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D'Alembert's functional equation and Chebyshev polynomials

Abstract. We consider D'Alembert's functional equation (1) where the domain of the function f is the additive group of the integers and the codomain is an arbitrary commutative ring with identity. We show that if $f(0) = 1$ then $f(n)$ is the value of the Chebyshev polynomial $T_{|n|}$ evaluated at $f(1)$.

1. Introduction

In 1750 d'Alembert introduced the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{1}$$

This arose in modelling the motion of a stretched string and in the foundations of mechanics. See Aczél & Dhombres [1; Chs. 1 & 8] for more details. This equation is also called the cosine equation since the cosine certainly satisfies it.

To see how one can "find" the cosine in (1) assume that the not identically zero solution f is twice continuously differentiable. Then from

$$\frac{f(x+y) + f(x-y) - 2f(x)}{y^2} = f(x)\,\frac{f(y) + f(-y) - 2f(0)}{y^2}, \tag{2}$$

using $f(-y) = f(y)$ and $f(0) = 1$ which follow from (1), and taking the limit as y tends to 0, one obtains

$$f''(x) = f(x)f''(0). \tag{3}$$

Hence

$$f(x) = \begin{cases} \cos(cx) & \text{if } f''(0) \leqslant 0 \\ \cosh(cx) & \text{if } f''(0) > 0 \end{cases} \tag{4}$$

where $c := \sqrt{|\, f''(0)\, |}$.

It is worth remarking that d'Alembert was among those calling for a theory of limits that would justify the argument just given. It is also worth remarking that the technique of reducing a functional equation (such as (1))to a differential

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equation (such as (3)) has been a mainstay for the past 250 years. Indeed Hilbert [1; p. 375], in proposing his fifth problem at the beginning of the 20^{th} century, said

> Specifically, we come to the broad and not uninteresting field of functional equations, hitherto largely investigated by assuming differentiability of the occurring functions. Equations treated in the literature, particularly the functional equations treated by Abel with such incisiveness, show no intrinsic characteristics that require the assumption of differentiability of the occurring functions...

Indeed Kannappan [3] solved (1) in great generality, in particular where x, y are elements of an additive abelian group and $f(x)$ is a complex number, without assuming any regularity (e.g. continuity) in the function. Kannappan proved that given a solution of d'Alembert's functional equation with $f(0) = 1$ there is a function $e : \mathrm{dom}\,(f) \to \mathbb{C}$ such that $e(0) = 1$ and $e(x+y) = e(x)e(y)$ and $2f(x) = e(x) + e(-x)$ for all $x \in \mathrm{dom}\,(f)$. In the classical cases $e(x) = e^{icx}$ for $\cos(cx)$ and $e(x) = e^{cx}$ for $\cosh(cx)$.

In this paper equation (1) (d'Alembert's equation) is solved when the domain of f is the additive group of the integers and the codomain of f is a commutative ring R. It is here that, perhaps surprisingly, the Chebyshev polynomials show up.

Theorem

Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. Then

$$f(m+n) + f(m-n) = 2f(m)f(n); \quad (m,n) \in \mathbb{Z}^2, \tag{5}$$

if, and only if

$$f(n) = T_{|n|}(f(1)); \quad n \in \mathbb{Z}. \tag{6}$$

Definition 1

$T_m \in \mathbb{Z}[X]$ is given by, for $m \in \mathbb{N}_o$,

$$T_m(X) = \sum_{k=0}^{q} \binom{m}{2k} X^{m-2k} \left(X^2 - 1\right)^k, \tag{7}$$

where q is the largest integer with $2q \leqslant m$. For equation (7) see Temme [4] eq. (6.39).

If $p \in \mathbb{Z}[X]$, say

$$p(X) = p_0 + p_1 X + \cdots + p_d X^d$$

and if $r \in R$ then, as usual,

$$p(r) := p_0 + p_1 r + \cdots + p_d r^d.$$

The general reference for Chebyshev polynomials (Tchebycheff — hence T) is

Rivlin [3]. The occurrence of T_n here is really as a polynomial not a polynomial function as in Rivlin generally.

The necessity ((5) implies (6)) is proved in Proposition 2. The sufficiency is proved in Proposition 3. Both use Proposition 1 that reduces the d'Alembert equation to a second order linear difference equation.

The identically zero function satisfies (5) but is not of the form $T_{(n)}(f(1))$: this is why $f(0) = 1$ is a constant assumption.

It is important to note that the domain of an equation must always be made clear: the equations

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (x,y) \in \mathbb{R}^2$$

and

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (x,y) \in \mathbb{R}_+^2 \ \ (\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\})$$

are different even though for both of them dom $(f) = \mathbb{R}$.

2. Reduction to a difference equation

The result below shows that the two variables m, n in (5) can, over $\mathbb{Z}$, be replaced by a single variable equation.

PROPOSITION 1

Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. Then f satisfies equation (5) if, and only if,

$$f(n+2) + f(n) = 2f(1)f(n+1) \quad n \in \mathbb{Z}. \tag{8}$$

Proof. Assume f satisfies equation (5). Then

$$f(n+1+1) + f(n+1-1) = 2f(n+1)f(1);$$

this is equation (8).

Assume conversely that f satisfies equation (8). Then f is even — that is

$$f(-n) = f(n) \quad n \in \mathbb{Z}. \tag{9}$$

It clearly suffices to prove (9) for all $n \in \mathbb{N}_0$. This is proved by induction on $n \in \mathbb{N}_0$. It is trivially true for $n = 0$. Also $f(-1) = f(1)$ since $f(0) = 1$. Assume it is true for all $n \in \mathbb{N}_0$ with $0 \leqslant n \leqslant N$, where $N \geqslant 1$. Then it is true for $n = N + 1$: using (8)

$$f(N+1) + f(N-1) = 2f(1)f(N).$$

Since $f(N-1) = f(-(N-1))$ and $f(N) = f(-N)$ by the induction hypothesis

$$f(N+1) + f(1-N) = 2f(1)f(-N). \tag{10}$$

But again using equation (8) with $n = -N - 1$

$$f(-N+1)+f(-N-1)=2f(1)f(-N). \tag{11}$$

Comparing (10) and (11) yields $f(-N-1)=f(N+1)$. Thus (9) is true for all $n \in \mathbb{N}_0$ by induction.

To show that f satisfies equation (5) the function $F:\mathbb{Z}^2 \to R$ defined next must be identically zero:

$$F(k,\ell) := f(k+\ell)+f(k-\ell)-2f(k)f(\ell) \quad (k,\ell)\in\mathbb{Z}^2. \tag{12}$$

Now since f, assumed to satisfy equation (8), has been shown to be even

$$F(k,\ell)=F(\ell,k)=F(-k,\ell) \quad (k,\ell)\in\mathbb{Z}^2. \tag{13}$$

So iterating the involutions $(k,\ell)\to(\ell,k)$ and $(k,\ell)\to(-k,\ell)$ it follows that F is identically zero on $\mathbb{Z}^2$ if, and only if, F is zero on

$$X := \left\{(k,\ell)\in\mathbb{Z}^2 :\ 0\leqslant \ell \leqslant k\right\}. \tag{14}$$

Using equation (8) to express $f(k+\ell)$ in terms of $f(k+\ell-1)$ and $f(k+\ell-2)$ and similarly $f(k-\ell)$ in terms of $f(k-\ell-1)$ and $f(k-\ell-2)$ it follows that

$$F(k,\ell)=2f(1)F(k-1,\ell)-F(k-2,\ell); \quad (k,\ell)\in\mathbb{Z}^2. \tag{15}$$

The 'size' of $(k,\ell)\in X$ is $k+\ell$ — the taxicab distance from $(0,0)$ to (k,ℓ) in X. By induction on the 'size' of $(k,\ell)\in X$ it is easy to show, using equation (15) that F is zero on X. [$F(0,0)=0$ is true since $f(0)=1$, as is $F(1,0)=0$. $F(1,1)=f(2)+f(0)-2f(1)f(1)=0$, since f satisfies (8)].

This completes the proof that (8) implies (5).

Corollary 1

Let $f:\mathbb{Z}\to R$ with $f(0)=1$. Then f satisfies equation (8) if, and only if, it is even (equation (9)) and

$$f(n+2)+f(n)=2f(1)f(n+1); \quad n\in\mathbb{N}_0. \tag{16}$$

Proof. Assume f satisfies (8). Then as above f must be even. Clearly f satisfies (16) as the domain of equation (8) includes the domain of equation (16). So this direction is proved.

Assume, conversely, that f satisfies (16) and is even. Then

$$\begin{aligned} f(-1)+f(1) &= f(1)+f(1)\ (f(-1)=f(1)) \\ &= 2f(1)f(0) \quad (f(0)=1). \end{aligned}$$

So f satisfies equation (8) for $n=-1$. Now let $n\in\mathbb{Z}$ with $n\leqslant -2$. Then

$$\begin{aligned} f(n+2)+f(n) &= f(n)+f(n+2) \\ &= f(-n)+f(-n-2) \quad (-n-2,\ -n\in\mathbb{N}_0) \\ &= 2f(1)f(-n-1) \quad (-n-1\in\mathbb{N}_0) \\ &= 2f(1)f(n+1). \end{aligned}$$

So f satisfies equation (8) for all integers $n \leqslant -1$. Thus

$$f(n+2) + f(n) = 2f(1)f(n+1); \quad n \in \mathbb{Z}$$

as claimed.

Equation (16) is a linear difference equation of the second order: since $f(0) = 1$ if $f(1)$ is given then $f(2)$, and recursively $f(3), f(4) \ldots$ are determined.

3. The universal solution

DEFINITION 2

$T : \mathbb{Z} \to \mathbb{Z}[X]$ is given by $T(0) = 1$, $T(1) = X$, $T(-n) = T(n)$; $n \in \mathbb{N}_0$ and

$$T(n+2) + T(n) = 2XT(n+1); \quad n \in \mathbb{N}_0. \tag{17}$$

It is customary to write $T(n)$ as $T_n(X)$. Thus $T_2(X) = 2X^2 - 1$, $T_3(X) = 4X^3 - 3X$, $T_4(X) = 8X^4 - 8X^2 + 1$ follow immediately from equation (17). By Proposition 1 $T : \mathbb{Z} \to \mathbb{Z}(X)$ is a solution of d'Alembert's functional equation (5). Indeed more is true!

PROPOSITION 2

Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. If f satisfies equation (5) then

$$f(n) = T_n(f(1)); \quad n \in \mathbb{Z}. \tag{18}$$

where $n \longmapsto T_n(X)$ is the family of polynomials from Definition 2 above.

Proof. Since both f and T are even (one by virtue of satisfying equation (5), the other by definition), it suffices to prove (18) for all $n \in \mathbb{N}_0$. Now $f(0) = 1 = T_0(f(1))$, and $f(1) = T_1(f(1))$. So assume it has been shown that $f(n) = T_n(f(1))$ for all $n \in \mathbb{N}_0$ with $n \leqslant N$ where $N \in \mathbb{N}_0$ and $N \geqslant 1$. Then

$$\begin{aligned} f(N+1) &= 2f(1)f(N) - f(N-1) \quad (f \text{ satisfies } (5)) \\ &= 2T_1(f(1))T_N(f(1)) - T_{N-1}(f(1)) \quad (\text{induction hypothesis}) \\ &= T_{N+1}(f(1)) \quad (T \text{ satisfies } (17)). \end{aligned}$$

Thus the result is true for $n \leqslant N + 1$. So the result follows for all $n \in \mathbb{N}_0$.

Note that what makes the preceding proof work is that for each $r \in R$ the evaluation ev_r of $p \in \mathbb{Z}(X)$ at r is a homomorphism from $\mathbb{Z}(X)$ to R:

$$ev_r(p+q) = ev_r(p) + ev_r(q) \;\; [(p+q)(r) = p(r) + q(r)]$$

$$ev_r(p \cdot q) = ev_r(p) + ev_r(q) \;\; [(pq)(r) = p(r)q(r)] .$$

Given f satisfying equation (5) there is a unique homomorphism (of commutative rings) $ev_{f(1)}$ such that $f = ev_{f(1)} \circ T$. Also $ev_{f(1)}$ is completely specified by

$$ev_{f(1)}(1) = 1 \quad \text{and} \quad ev_{f(1)}(X) = f(1):$$

there is one, and only one, ring homomorphism that sends 1 (of $\mathbb{Z}$) to 1 (of R), and X of $\mathbb{Z}[X]$ to $r \in R$.

4. Identification of the universal solution

The difference equation for T is

$$T(n+2) - 2XT(n+1) + T(n) = 0; \quad n \in \mathbb{N}_0. \tag{19}$$

The (quadratic) indicial equation for this is

$$\lambda^2 - 2X\lambda + 1 = 0. \tag{20}$$

This has roots

$$\lambda_1 = X + \sqrt{X^2 - 1},\ \lambda_2 = X - \sqrt{X^2 - 1}. \tag{21}$$

These roots lie in the quadratic extension A of $\mathbb{Z}[X]$ where

$$A = \left\{ \begin{bmatrix} p & q \\ (X^2-1)\,q & p \end{bmatrix} : p, q \in \mathbb{Z}[X] \right\}$$

so that, since $\begin{bmatrix} 0 & 1 \\ (X^2-1) & 0 \end{bmatrix}^2 = \begin{bmatrix} X^2-1 & 0 \\ 0 & X^2-1 \end{bmatrix}$,

$$\lambda_1 = \begin{bmatrix} X & 1 \\ X^2-1 & X \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} X & -1 \\ 1-X^2 & X \end{bmatrix}.$$

[Note that the characteristic polynomial of λ_1 is $t^2 - 2Xt + 1$.]

Hence, for some α_1 and $\alpha_2 \in A$

$$2T(n) = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n; \quad n \in \mathbb{N}_0. \tag{22}$$

From the initial conditions $T(0) = 1$, $T(1) = X$ and so $\alpha_1 = 1$ and $\alpha_2 = 1$. Thus

$$\begin{aligned} 2T_n(X) &= \sum_{j=0}^{n} \binom{n}{j} X^{n-j} \left[\left(\sqrt{X^2-1}\right)^j + (-1)^j \left(\sqrt{X^2-1}\right)^j \right] \\ &= \sum_{k=0}^{q} \binom{n}{2k} X^{n-2k}\, 2\left(X^2-1\right)^k \end{aligned}$$

where $j = 2k$ since for j odd $1 + (-1)^j = 0$. Hence the following result has been proved.

PROPOSITION 3

Suppose $T : \mathbb{Z} \to \mathbb{Z}[X]$ is given by Definition 2 (basically equation (17). Then $T(n)(= T_n(X))$ given by Definition 1. In other words: the universal *solution to the d'Alembert equation over $\mathbb{Z}$ is given by the family of Chebyshev polynomials.*

Since

$$\lambda_2 = \lambda_1^{-1}, \quad \left(\begin{bmatrix} X & 1 \\ X^2 - 1 & X \end{bmatrix}^{-1} = \begin{bmatrix} X & -1 \\ 1 - X^2 & X \end{bmatrix} \right)$$

the solution is seen to agree with Kannappan's general description. Define $E(n) := \lambda_1^n$. Then $E(m+n) = E(m)E(n)$, and $E(0) = 1$ and

$$2T(n) = E(n) + E(-n). \tag{23}$$

5. Concluding remarks

One direction of the Theorem in section 1 says: if $f : \mathbb{Z} \to R$ satisfies d'Alembert's equation then $f(n) = T_{|n|}(f(1))$ for all integers n. This has been proved: Proposition 2 gives this for the universal T, but Proposition 3 identifies the universal T as the Chebyshev family.

The other direction of the Theorem is just as easy now: the Chebyshev family is given by $(E(n) + E(-n))/2$ and so satisfies d'Alembert's equation, and consequently so does any homomorphic image via evaluation maps $T_n(X) \to T_n(f(1))$.

Thus the Theorem has been proved.

Finally, the well-known definition of the Chebyshev polynomial [see Rivlin [3; eq 1.2]] is a consequence of the Theorem: define for $\theta \in \mathbb{R}$ the function $f : \mathbb{Z} \to \mathbb{R}$ by $n \mapsto \cos(n\theta)$. Then f satisfies d'Alembert's functional equation as was noted in the introduction. Hence, by the Theorem

$$\cos(n\theta) = f(n) = T_n(f(1)) = T_n(\cos\theta). \tag{24}$$

In a similar way it can be shown that

$$\cosh(nt) = T_n(\cosh t) \quad n \in \mathbb{Z},\ t \in \mathbb{R}. \tag{25}$$

Equations (22) and (23) can be subsumed under the general result

$$\frac{X^n + X^{-n}}{2} = T_n\left(\frac{X + X^{-1}}{2}\right) \quad n \in \mathbb{Z}. \tag{26}$$

where the theorem has been applied to the function

$$n \mapsto \frac{X^n + X^{-n}}{2} \in \mathbb{Q}\left[X, X^{-1}\right].$$

So equation (22) follows from (24) by evaluating X at $e^{i\theta}$, as does (23) with evaluation at e^t.

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