of the semigroup of endomorphisms of a vector space, but it is more varied. Moreover, we shall give a certain clear characterization of the structure of the inverses in the ternary semigroup of linear mappings.

The results concerning a ternary semigroup of matrices will be immediate consequences of those obtained for the ternary semigroup of linear mappings.

2. Some definitions and results on ternary semigroups

A ternary semigroup is a particular case of the m-semigroup (cf. [5], [6]). We will list some basic definitions and results concerning ternary semigroups which will be needed in this paper.

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Definition 2.1

A *ternary semigroup* is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \longrightarrow A$ is a ternary operation satisfying the following associativity law:

$$\begin{aligned} f(f(x_1, x_2, x_3), x_4, x_5) &= f(x_1, f(x_2, x_3, x_4), x_5) \\ &= f(x_1, x_2, f(x_3, x_4, x_5)) \end{aligned} \tag{2.1}$$

for all $x_1, \ldots, x_5 \in A$.

Because of (2.1) we may write $f(x_1, \ldots, x_5)$ for $x_1, \ldots, x_5 \in A$. If $X_i \subseteq A$ for $i = 1, 2, 3$, then we set

$$f(X_1, X_2, X_3) = \{f(x_1, x_2, x_3) \in A : \ x_i \in X_i \text{ for } i = 1, 2, 3\}.$$

For simplicity we will write $f(A^2, a, A^2) = f(A, A, a, A, A)$. Throughout this paper the letter f will be reserved to denote the ternary operation in a ternary semigroup.

Definition 2.2

Let (A, f) be a ternary semigroup. A nonempty subset $I \subseteq A$ is called:

(a) *a left ideal* if $f(A, A, I) \subseteq I$,

(b) *a right ideal* if $f(I, A, A) \subseteq I$,

(c) *a lateral ideal* if $f(A, I, A) \subseteq I$,

(d) *a two-sided ideal* if I is both a left and right ideal,

(e) *an ideal* if I is a left, right, and lateral ideal.

Let $a \in A$ be an arbitrary fixed element of a ternary semigroup (A, f). The symbols $I_l(a)$, $I_r(a)$, $I_c(a)$, $I_j(a)$, $I(a)$ denote the principal left ideal, right ideal, lateral ideal, two-sided ideal, and ideal generated by the element a, respectively.

A straightforward reasoning yields the following

Proposition 2.3

Let (A, f) be a ternary semigroup. Let a be an arbitrarily fixed element of A. Then

(a) $I_l(a) = a \cup f(A, A, a)$,

(b) $I_r(a) = a \cup f(a, A, A)$,

(c) $I_c(a) = a \cup f(A, a, A) \cup f(A^2, a, A^2)$,

(d) $I_j(a) = a \cup f(A, A, a) \cup f(a, A, A) \cup f(A^2, a, A^2)$,

(e) $I(a) = a \cup f(A, A, a) \cup f(a, A, A) \cup f(A, a, A) \cup f(A^2, a, A^2)$.

Definition 2.4

Let (A, f) be a ternary semigroup. We define the following relations on the set A:

(a) $aLb \iff I_l(a) = I_l(b)$,

(b) $aRb \iff I_r(a) = I_r(b)$,

(c) $aCb \iff I_c(a) = I_c(b)$,

(d) $aJb \iff I_j(a) = I_j(b)$,

(e) $aTb \iff I(a) = I(b)$,

(f) $H = L \cap R$,

(g) $D = L \circ R$.

Applying a similar argument as in the theory of semigroups we can prove that $L \circ R = R \circ L$ and $L \subseteq J$, $R \subseteq J$, $H \subseteq J$, $D \subseteq J$. Thus, all the above relations are equivalences.

Definition 2.5 (cf. [6])

A ternary semigruop (A, f) is said to be *regular* if

$$\forall\, a \in A \;\exists\, x, y \in A \;\; [f(a, x, a, y, a) = a].$$

Let X and Y be nonempty sets. Let $T(X, Y)$ be the set of all mappings of X into Y. Put $T[X, Y] = T(X, Y) \times T(Y, X)$. Define the ternary operation $f : T[X, Y]^3 \longrightarrow T[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3) \qquad (*)$$

for all $(p_i, q_i) \in T[X, Y]$, where $i = 1, 2, 3$.

The algebraic structure $(T[X, Y], f)$ is a ternary semigroup.

Definition 2.6

The ternary semigroup $(T[X, Y], f)$ is called the *ternary semigroup of mappings of sets X and Y*. If $X \cap Y = \emptyset$, then $(T[X, Y], f)$ is called the *disjoint* ternary semigroup of mappings of sets X and Y.

It is easy to check that the ternary semigroups $(T[X, Y], f)$ and $(T[Y, X], f)$ are isomorphic.

A slightly modified argument applied in the proof of Theorem 3 in [5] yields the following theorem.

Theorem 2.7

Every ternary semigroup (A, f) is embeddable into a disjoint ternary semigroup $(T[X, Y], f)$ of mappings of sets X and Y.

In many areas of mathematics mutual connections between algebraic, ordered, topological structures and semigroups (groups) of some morphisms of these structures are studied. For characterizing two structures S_1 and S_2 by means of their morphisms we should consider morphisms from S_1 into S_2, and conversely. The ternary semigroups of morphisms of the structures S_1 and S_2 meet above requirements, and they are useful to achieve the desirable aim. For many structures (e.g. ordered sets, lattices, affine spaces, topological spaces) using ternary semigroups of morphisms we can obtain some clear information about a degree of characterization of these structures by means of their morphisms (cf. [1], [2], [3]). Taking into account the above justification and the remarks contained in the Introduction, it is well-founded to investigate the ternary semigroups of morphisms of various structures.

3. A ternary semigroup of linear mappings

The following two theorems concerning the linear mappings will be needed in this paper:

THEOREM 3.1

Let X and Y be vector spaces over a field K. Let $p : X \longrightarrow Y$ be a linear mapping. Then there exists a subspace X_0 of X such that:

(i) $\mathrm{Ker}\,(p) \oplus X_0 = X$,

(ii) $p|_{X_0} : X_0 \longrightarrow \mathrm{Im}\,(p)$ *is an isomorphism of the vector spaces* X_0 *and* $\mathrm{Im}\,(p)$.

THEOREM 3.2 (cf. [4], Th. 2, p. 83)

Let X and Y be vector spaces over a field K. Let X_0 be a subspace of the space X. Then every linear mapping $p_0 : X_0 \longrightarrow Y$ can be extended to a linear mapping $p : X \longrightarrow Y$, i.e. $p|_{X_0} = p_0$.

Let X and Y be vector spaces over a field K. Let $L(X, Y)$ be the set of all linear mappings of the space X into the space Y. Let us put $L[X, Y] = L(X, Y) \times L(Y, X)$. Define the ternary operation $f : L[X, Y]^3 \longrightarrow L[X, Y]$ by the formula $(*)$ for all $(p_i, q_i) \in L[X, Y]$, where $i = 1, 2, 3$.

The algebraic structure $(L[X, Y], f)$ is a ternary semigroup.

DEFINITION 3.3

The ternary semigroup $(L[X, Y], f)$ is called the *ternary semigroup of linear mappings of vector spaces X and Y over a field K.*

Throughout this paper we shall consider vector spaces over a field K. Suppose that $p \in L(X, Y)$. Put $r(p) = \dim \mathrm{Im}\,(p)$.

LEMMA 3.4

Let X and Y be vector spaces. For arbitrary $p, p' \in L(X, Y)$, $q \in L(Y, X)$ the following conditions are satisfied:

(a) $\mathrm{Im}\,(p) \subseteq \mathrm{Im}\,(p') \iff \exists\, p_1 \in L(X,Y)\, \exists\, q_1 \in L(Y,X)\ [p = p' \circ q_1 \circ p_1]$;

(b) $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p') \iff \exists\, p_1 \in L(X,Y)\, \exists\, q_1 \in L(Y,X)\ [p' = p_1 \circ q_1 \circ p]$;

(c) $r(p) \leqslant r(p') \iff \exists\, p_1, p_2 \in L(X,Y)\, \exists\, q_1, q_2 \in L(Y,X)\ [p = p_1 \circ q_1 \circ p' \circ q_2 \circ p_2]$;

(d) $r(p) \leqslant r(q) \iff \exists\, p_1, p_2 \in L(X,Y)\ [p = p_1 \circ q \circ p_2]$.

Proof. The implications ($\Longleftarrow$) for equivalences (a)-(d) are evident. We shall prove the implications ($\Longrightarrow$).

(a) By Theorem 3.1 it follows that there exists a subspace X_0 of the space X such that:

(i) $\mathrm{Ker}\,(p') \oplus X_0 = X$,

(ii) $p'|_{X_0} : X_0 \longrightarrow \mathrm{Im}\,(p')$ is an isomorphism.

Let q_1 be an extension onto Y of the isomophism $(p'|_{X_0})^{-1} : \mathrm{Im}\,(p') \longrightarrow X_0$ (see Th. 3.2). The implication ($\Longrightarrow$) for condition (a) is a direct consequence of the equality $p = p' \circ q_1 \circ p$.

(b) Put $\mathrm{Ker}\,(p) \oplus X_0 = X$. Let $q_1 \in L(Y,X)$ be an extension onto Y of the isomorphism $(p|_{X_0})^{-1} : \mathrm{Im}\,(p) \longrightarrow X_0$. For every $x \in X$ we have $x = x' + x_0$, where $x' \in \mathrm{Ker}\,(p)$ and $x_0 \in X_0$. Let us notice that $p(x) = p(x_0)$. Since $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p')$, it follows that $p'(x) = p'(x_0)$. Then $(p' \circ q_1 \circ p)(x) = p'(q_1(p(x_0))) = p'(x_0) = p'(x)$, consequently $p' = p' \circ q_1 \circ p$. The implication ($\Longrightarrow$) for condition (b) is satisfied.

(c) Since $r(p) \leqslant r(p')$, there exists a monomorphism $s : \mathrm{Im}\,(p) \longrightarrow \mathrm{Im}\,(p')$. Put $h = s \circ p$. Notice that $\mathrm{Ker}\,(h) = \mathrm{Ker}\,(p)$ and $\mathrm{Im}\,(h) \subseteq \mathrm{Im}\,(p')$. Applying conditions (a) and (b) we get $p = p_1 \circ q_1 \circ h$ and $h = p' \circ q_2 \circ p_2$ for some $p_1, p_2 \in L(X,Y)$ and $q_1, q_2 \in L(Y,X)$. Therefore $p = p_1 \circ q_1 \circ p' \circ q_2 \circ p_2$ for some $p_1, p_2 \in L(X,Y)$ and $q_1, q_2 \in L(Y,X)$.

(d) Put $\mathrm{Ker}\,(p) \oplus X_0 = X$ and $\mathrm{Ker}\,(q) \oplus Y_0 = Y$. In view of the inequality $r(p) \leqslant r(q)$ and Theorem 3.1(ii) we have $\dim X_0 \leqslant \dim Y_0$. Consider the following mappings:

π — the projection of X onto X_0;

η — a monomorphism of X_0 into Y_0;

$p_2 = \eta \circ \pi$;

σ — an extension of the isomorphism $(q \circ \eta)^{-1} : q(\eta(X_0)) \longrightarrow X_0$ onto X;

$p_1 = p \circ \sigma$.

Notice that $p = (p|_{X_0}) \circ \pi$ and $p_1, p_2 \in L(X,Y)$. Therefore we have:

$$\begin{aligned} p_1 \circ q \circ p_2 &= p \circ \sigma \circ q \circ \eta \circ \pi = (p|_{X_0}) \circ \pi \circ \sigma \circ q \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ (\pi|_{X_0}) \circ (\sigma|_{q(\eta(X_0))}) \circ (q|_{\eta(X_0)}) \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \eta^{-1} \circ (q|_{\eta(X_0)})^{-1} \circ (q|_{\eta(X_0)}) \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \eta^{-1} \circ \mathrm{id}_{\eta(X_0)} \circ \eta \circ \pi = (p|_{X_0}) \circ \eta^{-1} \circ \eta \circ \pi \\ &= (p|_{X_0}) \circ \mathrm{id}_{X_0} \circ \pi = (p|X_0) \circ \pi \\ &= p. \end{aligned}$$

Suppose that $(p, q), (p', q') \in L[X, Y]$. We set:

$\mathrm{Im}\,(p, q) = (\mathrm{Im}\,(p), \mathrm{Im}\,(q))$;

$\mathrm{Ker}\,(p, q) = (\mathrm{Ker}\,(p), \mathrm{Ker}\,(q))$;

$r(p, q) = (r(p), r(q))$;

$\mathrm{Im}\,(p, q) \subseteq \mathrm{Im}\,(p', q') \iff \mathrm{Im}\,(p) \subseteq \mathrm{Im}\,(p') \wedge \mathrm{Im}\,(q) \subseteq \mathrm{Im}\,(q')$;

$\mathrm{Ker}\,(p, q) \subseteq \mathrm{Ker}\,(p', q') \iff \mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p') \wedge \mathrm{Ker}\,(q) \subseteq \mathrm{Ker}\,(q')$;

$r(p, q) \leqslant r(p', q') \iff r(p) \leqslant r(p') \wedge r(q) \leqslant r(q')$;

$r(p, q) \leqslant^* r(p', q') \iff r(p) \leqslant r(q') \wedge r(q) \leqslant r(p')$.

According to Lemma 3.4 we have the following

THEOREM 3.5

Assume that $(p, q), (p', q') \in L[X, Y]$. *Then:*

(i) $\mathrm{Im}\,(p, q) \subseteq \mathrm{Im}\,(p', q') \iff \exists\,(p_1, q_1), (p_2, q_2) \in L[X, Y]$ $[(p, q) = f((p', q'), (p_1, q_1), (p_2, q_2))]$;

(ii) $\mathrm{Ker}\,(p', q') \subseteq \mathrm{Ker}\,(p, q) \iff \exists\,(p_1, q_1), (p_2, q_2) \in L[X, Y]$ $[(p, q) = f((p_1, q_1), (p_2, q_2), (p', q'))]$;

(iii) $r(p, q) \leqslant r(p', q') \iff \exists\,(p_i, q_i) \in L[X, Y]\ (i = 1, ..., 4)$ $[(p, q) = f((p_1, q_1), (p_2, q_2), (p', q'), (p_3, q_3), (p_4, q_4))]$;

(iv) $r(p, q) \leqslant^* r(p', q') \iff \exists\,(p_1, q_1), (p_2, q_2) \in L[X, Y]$ $[(p, q) = f((p_1, q_1), (p', q'), (p_2, q_2))]$.

In view of Theorem 3.5, Proposition 2.3, and Definition 2.4 we can formulate the following

COROLLARY 3.6

Assume that $(p, q), (p', q') \in L[X, Y]$. *The following conditions are satisfied:*

(i) $(p', q') \in I_r(p, q) \iff \mathrm{Im}\,(p', q') \subseteq \mathrm{Im}\,(p, q)$;

(ii) $I_r(p, q) = I_r(p', q') \iff \mathrm{Im}\,(p, q) = \mathrm{Im}\,(p', q')$;

(iii) $(p,q)\ R\ (p',q') \iff \mathrm{Im}\,(p,q) = \mathrm{Im}\,(p',q')$;

(iv) $(p',q') \in I_l(p,q) \iff \mathrm{Ker}\,(p,q) \subseteq \mathrm{Ker}\,(p',q')$;

(v) $I_l(p,q) = I_l(p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q')$;

(vi) $(p,q)\ L\ (p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q')$;

(vii) $(p,q)\ H\ (p',q') \iff \mathrm{Ker}\,(p,q) = \mathrm{Ker}\,(p',q') \wedge \mathrm{Im}\,(p,q) = \mathrm{Im}\,(p',q')$.

The following known fact concerning vector spaces will be useful in the proof of the next theorem.

Let X_1 and X_2 be subspaces of a space X such that $X_1 \subseteq X_2$. Then

$$\dim(X/X_2) \leqslant \dim(X/X_1). \tag{3.2}$$

THEOREM 3.7

Let $(p,q),(p',q') \in L[X,Y]$, *then:*

(i) $(p',q') \in I_j(p,q) \iff r(p',q') \leqslant r(p,q)$;

(ii) $I_j(p,q) = I_j(p',q') \iff r(p,q) = r(p',q')$;

(iii) $(p,q)\ J\ (p',q') \iff r(p,q) = r(p',q')$.

Proof. First we will prove (i). Assume that $(p',q') \in I_j(p,q)$. According to Proposition 2.3(d) we consider the following cases:

(a) If $(p',q') = (p,q)$, then $r(p',q') = r(p,q)$.

(b) Suppose that

$$(p',q') = f((p_1,q_1),(p_2,q_2),(p,q)) \text{ for some } (p_1,q_1),(p_2,q_2) \in L[X,Y].$$

It follows from Theorem 3.5(ii) that $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(p')$ and $\mathrm{Ker}\,(q) \subseteq \mathrm{Ker}\,(q')$. By formula (3.2) we get

$$r(p') = \dim(X/\mathrm{Ker}\,(p')) \leqslant \dim(X/\mathrm{Ker}\,(p)) = r(p).$$

Similarly, $r(q') \leqslant r(q)$. Hence $r(p',q') \leqslant r(p,q)$.

(c) Suppose that

$$(p',q') = f((p,q),(p_1,q_1),(p_2,q_2)) \text{ for some } (p_1,q_1),(p_2,q_2) \in L[X,Y].$$

By Theorem 3.5(i), $\mathrm{Im}\,(p',q') \subseteq \mathrm{Im}\,(p,q)$, and therefore $r(p',q') \leqslant r(p,q)$.

(d) Suppose that

$$(p',q') = f((p_1,q_1),(p_2,q_2),(p,q),(p_3,q_3),(p_4,q_4)) \text{ for some } (p_i,q_i) \in L[X,Y],\ i = 1,\dots,4.$$

By Theorem 3.5(iii), $r(p',q') \leqslant r(p,q)$.

Conversely, assume that $r(p',q') \leqslant r(p,q)$. In view of Theorem 3.5(iii), $(p',q') \in I_j(p,q)$.

According to (i) and Definition 2.4(d) we get (ii) and (iii).

PROPOSITION 3.8

The relations D and J in the ternary semigroup $L[X,Y]$ are identical.

Proof. It is enough to prove that $J \subseteq D$. Suppose that $(p,q)J(p',q')$ for $(p,q),(p',q') \in L[X,Y]$. This means that $r(p,q) = r(p',q')$. Since $\dim(\mathrm{Im}\,(p)) = \dim(\mathrm{Im}\,(p'))$, there exists an isomorphism $b : \mathrm{Im}\,(p) \longrightarrow \mathrm{Im}\,(p')$. Put $p_1 = b \circ p$. Notice that $\mathrm{Ker}\,(p_1) = \mathrm{Ker}\,(p)$ and $\mathrm{Im}\,(p_1) = \mathrm{Im}\,(p')$. Similarly one can construct the linear mapping $q_1 \in L(Y,X)$ such that $\mathrm{Ker}\,(q_1) = \mathrm{Ker}\,(q)$ and $\mathrm{Im}\,(q_1) = \mathrm{Im}\,(q')$. Thus $(p,q)\,L\,(p_1,q_1)$ and $(p_1,q_1)\,R\,(p',q')$, and so $(p,q)\,D\,(p',q')$.

According to Proposition 3.8 and Theorem 3.7(iii) we obtain the following

COROLLARY 3.9

If $(p,q),(p',q') \in L[X,Y]$, then $(p,q)\,D\,(p',q')$ if and only if $r(p,q) = r(p',q')$.

The next result is an immediate consequence of Proposition 2.3, Theorems 3.5(iii) and 3.5(iv).

THEOREM 3.10

If $(p,q),(p',q') \in L[X,Y]$, then $(p',q') \in I_c(p,q)$ if and only if $r(p',q') \leqslant r(p,q)$ or $r(p',q') \leqslant^* r(p,q)$.

Assume that $(p,q),(p',q') \in L[X,Y]$. Notice that $r(p,q) \leqslant^* r(p',q')$ and $r(p',q') \leqslant^* r(p,q)$ iff $r(p) = r(q')$ and $r(q) = r(p')$. Therefore we set

$$r(p,q) \stackrel{*}{=} r(p',q') \Longleftrightarrow r(p) = r(q') \wedge r(q) = r(p').$$

LEMMA 3.11

If $r(p,q) \leqslant r(p',q')$ and $r(p',q') \leqslant^* r(p,q)$, then $r(p,q) = r(p',q')$ for $(p,q),(p',q') \in L[X,Y]$.

Proof. Since $r(p') \leqslant r(q) \leqslant r(q') \leqslant r(p)$ and $r(q') \leqslant r(p) \leqslant r(p') \leqslant r(q)$, it follows that $r(p',q') \leqslant r(p,q)$. Consequently $r(p,q) = r(p',q')$.

THEOREM 3.12

If $(p,q),(p',q') \in L[X,Y]$, then $I_c(p,q) = I_c(p',q')$ if and only if $r(p,q) = r(p',q')$ or $r(p,q) \stackrel{*}{=} r(p',q')$.

Proof. We have $I_c(p,q) = I_c(p',q')$ iff $(p,q) \in I_c(p',q')$ and $(p',q') \in I_c(p,q)$. In view of Theorem 3.10 and Lemma 3.11, applying a straightforward calculation we get the desired result.

The following corollary results from Definition 2.4(c) and Theorem 3.12.

COROLLARY 3.13
If $(p,q),(p',q') \in L[X,Y]$, then $(p,q)\ C\ (p',q')$ if and only if $r(p,q) = r(p',q')$ or $r(p,q) \stackrel{*}{=} r(p',q')$.

By Proposition 2.3(e) and Theorem 3.5(iv) applying an argument similar to that in the proof of Theorem 3.7(i) we get the following result:

THEOREM 3.14
If $(p,q),(p',q') \in L[X,Y]$, then $(p',q') \in I(p,q)$ if and only if $r(p',q') \leqslant r(p,q)$ or $r(p',q') \leqslant^* r(p,q)$.

PROPOSITION 3.15
If $(p,q) \in L[X,Y]$, then $I(p,q) = I_c(p,q)$.

The proof follows from Theorems 3.10 and 3.14.
By Proposition 3.15, Theorem 3.12, and Definition 2.4(e) the following corollaries hold.

COROLLARY 3.16
If $(p,q),(p',q') \in L[X,Y]$, then

(i) $I(p,q) = I(p',q')$ *iff* $r(p,q) = r(p',q')$ *or* $r(p,q) \stackrel{*}{=} r(p',q')$,

(ii) $(p,q)\ T\ (p',q')$ *iff* $r(p,q) = r(p',q')$ *or* $r(p,q) \stackrel{*}{=} r(p',q')$.

Corollaries 3.13 and 3.16 yield

COROLLARY 3.17
The relations C and T in the ternary semigroup $L[X,Y]$ are identical.

COROLLARY 3.18
The relations C and D in the ternary semigroup $L[X,Y]$ satisfy the set-inclusion $D \subseteq C$.

This statement follows from Corollaries 3.9 and 3.13.
Let S be an equivalence relation. The symbol $S(x)$ denotes the equivalence class of S containing x.

THEOREM 3.19
If $(p,q) \in L[X,Y]$ and $r(p) = r(q)$, then $C(p,q) = D(p,q)$.

Proof. By Corollary 3.18, $D(p,q) \subseteq C(p,q)$. Suppose that $(p',q') \in C(p,q)$. If $r(p',q') = r(p,q)$, then $(p',q') \in D(p,q)$. If $r(p',q') \stackrel{*}{=} r(p,q)$, then $r(p) = r(q) = r(p') = r(q')$, and so $r(p',q') = r(p,q)$. This means that $(p',q') \in D(p,q)$.

LEMMA 3.20

Assume that $(p,q) \in L[X,Y]$ and $r(p) \neq r(q)$. Then there exists a pair of linear mappings $(p',q') \in L[X,Y]$ such that:

(i) $r(p,q) \neq r(p',q')$,

(ii) $r(p,q) \stackrel{*}{=} r(p',q')$.

Proof. First we will construct $p' \in L(X,Y)$ such that $r(p') = r(q)$. Consider $\mathrm{Ker}\,(q) \oplus Y_0 = Y$ and put $g = q|_{Y_0}$. There exists an epimorphism $s : X \longrightarrow \mathrm{Im}\,(q)$. Put $p' = g^{-1} \circ s$. Thus $p' \in L(X,Y)$ and $r(p') = r(q)$. Similarly one can construct $q' \in L(Y,X)$ such that $r(q') = r(p)$. Therefore the conditions (i) and (ii) hold.

THEOREM 3.21

Assume that $(p,q) \in L[X,Y]$ and $r(p) \neq r(q)$. Then the C-class $C(p,q)$ is the union of the two distinct D-classes D_1 and D_2 defined by the formulas:

$$D_1 = \{(p',q') \in L[X,Y] : \ r(p',q') = r(p,q)\}, \tag{3.3}$$

$$D_2 = \{(p',q') \in L[X,Y] : \ r(p',q') \stackrel{*}{=} r(p,q)\}. \tag{3.4}$$

Proof. Since $r(p) \neq r(q)$, it follows from Lemma 3.20 and C orollary 3.18 that the C-class $C(p,q)$ contains at least two distinct D-classes. Suppose that the C-class $C(p,q)$ contains three pairwise distinct D-classes $D(p_1,q_1)$, $D(p_2,q_2), D(p_3,q_3)$. Thus $r(p_1,q_1) \stackrel{*}{=} r(p_2,q_2)$ and $r(p_2,q_2) \stackrel{*}{=} r(p_3,q_3)$. Consequently $r(p_1) = r(q_2)$, $r(q_1) = r(p_2)$, $r(p_2) = r(q_3)$, $r(q_2) = r(p_3)$, and so $r(p_1) = r(p_3)$ and $r(q_1) = r(q_3)$. Therefore $D(p_1,q_1) = D(p_3,q_3)$. This contradicts our assumption.

We can extend the notion of an inverse in a binary semigroup to the ternary semigroup $L[X,Y]$. A pair $(p',q') \in L[X,Y]$ is called an *inverse* of a pair $(p,q) \in L[X,Y]$ if

$$f((p,q),(p',q'),(p,q)) = (p,q) \quad \text{and} \quad f((p',q'),(p,q),(p',q')) = (p',q').$$

THEOREM 3.22

For every pair $(p,q) \in L[X,Y]$ there exists an inverse $(p',q') \in L[X,Y]$.

Proof. Let X_0, Y_0 be such that $\mathrm{Ker}\,(p) \oplus X_0 = X$ and $\mathrm{Ker}\,(q) \oplus Y_0 = Y$. The mappings $g_1 : X_0 \longrightarrow \mathrm{Im}\,(p)$ and $g_2 : Y_0 \longrightarrow \mathrm{Im}\,(q)$ such that $g_1 = p|_{X_0}$ and $g_2 = q|_{Y_0}$ are isomorphisms.Let $s_1 : X \longrightarrow \mathrm{Im}\,(q)$ and $s_2 : Y \longrightarrow \mathrm{Im}\,(p)$

be epimorphisms such that $s_1|_{\mathrm{Im}\,(q)} = \mathrm{id}_{\mathrm{Im}\,(q)}$ and $s_2|_{\mathrm{Im}\,(p)} = \mathrm{id}_{\mathrm{Im}\,(p)}$. Set $p' = g_2^{-1} \circ s_1$ and $q' = g_1^{-1} \circ s_2$. Evidently $(p', q') \in L[X, Y]$. First we will prove that $f((p, q), (p', q'), (p, q)) = (p, q)$. We have $f((p, q), (p', q'), (p, q)) = (p \circ q' \circ p,\ q \circ p' \circ q)$. Observe that $(p \circ q' \circ p)(x) = (p \circ g_1^{-1} \circ s_2 \circ p)(x) = p(x)$ for every $x \in X$. Similarly, $(q \circ p' \circ q)(y) = q(y)$ for every $y \in Y$. Next we will show that $f((p', q'), (p, q), (p', q')) = (p', q')$. We have $f((p', q'), (p, q), (p', q')) = (p' \circ q \circ p',\ q' \circ p \circ q')$. Notice that

$$\begin{aligned}(p' \circ q \circ p')(x) &= (p' \circ q \circ g_2^{-1} \circ s_1)(x) = p'(s_1(x)) \\ &= g_2^{-1}(s_1(s_1(x))) = g_2^{-1}(s_1(x)) \\ &= p'(x)\end{aligned}$$

for every $x \in X$. Similarly, $(q' \circ p \circ q')(y) = q'(y)$ for every $y \in Y$. Therefore (p', q') is an inverse of (p, q) in $L[X, Y]$.

From Definition 2.5 and Theorem 3.22 it follows

COROLLARY 3.23

The ternary semigroup $L[X, Y]$ is regular.

PROPOSITION 3.24

If $(p', q') \in L[X, Y]$ is an inverse of $(p, q) \in L[X, Y]$, then $r(p, q) \overset{*}{=} r(p', q')$.

This fact follows immediately from Theorem 3.5(iv).

Taking into account Corollary 3.13 and Proposition 3.24 we get

COROLLARY 3.25

If $(p', q') \in L[X, Y]$ is an inverse of $(p, q) \in L[X, Y]$, then $(p, q)\ C\ (p', q')$.

Assume that $E = \{(p, q) \in L[X, Y] : r(p) = r(q)\}$ and $E^* = L[X, Y] \setminus E$. From Corollary 3.9 it follows that $D(p, q) \subseteq E$ for every $(p, q) \in E$. Therefore $E = \bigcup\{D(p, q) : (p, q) \in E\}$.

PROPOSITION 3.26

For every C-class $C_0 \subseteq L[X, Y]$ precisely one of the following two conditions holds:

(i) $C_0 \subseteq E$,

(ii) $C_0 \subseteq E^*$.

Proof. Suppose that there exists a C-class $C_0 \subseteq L[X, Y]$ such that (p_1, q_1), $(p_2, q_2) \in C_0$, $(p_1, q_1) \in E$, and $(p_2, q_2) \in E^*$ for some $(p_1, q_1), (p_2, q_2) \in L[X, Y]$. From the foregoing and Theorem 3.19 it follows that $C_0 = C(p_1, q_1) = D(p_1, q_1) \subseteq E$. We have obtained a contradiction.

Summarizing we get the following theorem.

THEOREM 3.27

Given the ternary semigroup $L[X,Y]$.

(A) *Assume that a C-class $C_0 \subseteq E$. Then every inverse (p',q') of $(p,q) \in C_0$ is an element of the C-class C_0 (C_0 is a D-class).*

(B) *Assume that a C-class $C_0 \subseteq E^*$. Then $C_0 = D_1 \cup D_2$, where the D-classes D_1 and D_2 are defined by the formulas (3.3) and (3.4). Every inverse (p',q') of $(p,q) \in D_1$ is an element of D_2. Every inverse (p',q') of $(p,q) \in D_2$ is an element of D_1.*

Proof. The condition (A) is an immediate consequence of Corollary 3.25. To prove (B), assume that $C_0 = C(p_0,q_0)$. Therefore

$$D_1 = \{(p,q) \in L[X,Y] : r(p,q) = r(p_0,q_0)\}$$

and

$$D_2 = \{(p,q) \in L[X,Y] : r(p,q) \stackrel{*}{=} r(p_0,q_0)\}.$$

Suppose that $(p,q) \in D_1$ and (p',q') is an inverse of (p,q). In view of Proposition 3.24 we get $r(p,q) \stackrel{*}{=} r(p',q')$, and so $r(p_0,q_0) \stackrel{*}{=} r(p',q')$. Consequently $(p',q') \in D_2$. Suppose that $(p,q) \in D_2$ and (p',q') is an inverse of (p,q). By Proposition 3.24, $r(p,q) \stackrel{*}{=} r(p',q')$, and so $r(p_0,q_0) = r(p',q')$. Consequently $(p',q') \in D_1$.

4. A ternary semigroup of matrices

Let K be a field. Let $M(m,n)$ denote the set of all $m \times n$ matrices over K. Put $M[m,n] = M(m,n) \times M(n,m)$. Define the ternary operation $f : M[m,n]^3 \longrightarrow M[m,n]$ by the formula:

$$f((A_1,B_1),(A_2,B_2),(A_3,B_3)) = (A_1B_2A_3,\ B_1A_2B_3)$$

for all $(A_i,B_i) \in M[m,n]$, where $i = 1,2,3$.

The algebraic structure $(M[m,n],f)$ is a ternary semigroup.

DEFINITION 4.1

The ternary semigroup $(M[m,n],f)$ is called the *ternary semigroup of $m \times n$ matrices over a field K.*

Assume that $A \in M(m,n)$. Let $I(A)$ denote the subspace of the vector space K^m spanned by all the columns of the matrix A. Consider the homogeneous system of linear equations

$$A\,X = 0. \tag{4.5}$$

Let $K(A)$ denote the subspace of the vector space K^n of all the solutions of the system (4.5). Consider the linear mapping $p_A \in L(K^n,K^m)$ determined by the matrix A with respect to the canonical bases $(e_1,...,e_n)$ and $(\hat{e}_1,...,\hat{e}_m)$

in the vector spaces K^n and K^m, respectively. It is easy to notice that $K(A) = \mathrm{Ker}\,(p_A)$ and $I(A) = \mathrm{Im}\,(p_A)$. The rank $r(A)$ of the matrix A is identical with the rank of the linear mapping p_A, i.e. $r(A) = r(p_A)$. Assume that $(A,B) \in M[m,n]$. We set $K(A,B) = (K(A),K(B))$, $I(A,B) = (I(A),I(B))$, $r(A,B) = (r(A),r(B))$. The pair of matrices $(A,B) \in M[m,n]$ represents the pair of linear mappings $(p_A,q_B) \in L[K^n,K^m]$. Consider the pairs of matrices $(A_i,B_i) \in M[m,n]$, where $i = 1,2,3$. Then the pair of matrices $(A,B) = f((A_1,B_1),(A_2,B_2),(A_3,B_3))$ represents the pair of linear mappings $(p_A,p_B) = f((p_{A_1},q_{B_1}),(p_{A_2},q_{B_2}),(p_{A_3},q_{B_3}))$.

Taking into account the foregoing considerations we can formulate all the results obtained for the ternary semigroup of linear mappings to get the similar results for the ternary semigroup of matrices.

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