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General continuous solution of a nonlinear functional inequality

Dedicated to Professor Zenon Moszner on the occasion of his 70th birthday

Abstract. In this paper we present theorems on the existence of continuous solutions of the functional inequality (1) in the case where the continuous solution of the corresponding functional equation (2) is not unique.

In the present paper we shall deal with the problem of existence of continuous solutions ψ of the functional inequality

$$\psi[f(x)] \leqslant G(x, \psi(x)), \tag{1}$$

in the case where the continuous solution φ of the corresponding functional equation

$$\varphi[f(x)] = G(x, \varphi(x)) \tag{2}$$

depends on an arbitrary function.

Some problems connected with continuous solutions of nonlinear functional inequalities have been investigated by D. Brydak in [3], [4], [5] and also by K. Baron in [2]. But these results concerned the case of uniqueness of continuous solutions of (2).

1. Let $I = (\xi, a)$, where $\xi < a \leqslant \infty$. We assume that

(i) The function $f : I \longrightarrow \mathbb{R}$ is continuous and strictly increasing in I. Moreover

$$\xi < f(x) < x, \quad x \in I.$$

Remark 1

Hypothesis (i) implies that $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$ (cf. [6], p. 21). Here f^n denotes the n-th iterate of the function f.

As to the function G we assume

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(ii) $G : \Omega \longrightarrow \mathbb{R}$ is continuous, where $\Omega \subset I \times \mathbb{R}$ is an open set.

(iii) For every $x \in I$ the set

$$\Omega_x := \{y : \ (x,y) \in \Omega\} \tag{3}$$

is a nonempty open interval and

$$G(x, \Omega_x) \subset \Omega_{f(x)}. \tag{4}$$

Let $J \subset I$ be an interval such that $\xi \in \operatorname{cl} J$. We shall consider the solutions ψ of inequality (1) and those φ of equation (2) such that their graphs lie in Ω, i.e.

$$\psi(x), \varphi(x) \in \Omega_x \quad \text{for } x \in J \subset I.$$

The class of this solutions we shall denote by $\mathbf{\Psi}(J)$ and $\mathbf{\Phi}(J)$ respectively. Moreover

$$I_k := [f^{k+1}(x_0), f^k(x_0)] \quad \text{for } x_0 \in I,\ k \in \mathbb{N} \cup \{0\}.$$

At first we shall prove an important property of the set Ω which is implied by the above condition.

LEMMA 1

Let us assume that the open set $\Omega \subset I \times \mathbb{R}$ is such that (iii) *holds. Then for two arbitrary points $(x_i, y_i) \in \Omega$, $i = 1, 2$ such that $x_1 < x_2$ there exists a continuous function φ defined in $[x_1, x_2]$ such that $\varphi(x) \in \Omega_x$ for $x \in [x_1, x_2]$ and $\varphi(x_i) = y_i$ for $i = 1, 2$.*

Proof. The lemma results from known facts from the theory of multivalued functions, cf. Propositions 3 and 2 on p. 81 of [1]:

The multifunction $F : I \longrightarrow n(\mathbb{R})$ (the family of all nonempty subsets of $\mathbb{R}$) which has the open graph admits a local selection, whence so does the function $\Phi : [x_1, x_2] \longrightarrow n(\mathbb{R})$ defined by

$$\Phi(x) = \begin{cases} F(x), & x \in (x_1, x_2), \\ \{y_1\}, & x = x_1, \\ \{y_2\}, & x = x_2. \end{cases}$$

Thus there exists a continuous selection $\varphi : [x_1, x_2] \longrightarrow \mathbb{R}$, having the properties stated in the lemma.

It is known (see [6], p. 68) that if the given functions f and G fulfil hypotheses (i)-(iii), then the continuous solution of equation (2) depends on an arbitrary function. It means that for an arbitrary $x_0 \in I$ and an arbitrary continuous function $\varphi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions:

$$\varphi_0(x) \in \Omega_x, \tag{5}$$

$$\varphi_0[f(x_0)] = G(x_0, \varphi_0(x_0)) \tag{6}$$

there exists exactly one continuous solution $\varphi \in \mathbf{\Phi}((\xi, x_0])$ of equation (2) such that

$$\varphi(x) = \varphi_0(x) \quad \text{for } x \in I_0. \tag{7}$$

If we assume additionally that

(iv) For every $x \in I$ the function G is invertible with respect to the second variable,

(v) The function f fulfils condition $f(I) = I$,

(vi) For every $x \in I$ the following condition is fulfilled

$$G(x, \Omega_x) = \Omega_{f(x)}, \tag{8}$$

where Ω_x is defined by (3),

then for an arbitrary $x_0 \in I$ every continuous function $\varphi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (5) and (6) may be extended to a continuous solution $\varphi \in \mathbf{\Phi}(I)$ of (2) (see Theorem 3.1 of [6]).

We are going to present corresponding results for inequality (1).

2. Let us assume (i)-(iv). Hypothesis (iv) guarantees the existence of the function $G^{-1}(x, \cdot)$ inverse to the function G with respect to the second variable.

We introduce the sequence $\{g_k\}$ defined on Ω by the formula

$$\begin{cases} g_0(x,y) = y, \\ g_{k+1}(x,y) = G(f^k(x), g_k(x,y)), \quad k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{9}$$

If we assume (v) and (vi) additionally, then we may put

$$g_{-k-1}(x,y) = G^{-1}(f^{-k-1}(x), g_{-k}(x,y)), \quad k \in \mathbb{N} \cup \{0\}.$$

It is obvious (by virtue of (4) and (8)) that the above sequences are well defined. By induction we see that

$$g_k(x,y) \in \Omega_{f^k(x)}, \quad k \in \mathbb{Z}.$$

Moreover, if φ is a solution of equation (2) then

$$\varphi[f^k(x)] = g_k(x, \varphi(x)), \quad k \in \mathbb{N} \cup \{0\}. \tag{10}$$

We omit elementary proofs of the above properties.

Now, we shall prove the following:

THEOREM 1

Let hypotheses (i) - (iii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions*

$$\psi_0[f(x_0)] \leqslant G(x_0, \psi_0(x_0)), \tag{11}$$

$$\psi_0(x) \in \Omega_x, \quad x \in I_0 \tag{12}$$

there exists a continuous solution $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ *of inequality* (1) *such that*

$$\psi(x) = \psi_0(x), \quad x \in I_0, \tag{13}$$

This solution is given by the formula

$$\psi[f^k(x)] = \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \quad \text{for } x \in I_0,\ k \in \mathbb{N} \cup \{0\} \tag{14}$$

where $\lambda_k : I_k \longrightarrow \mathbb{R}$ *are arbitrary continuous functions fulfilling the following conditions*

$$\lambda_0(x) = 0, \quad x \in I_0, \tag{15}$$

$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}, \quad x \in I_0,\ k \in \mathbb{N} \cup \{0\}, \tag{16}$$

$$\begin{aligned} \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \leqslant G(&f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] \\ &+ g_{k-1}(x, \psi_0(x))), \quad x \in I_0,\ k \in \mathbb{N}, \end{aligned} \tag{17}$$

$$\begin{aligned} \lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) = \lambda_{k-1}&[f^k(x_0)] \\ &+ g_{k-1}(f(x_0), \psi_0[f(x_0)]), \quad k \in \mathbb{N}. \end{aligned} \tag{18}$$

Moreover, all continuous solutions $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Proof. Let us fix $x_0 \in I$ and an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions (11) and (12). Moreover let us fix an arbitrarily chosen sequence of continuous functions $\lambda_k : I_k \longrightarrow \mathbb{R}$ fulfilling conditions (15) - (18)[1]. If we define the function ψ by formula (14), then we have (13) from (15) and $\psi(x) \in \Omega_x$, for $x \in (\xi, x_0]$ by virtue of (16).

Now, let $x \in (\xi, f(x_0))$. If $k \in \mathbb{N}$ and $t \in I_0$ are such that $x = f^k(t)$, then (17) implies

$$\begin{aligned} \psi[f(x)] &= \psi[f^{k+1}(t)] = \lambda_{k+1}[f^{k+1}(t)] + g_{k+1}(t, \psi_0(t)) \\ &\leqslant G(f^k(t), \lambda_k[f^k(t)] + g_k(t, \psi_0(t))) \\ &= G(x, \psi(x)). \end{aligned}$$

Consequently formula (14) defines a solution of (1) in $(\xi, x_0]$. Now, we shall prove that ψ is continuous in $(\xi, x_0]$.

The function ψ is continuous in every interval $(f^{i+1}(x_0), f^i(x_0))$, $i \in \mathbb{N}$. By (13), (14), (18) and the continuity of the functions f, G, λ_k it follows that

$$\lim_{x \to f^i(x_0)} \psi(x) = \psi[f^i(x_0)], \quad i \in \mathbb{N}. \tag{19}$$

Indeed, we have

[1] As to the construction of $\{\lambda_k\}$, cf. the Remark 2

$$\begin{aligned}\lim_{x\to f^{k+1}(x_0)^+}\psi(x) &= \lim_{x\to f(x_0)^+}\psi[f^k(x)]\\ &= \lambda_k[f^{k+1}(x_0)] + g_k(f(x_0),\psi_0[f(x_0)])\\ &= \lambda_{k+1}[f^{k+1}(x_0)] + g_{k+1}(x_0,\psi_0(x_0))\\ &= \psi[f^{k+1}(x_0)],\end{aligned}$$

$$\begin{aligned}\lim_{x\to f^{k+1}(x_0)^-}\psi(x) &= \lim_{x\to x_0^-}\psi[f^k(x)]\\ &= \lambda_{k+1}[f^{k+1}(x_0)] + g_{k+1}(x_0,\psi_0(x_0))\\ &= \psi[f^{k+1}(x_0)].\end{aligned}$$

This completes the proof of (19).

Let us now assume that $\psi \in \mathbf{\Psi}((\xi, x_0])$ fulfils (1). It is sufficient to put

$$\psi_0(x) := \psi(x) \quad \text{for } x \in I_0, \tag{20}$$

$$\lambda_k[f^k(x)] := \psi[f^k(x)] - g_k(x,\psi(x)) \quad \text{for } x \in I_0,\ k \in \mathbb{N}\cup\{0\}, \tag{21}$$

to see that conditions (11), (12), (15)-(18) hold and that the solution ψ is represented by formula (14). This ends the proof of the theorem.

If we assume (iv)-(vi) additionally, then we may prove the following:

THEOREM 2

Let hypotheses (i)-(vi) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling* (11) *and* (12), *there exists a continuous solution $\psi \in \mathbf{\Psi}(I)$ of inequality* (1) *such that* (13) *holds.*

This solution is given by formulas (14) *and*

$$\psi[f^{-k}(x)] = l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x)) \quad \textit{for } x \in I_0,\ k \in \mathbb{N} \tag{22}$$

where $\lambda_k : I_k \longrightarrow \mathbb{R}$, $l_k : I_{-k} \longrightarrow \mathbb{R}$ are arbitrarily chosen continuous functions fulfilling conditions (15)-(18) *and moreover the conditions*

$$l_0(x) = 0 \quad \textit{for } x \in I_0, \tag{23}$$

$$l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x)) \in \Omega_{f^{-k}(x)}, \quad x \in I_0,\ k \in \mathbb{N}, \tag{24}$$

$$\begin{aligned}&l_{k+1}[f^{-k-1}(x)] + g_{-k-1}(x,\psi_0(x))\\ &\quad\leqslant G(f^{-k}(x), l_k[f^{-k}(x)] + g_{-k}(x,\psi_0(x))), \quad x \in I_0,\ k \in \mathbb{N},\end{aligned} \tag{25}$$

$$\begin{aligned}&l_{k+1}[f^{-k-1}(x_0)] + g_{-k-1}(x_0,\psi_0(x_0))\\ &\quad= l_k[f^{-k-1}(x_0)] + g_{-k}(f(x_0),\psi_0[f(x_0)]), \quad k \in \mathbb{N}.\end{aligned} \tag{26}$$

Moreover, all continuous solutions $\psi \in \mathbf{\Psi}(I)$ of inequality (1) *may be obtained in this way.*

The proof of the above theorem runs analogously to that of Theorem 1 and will be omitted here.

REMARK 2

Contrary to the situation with continuous solutions of equation (2) in I, a continuous function ψ_0 fulfilling (11), (12) cannot be extended uniquely to a continuous solution of inequality (1) in I. This follows from the fact that the sequences of continuous functions $\{\lambda_k\}$, $\{l_k\}$ fulfilling (15)-(18) and (23)-(26) may be chosen in various ways.

We show a construction of a sequence of continuous functions $\lambda_k : I_k \longrightarrow \mathbb{R}$ such that conditions (15)-(18) hold.

Let us take a continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (11) and (12). We put

$$y_{1,0} := \psi_0[f(x_0)] - G(x_0, \psi_0(x_0)).$$

Let us fix a $y_{1,1} \leqslant 0$ fulfilling additionally the condition

$$y_{1,1} + G(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^2(x_0)}.$$

It is possible since $\Omega_{f^2(x_0)}$ is a nonempty open interval and

$$G(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^2(x_0)}.$$

Thus we may take (by virtue of Lemma 1) a continuous function $\mu_1 : I_1 \to \mathbb{R}$ such that the conditions

$$\mu_1[f(x_0)] = \psi_0[f(x_0)], \qquad \mu_1[f^2(x_0)] = y_{1,1} + G(f(x_0), \psi_0[f(x_0)]),$$
$$\mu_1[f(x)] \in \Omega_{f(x)}, \quad x \in I_0$$

hold. For the function

$$\lambda[f(x)] := \mu_1[f(x)] - G(x, \psi_0(x)), \quad x \in I_0$$

we now put

$$\lambda_1(x) := \frac{1}{2}\left(\lambda(x) - |\lambda(x)|\right), \quad x \in I_1 .$$

It is obvious that $\lambda_1 : I_1 \longrightarrow \mathbb{R}$ is a continuous function such that the conditions

$$\lambda_1[f(x_0)] = y_{1,0}, \quad \lambda_1[f^2(x_0)] = y_{1,1}, \quad \lambda_1(x) \leqslant 0, \quad x \in I_1,$$
$$\lambda_1[f(x)] + G(x, \psi_0(x)) \in \Omega_{f(x)}, \quad x \in I_0$$

hold. If we assume that we have continuous functions $\lambda_0, \ldots, \lambda_{k-1}$ defined on I_j, $j = 0, \ldots, k-1$, respectively, and fulfilling

$$\lambda_i[f^i(x)] + g_i(x, \psi_0(x)) \leqslant G(f^{i-1}(x), \lambda_{i-1}[f^{i-1}(x)] + g_{i-1}(x, \psi_0(x))),$$
$$x \in I_0, i = 1, \ldots, k-1$$

$$\lambda_i[f^i(x)] + g_i(x, \psi_0(x)) \in \Omega_{f^i(x)}, \quad x \in I_0 , \; i = 1, \ldots, k-1$$

$$\lambda_i[f^i(x_0)] + g_i(x_0, \psi_0(x_0)) = \lambda_{i-1}[f^i(x_0)] + g_{i-1}(f(x_0), \psi_0[f(x_0)]),$$
$$i = 1, \dots, k-1$$

then it is sufficient to put

$$y_{k,0} := \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]) - g_k(x_0, \psi_0(x_0))$$

and fix a $y_{k,1}$,

$$y_{k,1} \leqslant G(f^k(x_0), \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)])) - g_k(f(x_0), \psi_0[f(x_0)])$$

fulfilling the condition

$$y_{k,1} + g_k(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^{k+1}(x_0)}.$$

It is possible because of the relations

$$G(f^k(x_0), \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)])) \in \Omega_{f^{k+1}(x_0)},$$
$$g_k(f(x_0), \psi_0[f(x_0)]) \in \Omega_{f^{k+1}(x_0)}.$$

Thus we may take, again by Lemma 1, a continuous function $\mu_k : I_k \longrightarrow \mathbb{R}$ such that the condition

$$\mu_k[f^k(x_0)] = y_{k,0} + g_k(x_0, \psi_0(x_0)), \quad \mu_k[f^{k+1}(x_0)] = y_{k,1} + g_k(f(x_0), \psi_0[f(x_0)]),$$
$$\mu_k[f^k(x)] \in \Omega_{f^k(x)}, \quad x \in I_0$$

Now, for the functions γ, H defined by formulas:

$$\gamma[f^k(x)] := \mu_k[f^k(x)] - g_k(x, \psi_0(x)), \quad x \in I_0,$$

$$H[f^k(x)] := G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))) - g_k(x, \psi_0(x)), \; x \in I_0,$$

we put

$$\lambda_k(x) := \frac{1}{2}\left(\gamma(x) + H(x) - |\gamma(x) - H(x)|\right), \quad x \in I_k.$$

It is easy to notice that $\lambda_k : I_k \longrightarrow \mathbb{R}$ is a continuous function such that the conditions

$$\lambda_k[f^k(x_0)] = y_{k,0}, \quad \lambda_k[f^{k+1}(x_0)] = y_{k,1},$$
$$\lambda_k[f^k(x)] \leqslant G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))) - g_k(x, \psi_0(x)), \quad x \in I_0,$$
$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}, \quad x \in I_0.$$

hold. This ends the inductive construction of the sequence $\{\lambda_k\}$. In a similar way we may construct a sequence $\{l_k\}$ fulfilling (23)-(26).

3. Let us assume (i)-(iii) again. We will consider the following assumption stronger than (iv):

(vii) For every $x \in I$ the function G is strictly increasing with respect to the second variable.

In this section we shall characterize continuous solutions ψ of inequality (1) which fulfil additionaly the following condition

$$L_k^{\psi}[f(x)] \in G(x, \Omega_x), \quad x \in I,\ k \in \mathbb{N}. \tag{27}$$

where the sequence $\{L_k^{\psi}\}$ is defined by the reccurence formula

$$\begin{cases} L_0^{\psi}(x) = \psi(x), \\ L_{k+1}^{\psi}(x) = G^{-1}(x, L_k^{\psi}[f(x)]), \quad k \in \mathbb{N}. \end{cases} \tag{28}$$

It is easy to notice that (cf. (vii)) the sequence $\{L_k^{\psi}\}$ is decreasing and if φ is a solution of (2), then $L_k^{\varphi}(x) = \varphi(x)$, $k \in N$.

Moreover if ψ is a solution of inequality (1) then we obtain by induction that

$$\psi[f^k(x)] \leqslant g_k(x, \psi(x)), \quad k \in \mathbb{N}. \tag{29}$$

and that the function $g_k(x, \cdot)$ is also strictly increasing.

Now, we may formulate the following

THEOREM 3

Let hypotheses (i)-(iii), (vii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0 : I_0 \longrightarrow \mathbb{R}$ *fulfilling* (11), (12) *and, moreover, the condition*

$$\psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}) \tag{30}$$

there exists a continuous solution $\psi \in \mathbf{\Psi}((\xi, x_0])$ *of inequality* (1) *fulfilling* (13) *and such that*

$$L_k^{\psi}[f(x)] \in G(x, \Omega_x), \quad x \in (\xi, x_0],\ k \in \mathbb{N} \cup \{0\}. \tag{31}$$

This solution is given by the formula

$$\psi[f^k(x)] = g_k(x, \gamma_k(x) + \psi_0(x)) \quad \text{for } x \in I_0,\ k \in \mathbb{N} \cup \{0\}, \tag{32}$$

where γ_k *are arbitrary continuous functions defined in* I_0 *and fulfilling the conditions:*

$$\gamma_0(x) = 0 \quad \text{for } x \in I_0, \tag{33}$$

$$\text{the sequence } \{\gamma_k\} \text{ is decreasing in } I_0, \tag{34}$$

$$\gamma_k(x) + \psi_0(x) \in \Omega_x \quad \text{for } x \in (f(x_0), x_0],\ k \in \mathbb{N} \cup \{0\}, \tag{35}$$

$$\gamma_k[f(x_0)] + \psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}), \quad k \in \mathbb{N}, \tag{36}$$

$$g_k(x_0, \gamma_k(x_0) + \psi_0(x_0)) = g_{k-1}(f(x_0), \gamma_{k-1}[f(x_0)] + \psi_0[f(x_0)]), \quad k \in \mathbb{N}. \tag{37}$$

Moreover, all continuous solutions $\psi \in \mathbf{\Psi}((\xi, x_0])$ *of inequality* (1), *fulfilling* (31) *may be obtained in this manner.*

Proof. Let us fix $x_0 \in I$ and an arbitrary continuous function $\psi_0 : I_0 \longrightarrow \mathbb{R}$ fulfilling (11), (12) and (30). Moreover let us fix an arbitrarily chosen sequence of continuous functions $\{\gamma_k\}$ defined in I_0 and fulfilling conditions (33)-(37) [2]

If we define the function ψ by formula (32) then we have (13) from (33) and we obtain (31) from (30), (32), (35), (36). Indeed, let $x \in (\xi, x_0]$. If $k \in \mathbb{N} \cup \{0\}$ and $t \in I_0$ are such that $x = f^k(t)$, then formulas (28), (32) imply

$$\begin{aligned} L_0^{\psi}[f(x)] &= \psi[f(x)] = \psi[f^{k+1}(t)] = g_{k+1}(t, \gamma_{k+1}(t) + \psi_0(t)) \in g_{k+1}(t, \Omega_t) \\ &= G(f^k(t), g_k(t, \Omega_t)) \\ &\subset G(f^k(t), \Omega_{f^k(t)}) \\ &= G(x, \Omega_x). \end{aligned}$$

Thus (31) holds for $k = 0$. Now, we introduce the sequence $\{h_k\}$ defined by the formula

$$\begin{cases} h_0(x,y) = y, \\ h_{k+1}(x,y) = G^{-1}(f(x), h_k(f(x), y)), \quad k \in \mathbb{N}. \end{cases} \tag{38}$$

It is easy to prove (by induction) that

$$L_k^{\psi}[f(x)] = h_k(x, \psi[f^{k+1}(x)]), \quad x \in (\xi, x_0], \ k \in \mathbb{N} \cup \{0\} \tag{39}$$

and (9) with (38) imply

$$h_k(x, g_{k+1}(x, \Omega_x)) = G(x, \Omega_x), \quad x \in (\xi, x_0], \ k \in \mathbb{N} \cup \{0\}. \tag{40}$$

Let us fix a $k \in \mathbb{N}$. From (39) and (40) we have

$$L_k^{\psi}[f(x)] = h_k(x, \psi[f^{k+1}(x)]) \in h_k(x, g_{k+1}(x, \Omega_x)) = G(x, \Omega_x), \quad x \in (\xi, x_0].$$

Consequently condition (31) holds.

Now, let $x \in (\xi, f(x_0))$. If $k \in \mathbb{N}$ and $t \in I_0$ are such that $x = f^k(t)$ then the monotonicity of $g_k(x, \cdot)$ and (34) imply

$$\begin{aligned} \psi[f(x)] &= \psi[f^{k+1}(t)] = g_{k+1}(t, \gamma_{k+1}(t) + \psi_0(t)) \\ &\leqslant g_{k+1}(t, \gamma_k(t) + \psi_0(t)) \\ &= G(f^k(t), g_k(t, \gamma_k(t) + \psi_0(t))) \\ &= G(x, \psi(x)). \end{aligned}$$

Consequently, formula (32) defines the solution of (1) in $(\xi, x_0]$. The function ψ is continuous in every interval $(f^{i+1}(x_0), f^i(x_0))$, $i \in \mathbb{N}$, and it is sufficient to show that (19) holds. The proof of (19) runs analogously as that in the proof of Theorem 1 and it will be omitted here.

Now, let us assume that $\psi \in \boldsymbol{\Psi}((\xi, x_0])$ is a continuous solution of (1) fulfilling (31). It is sufficient to define ψ_0 by (20) and to put

[2] see Remark 3

$$\gamma_k(x) := L_k^{\psi}(x) - \psi_0(x) \quad \text{for } x \in I_0,\ k \in \mathbb{N} \cup \{0\}. \tag{41}$$

Let us notice that (33), (35)-(37) hold. Inequality (1) implies also condition (34). We may prove by induction on p that:

$$L_{k-p}^{\psi}[f^p(x)] = g_p(x, L_k^{\psi}(x)), \quad \textit{for } x \in I_0,\ p = 0, 1, \ldots, k,\ k \in \mathbb{N}.$$

This implies that ψ may be represented by formula (32) and ends the proof of the theorem.

We have also the following theorem corresponding to Theorem 2. Its simple proof is omitted.

THEOREM 4

Let hypotheses (i)-(iii), (v)-(vii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0 : I_0 \longrightarrow \mathbb{R}$ *fulfilling* (11) *and* (12), *there exists a continuous solution* $\psi \in \mathbf{\Psi}(I)$ *of inequality* (1) *such that* (13) *and* (27) *holds. This solution is given by formula* (32) *and*

$$\psi[f^{-k}(x)] = g_{-k}(x, \eta_k(x) + \psi_0(x)), \quad x \in I_0,\ k \in \mathbb{N}$$

where γ_k, η_k *are arbitrary chosen sequences of continuous functions defined in* I_0 *such that* (33)-(37) *and, moreover, the conditions*

$$\eta_0(x) = 0 \quad \textit{for } x \in I_0,$$

$$\textit{the sequence } \{\eta_k\} \textit{ is decreasing in } I_0,$$

$$\eta_k(x) + \psi_0(x) \in \Omega_x \quad \textit{for } x \in I_0,\ k \in \mathbb{N},$$

$$g_{-k}(f(x_0), \eta_k[f(x_0)] + \psi_0[f(x_0)]) = g_{-k+1}(x_0, \eta_k(x_0) + \psi_0(x_0)), \quad k \in \mathbb{N},$$

are satisfied. Moreover, all continuous solutions $\psi \in \mathbf{\Psi}(I)$ *of inequality* (1) *fulfilling* (27) *may be obtained in this manner.*

REMARK 3

We may construct a sequence $\{\gamma_k\}$ fulfilling conditions (33)-(37) in a similar way as in Remark 2. However, having taken a solution $\psi \in \mathbf{\Psi}((\xi, x_0])$ of (1) defined by formula (14) and fulfilling (31) we may also define a sequence $\{\gamma_k\}$ by (41), cf. (28).

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