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On a Banach space automorphism and its connections to functional equations and continuous nowhere differentiable functions

Abstract. Denote by $\mathcal{H}$ the Banach space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even. It turns out that $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

is a Banach space automorphism. Important properties of F are closely related to a de Rham type functional equation for $F[\varphi]$.

Many continuous nowhere differentiable functions are of the form $F[\varphi]$. A large part of them can be identified by simple properties of the generating function φ.

1. Introduction

The set $\mathcal{H}$ of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even, equipped with the uniform norm on $\mathbb{R}$, is a Banach space. Several prominent continuous but nowhere differentiable (cnd) functions can be generated from functions $\varphi \in \mathcal{H}$ via the linear operator $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x). \tag{1}$$

As examples we mention the Takagi function $T : \mathbb{R} \to \mathbb{R}$, given by

$$T(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} D(2^k x), \quad D(y) := \operatorname{dist}(y, \mathbb{Z}) \tag{2}$$

and the Weierstrass type function $W : \mathbb{R} \to \mathbb{R}$ given by

$$W(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} C(2^k x), \quad C(y) := \cos 2\pi y. \tag{3}$$

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The *cnd* property of W has been proved by Hardy [4] 1916, the *cnd* property of T by Takagi [11] 1903 and later by many other authors. The *cnd* property of functions defined by general series of type (1) has been investigated by Knopp [8] 1918, Behrend [1] 1949, Mikolás[9] 1956 and Girgensohn [2] 1993, [3] 1994. These authors stated properties of the generating function $\varphi : \mathbb{R} \to \mathbb{R}$ which imply the nondifferentiability of $F[\varphi]$. We mention two results which can be deduced from [1] respectively [9]:

THEOREM 1 (Behrend)
Assume that $\varphi \in \mathcal{H}$ is polygonal with a finite number of vertices in $[0,1]$ all of which have rational abscissae with $\varphi'_+(0) \neq 0$. Then $F[\varphi]$ is cnd.

THEOREM 2 (Mikolás)
Assume that $\varphi \in \mathcal{H}$ is convex on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$ (or concave on both intervals) with $\varphi(0) \neq \varphi(\frac{1}{2})$. Then $F[\varphi]$ is cnd.

These results show that there is an ample supply of *cnd* functions in $F[\mathcal{H}]$. In this note we pay special attention to the operator $F : \mathcal{H} \to \mathcal{H}$ given by (1). A detailed analysis of F is the subject of Section 2. It turns out that $F : \mathcal{H} \to \mathcal{H}$ is a Banach space automorphism. In proving this and other properties of F, a simple functional equation for $F[\varphi]$ is very useful. It can be shown in a few lines: for any $x \in \mathbb{R}$ we have (with $\psi := F[\varphi]$) $\frac{1}{2}\psi(2x) = \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \cdots$, hence

$$\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x). \tag{4}$$

Equation (4) has been investigated by de Rham [10] 1957 for $\varphi = D$, the distance function defined in (2). The general case and other functional equations for $F[\varphi]$ have been discussed by Kairies [5] 1997, [6] 1998, [7] 1999.

In Section 3 we derive the Fourier expansion of $F[\varphi]$. The Fourier coefficients of $F[\varphi]$ are connected to the Fourier coefficients of φ by means of a recursion formula which follows from (4) and can be interpreted in part as a discrete analog of (4). As an application we compute the Fourier series of Takagi's function T.

2. The Banach space automorphism F

It is straightforward to check that

$$\mathcal{H} := \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ continuous},\ 1-\text{periodic and even}\},$$

equipped with the uniform norm $\|\ldots\|_u$ on $\mathbb{R}$, is a real Banach space and that the operator F, given by (1): $F[\varphi](x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$, is linear and maps $\mathcal{H}$ into $\mathcal{H}$.

In the following statement we describe the interaction of F with the functional equation (4): $\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x)$.

Proposition 1

Let $\varphi \in \mathcal{H}$. Then $\psi = F[\varphi]$ iff ψ is a bounded solution of (4) on $\mathbb{R}$.

Proof. $\psi(x) = \varphi(x)+\frac{1}{2}\varphi(2x)+\frac{1}{4}\varphi(4x)+\cdots$ and $\varphi \in \mathcal{H}$ imply the boundedness of ψ and because of

$$\frac{1}{2}\psi(2x) = \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \cdots,$$

we get $\psi(x) - \frac{1}{2}\psi(2x) = \varphi(x)$ for every $x \in \mathbb{R}$.

On the other hand, $\varphi(x) = \psi(x) - \frac{1}{2}\psi(2x)$ implies

$$\begin{aligned}\psi(x) &= \frac{1}{2}\psi(2x) + \varphi(x)\\ &= \frac{1}{2}\{\frac{1}{2}\psi(4x) + \varphi(2x)\} + \varphi(x)\\ &= \frac{1}{4}\{\frac{1}{2}\psi(8x) + \varphi(4x)\} + \frac{1}{2}\varphi(2x) + \varphi(x)\\ &\vdots\\ &= \frac{1}{2^m}\psi(2^m x) + \sum_{k=0}^{m-1}\frac{1}{2^k}\varphi(2^k x) \quad \text{for every } x \in \mathbb{R},\ m \in \mathbb{N}.\end{aligned}$$

As ψ is bounded,

$$\psi(x) = \lim_{m\to\infty}\psi(x) = \sum_{k=0}^{\infty}\frac{1}{2^k}\varphi(2^k x) = F[\varphi](x).$$

Now we shall list some important properties of the operator F. As usual, $\| F \| := \sup\{\| F[\varphi] \|_u;\ \varphi \in \mathcal{H},\ \| \varphi \|_u \leqslant 1\}$ denotes the operator norm of F.

Theorem 3

$F : \mathcal{H} \to \mathcal{H}$ is a continuous Banach space automorphism with $\| F \| = 2$. The inverse operator F^{-1} is given by

$$F^{-1}[\psi](x) = \psi(x) - \frac{1}{2}\psi(2x)$$

and is continuous as well with $\| F^{-1} \| = \frac{3}{2}$.

Proof. The linearity of F was already stated. As we shall see, the bijectivity is an immediate consequence of Proposition 1.

Namely, to prove injectivity, observe that if $\varphi \in \mathcal{H}$ and $F[\varphi] = \mathbf{o}$ (the zero function) then, by Proposition 1, necessarily $\varphi = \mathbf{o}$.

To prove surjectivity, let $\psi \in \mathcal{H}$. Define $\varphi(x) := \psi(x) - \frac{1}{2}\psi(2x)$ for $x \in \mathbb{R}$. Then clearly $\varphi \in \mathcal{H}$. By Proposition 1, $\psi(x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x) = F[\varphi](x)$, hence $\psi = F[\varphi]$ for some $\varphi \in \mathcal{H}$.

For $\varphi \in \mathcal{H}$ with $\| \varphi \|_u \leqslant 1$ we obtain

$$\| F[\varphi] \|_u = \sup \{|\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)|;\ x \in \mathbb{R}\} \leqslant \sum_{k=0}^{\infty} 2^{-k} \cdot 1 = 2,$$

hence $\| F \| \leqslant 2$. On the other hand, for the constant function $\mathbf{1}$ ($\mathbf{1}(x) = 1$) we have $\mathbf{1} \in \mathcal{H}, \| \mathbf{1} \|_u = 1$ and $\| F[\mathbf{1}] \|_u = \sum\limits_{k=0}^{\infty} 2^{-k} \cdot 1 = 2$, hence $\| F \| \geqslant 2$.

The inverse operator F^{-1} can be explicitely given: By Proposition 1 it follows that $\psi = F[\varphi]$ iff $\varphi(x) = F^{-1}[\psi](x) = \psi(x) - \frac{1}{2}\psi(2x)$ for every $x \in \mathbb{R}$.

Consequently, for $\| \psi \|_u \leqslant 1$ we have $\| F^{-1}[\psi] \|_u = \sup\{|\psi(x) - \frac{1}{2}\psi(2x)|;\ x \in \mathbb{R}\} \leqslant 3/2$. On the other hand, let $\psi_0(x) := 4D(x) - 1$, i.e., $\psi_0 \in \mathcal{H}$ with $\psi_0(x) = 4x - 1$ for $0 \leqslant x \leqslant \frac{1}{2}$. Then $\| \psi_0 \|_u = 1$ and $\| F^{-1}[\psi_0] \|_u = \sup\{|\psi_0(x) - \frac{1}{2}\psi_0(2x)|;\ x \in \mathbb{R}\} \geqslant \psi_0(1/2) - \frac{1}{2}\psi_0(1) = 3/2$.

REMARK 1

a) Let $\mathcal{A} := \{\varphi \in \mathcal{H};\ \varphi$ real analytic on $\mathbb{R}\}$.

Clearly $\mathcal{A}$ and $F[\mathcal{A}] = \{F[\varphi]; \varphi \in \mathcal{A}\}$ are subspaces of $\mathcal{H}$.

The examples $\mathbf{1} \in \mathcal{A}$ and $C \in \mathcal{A}$ ($C(x) = \cos 2\pi x$) show that F does not preserve this kind of regularity: $F[\mathbf{1}] = 2\cdot\mathbf{1}$ is again analytic whereas $F[C] = W$ is *cnd* (this is in our context the worst possible regularity property which can occur).

In severe contrast, the operator F^{-1} obviously maps $\mathcal{A}$ into $\mathcal{A}$ and preserves similar types of regularity as well, e.g., differentiability of order $n \in \mathbb{N}$.

b) Let $\mathcal{B} := \{\varphi \in \mathcal{H};\ \varphi$ nowhere differentiable$\}$. The last observation in a) shows that F does not map any $\varphi \in \mathcal{B}$ to some $F[\varphi] \in \mathcal{H} \cap C^n(\mathbb{R})$ with $n \in \mathbb{N}$ or even to some $F[\varphi] \in \mathcal{H} \cap \mathrm{BV}\ [0,1]$.

c) The operator equation $F[\varphi] = \psi$ has for any given $\psi \in \mathcal{H}$ exactly one solution: $\varphi = F^{-1}[\psi], \varphi(x) = \psi(x) - \frac{1}{2}\psi(2x)$.

Similarly, $F^2[\varphi] = \psi$ if and only if $\varphi(x) = \psi(x) - \psi(2x) + \frac{1}{4}\psi(4x)$.

In this manner, $F^n[\varphi] = \psi$ can be explicitely solved in terms of the given $\psi \in \mathcal{H}$ for every $n \in \mathbb{N}$.

3. Fourier series of $F[\varphi]$

First we fix some notations. The Fourier series of a function $g \in L^1[0,1]$ will be denoted by $S[g]$. Throughout this section we assume $\varphi \in \mathcal{H}$ and write

$$S[\varphi](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi kx, \quad a_k = 2\int_0^1 \varphi(t) \cos 2\pi kt\ dt$$

and

$$S[F[\varphi]](x) = \frac{u_0}{2} + \sum_{k=1}^{\infty} u_k \cos 2\pi kx, \quad u_k = 2\int_0^1 F[\varphi](t)\cos 2\pi kt\, dt, \quad k \in \mathbb{N}_0.$$

REMARK 2

a) For $\varphi \in \mathcal{H}$ the Fourier coefficients a_k and u_k exist and we have

$$u_k = 2\int_0^1 \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n t)\cos 2\pi kt\, dt = 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(2^n t)\cos 2\pi kt\, dt,$$

in particular,

$$\begin{aligned} u_0 &= 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(2^n t)\, dt = 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^{2^n} \varphi(\tau_n)\frac{1}{2^n}\, d\tau_n \quad (\tau_n = 2^n t),\\ &= 2\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \varphi(\tau)\, d\tau = \sum_{n=0}^{\infty}\frac{1}{2^n}\cdot a_0\\ &= 2\cdot a_0. \end{aligned}$$

b) In general it is not true that $\varphi \in \mathcal{H}$ coincides with its Fourier series $S[\varphi]$: Fejér's famous example of a continuous function γ whose Fourier series is divergent at the point zero can be modified in such a way that the new function $\widetilde{\gamma}$ belongs to $\mathcal{H}$ and $S[\widetilde{\gamma}](0)$ diverges. However, if $(c_k) \in \ell^1$ and $\varphi(x) := \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos 2\pi kx$, then clearly $\varphi \in \mathcal{H}$ and $S[\varphi] = \varphi$.

THEOREM 4

a) *Let $\varphi \in \mathcal{H}$. Then*

$$u_{2k} - \frac{1}{2}u_k = a_{2k} \text{ and } u_{2k+1} = a_{2k+1} \text{ for every } k \in \mathbb{N}_0. \tag{5}$$

b) *The recursive system* (5) *has, for any given sequence (a_k) of real numbers, a unique solution (u_k), namely*

$$u_0 = 2a_0,\ u_k = u_{2^m(2j+1)} = \sum_{\lambda=0}^{m} \frac{1}{2^{m-\lambda}} a_{2^\lambda(2j+1)} \text{ for } k \in \mathbb{N}. \tag{6}$$

Proof. a) By Proposition 1 we have $F[\varphi](x) = \frac{1}{2}F[\varphi](2x) + \varphi(x)$. This implies

$$2\int_0^1 F[\varphi](x)\cos 2\pi kx\,dx = 2\int_0^1 \{\tfrac{1}{2}F[\varphi](2x)\cos 2\pi kx + \varphi(x)\cos 2\pi kx\}\,dx,$$

hence $(2x = t)$

$$u_k = \frac{1}{2}\int_0^2 F[\varphi](t)\cos \pi kt\,dt + a_k \quad (k \in \mathbb{N}_0).$$

In particular,

$$\begin{aligned} u_{2k} &= \frac{1}{2}\int_0^2 F[\varphi](t)\cos 2\pi kt\,dt + a_{2k} \\ &= \int_0^1 F[\varphi](t)\cos 2\pi kt\,dt + a_{2k} \\ &= \frac{1}{2}u_k + a_{2k}, \end{aligned}$$

because the integrand has period 1 and

$$u_{2k+1} = \frac{1}{2}\int_0^2 F[\varphi](t)\cos \pi(2k+1)t\,dt + a_{2k+1} = a_{2k+1},$$

because the integrand is odd with respect to $1/2$ in $[0,1]$ and odd with respect to $3/2$ in $[1,2]$.

b) $u_0 = 2a_0$ follows immediately from (5).
Every $k \in \mathbb{N}$ has a unique representation $2^m(2j+1)$ with some $m, j \in \mathbb{N}_0$.
For $m = 0$, the second equation of (5) gives $u_{2j+1} = a_{2j+1}$ for every $j \in \mathbb{N}_0$.
For $m \geqslant 1$, by repeated use of the first equation of (5), we get

$$\begin{aligned} u_k = u_{2^m(2j+1)} &= \frac{1}{2}u_{2^{m-1}(2j+1)} + a_{2^m(2j+1)} \\ &= \frac{1}{4}u_{2^{m-2}(2j+1)} + \frac{1}{2}a_{2^{m-1}(2j+1)} + a_{2^m(2j+1)} \\ &\ \vdots \\ &= \frac{1}{2^m}u_{2^0(2j+1)} + \sum_{\lambda=1}^{m}\frac{1}{2^{m-\lambda}}a_{2^\lambda(2j+1)} \\ &= \sum_{\lambda=0}^{m}\frac{1}{2^{m-\lambda}}a_{2^\lambda(2j+1)}. \end{aligned}$$

On the other hand, any sequence (u_k) given by (6) satisfies in fact (5): The case $k=0$ is trivial. For $k=2^m(2j+1)$ and $m=0$ we get immediately $u_{2j+1}=a_{2j+1}$, whereas for $m \geqslant 1$ we obtain

$$\begin{aligned}
u_{2k}-\frac{1}{2}u_k &= u_{2^{m+1}(2j+1)}-\frac{1}{2}u_{2^m(2j+1)} \\
&= \frac{1}{2}\sum_{\lambda=0}^{m+1}\frac{1}{2^{m-\lambda}}\,a_{2^\lambda(2j+1)}-\frac{1}{2}\sum_{\lambda=0}^{m}\frac{1}{2^{m-\lambda}}\,a_{2^\lambda(2j+1)} \\
&= \frac{1}{2}\,\frac{1}{2^{-1}}\,a_{2^{m+1}(2j+1)}=a_{2k}.
\end{aligned}$$

As a first useful consequence of Theorem 2 we note

PROPOSITION 2

Assume that $(c_k)\in\ell^1$ *and that*

$$\varphi(x):=\frac{1}{2}c_0+\sum_{k=1}^{\infty}c_k\cos 2\pi kx.$$

Then $\varphi\in\mathcal{H}$, $S[\varphi]=\varphi$ *and with*

$$S[F[\varphi]](x)=\frac{1}{2}v_0+\sum_{k=1}^{\infty}v_k\cos 2\pi kx$$

we have $(v_k)\in\ell^1$ *and* $S[F[\varphi]]=F[\varphi]$.

Proof. Clearly φ is continuous, 1-periodic and even, hence $\varphi\in\mathcal{H}$. The uniform convergence of the series representing φ implies that φ coincides with its Fourier series $S[\varphi]$. By Theorem 2 we have

$$|v_0|=2|c_0| \text{ and } |v_{2^m(2j+1)}|\leqslant\sum_{\lambda=0}^{m}2^{\lambda-m}|c_{2^\lambda(2j+1)}| \text{ for } m,j\in\mathbb{N}_0.$$

Consequently, for every $j\in\mathbb{N}_0$,

$$\begin{aligned}
\sum_{m=0}^{\infty}|v_{2^m(2j+1)}| &\leqslant \sum_{m=0}^{\infty}\sum_{\lambda=0}^{m}2^{\lambda-m}|c_{2^\lambda(2j+1)}| \\
&= |c_{2j+1}|+\big(\frac{1}{2}|c_{2j+1}|+|c_{2(2j+1)}|\big) \\
&\quad +\big(\frac{1}{4}|c_{2j+1}|+\frac{1}{2}|c_{2(2j+1)}|+|c_{2^2(2j+1)}|\big)+\cdots \\
&= 2\sum_{m=0}^{\infty}|c_{2^m(2j+1)}|.
\end{aligned}$$

Moreover,

$$\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}|v_{2^m(2j+1)}| \leqslant \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}|2c_{2^m(2j+1)}| = 2\sum_{k=1}^{\infty}|c_k| < \infty$$

because of $(c_k) \in \ell^1$. By the main rearrangement theorem, $\sum\limits_{k=1}^{\infty}|v_k| < \infty$, i.e., $(v_k) \in \ell^1$ as well.

This implies the convergence of $S[F[\varphi]](x)$ for every $x \in \mathbb{R}$, thus by Fejér's theorem $F[\varphi]$ coincides with its Fourier series $S[F[\varphi]]$.

As a second consequence of Theorem 2 we derive the Fourier series of Takagi's function $T = F[D]$.

PROPOSITION 3
If T is given by (2), then

$$S[T](x) = \frac{1}{2} + \sum_{k=1}^{\infty} u_k \cos 2\pi kx$$

with

$$u_k = u_{2^m(2j+1)} = \frac{-1}{2^{m-1}\pi^2(2j+1)^2} \quad \text{for } m, j \in \mathbb{N}_0.$$

This series is absolutely and uniformly convergent on $\mathbb{R}$ and $S[T] = T$.

Proof. It is well known that the distance function D has the Fourier series

$$S[D](x) = \frac{1}{4} - \frac{2}{\pi^2}\Big\{\frac{1}{1^2}\cos 2\pi x + \frac{1}{3^2}\cos(3\cdot 2\pi x) + \frac{1}{5^2}\cos(5\cdot 2\pi x) + \cdots\Big\}.$$

Hence $a_0 = \dfrac{1}{2}$, $a_{2k} = 0$ for $k \in \mathbb{N}$ and $a_{2j+1} = \dfrac{-2}{\pi^2(2j+1)^2}$ for $j \in \mathbb{N}_0$. Clearly $(a_n) \in \ell^1$.

By Theorem 2 we have $u_0 = 2a_0 = 1, u_{2j+1} = a_{2j+1} = \dfrac{-2}{\pi^2(2j+1)^2}$ for $j \in \mathbb{N}_0$ and, because of $a_{2k} = 0$,

$$u_{2^m(2j+1)} = \frac{1}{2^m}a_{2j+1} + \sum_{\lambda=1}^{m} a_{2^\lambda(2j+1)} = \frac{-1}{2^{m-1}\pi^2(2j+1)^2} \text{ for } j \in \mathbb{N}_0,\ m \in \mathbb{N}.$$

By Proposition 2, T coincides with its Fourier series $S[T]$ and $(u_k) \in \ell^1$.

Therefore we have the following representation of Takagi's function by an absolutely and uniformly (on $\mathbb{R}$) convergent trigonometric series:

$$
\begin{aligned}
T(x) = \frac{1}{2} - \frac{1}{\pi^2} \Big\{ & \frac{2}{1^2}\cos(1\cdot 2\pi x) + \frac{1}{1^2}\cos(2\cdot 2\pi x) + \frac{2}{3^2}\cos(3\cdot 2\pi x) \\
& + \frac{1}{2}\cos(4\cdot 2\pi x) + \frac{2}{5^2}\cos(5\cdot 2\pi x) + \frac{1}{3^2}\cos(6\cdot 2\pi x) \\
& + \frac{2}{7^2}\cos(7\cdot 2\pi x) + \frac{1}{2^2}\cos(8\cdot 2\pi x) + \frac{2}{9^2}\cos(9\cdot 2\pi x) \\
& + \frac{1}{5^2}\cos(10\cdot 2\pi x) + \frac{2}{11^2}\cos(11\cdot 2\pi x) + \frac{1}{2\cdot 3^2}\cos(12\cdot 2\pi x) \\
& + \frac{2}{13^2}\cos(13\cdot 2\pi x) + \frac{1}{7^2}\cos(14\cdot 2\pi x) + \frac{2}{15^2}\cos(15\cdot 2\pi x) \\
& + \frac{1}{2^3}\cos(16\cdot 2\pi x) + \cdots \Big\}.
\end{aligned}
$$

Note that in our approach we did not need an explicit calculation of the rather unpleasant series $u_k = 2\sum\limits_{n=0}^{\infty} \frac{1}{2^n}\int\limits_0^1 D(2^n x)\cos 2\pi kx\, dx,\ k \in \mathbb{N}_0$.

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