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On the stability of the generalized cosine functional equations

Abstract. The aim of this paper is to study the stability problem of the generalized cosine functional equations for complex and vector valued functions.

1. Introduction

One of the trigonometric functional equations studied extensively is the equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad \text{for } x,y\in G \tag{C}$$

(Wilson [8], Kannappan [6], Dacić [3]) known as the *cosine* or *d'Alembert's functional equation* where $f : G \to \mathbb{C}$, G, a group (not necessarily Abelian) and $\mathbb{C}$, the set of complex numbers. It is known (see [6]) that if f satisfies (C) and the condition

$$f(x+y+z)=f(x+z+y), \quad \text{for } x,y,z\in G, \tag{K}$$

then there is a homomorphism $m : G \to \mathbb{C}^*$ $(\mathbb{C}^* = \mathbb{C}\backslash\{0\})$

$$m(x+y)=m(x)m(y), \quad \text{for } x,y\in G, \tag{1}$$

such that f has the form

$$f(x)=\frac{1}{2}(m(x)+m(-x)), \quad \text{for } x\in G. \tag{2}$$

Ever since Ulam [7] in 1940 raised the stability problem of the Cauchy equation $f(x+y)=f(x)+f(y)$, many authors (see Hyers [5], Ger [4], etc.) treated the stability problem for many other functional equations. Baker [2] proved the result:

Let $\varepsilon \geqslant 0$ be a given number and let $G(+)$ be an Abelian group. Let $f : G \to \mathbb{C}$ be such that

$$|f(x+y)+f(x-y)-2f(x)f(y)| \leqslant \varepsilon, \quad \text{for } x,y\in G.$$

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Then either $|f(x)| \leqslant \frac{1}{2}(1+\sqrt{1+2\varepsilon})$ for all $x \in G$ or f is a solution of the cosine equation (C).

Badora [1] presented a new, short proof of Baker's result.

The aim of this paper is to investigate the stability problem of the functional equations

$$f(x+y)+f(x-y)=2g(x)f(y), \quad x,y \in G \tag{3}$$

and

$$f(x+y)+f(x-y)=2f(x)g(y), \quad x,y \in G \tag{4}$$

in the next two sections modelled after [1], where G is a group.

2. Stability of (3) and (4) for complex functions

In this section we will consider the stability of (3) and (4) and their variants. First we will take up (3) and prove the following theorem.

THEOREM 1

Let $\varepsilon \geqslant 0$ and $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x+y)+f(x-y)-2g(x)f(y)| \leqslant \varepsilon \tag{3$'$}$$

with f satisfying the (K) *condition, where $G(+)$ is a group. Then either f and g are bounded or g satisfies* (C) *and f and g satisfy* (3) *and* (4). *Further, in the latter case there exists a homomorphism $m : G \to \mathbb{C}^*$ satisfying* (1) *such that*

$$f(x)=\frac{b}{2}(m(x)+m(-x)) \quad \textit{and} \quad g(x)=\frac{1}{2}(m(x)+m(-x)), \tag{5}$$

for $x \in G$, where b is a constant.

Proof. We will consider only the nontrivial f (that is, $f \neq 0$). Put $y=0$ in (3)$'$ to get

$$|f(x)-g(x)f(0)| < \frac{\varepsilon}{2}, \quad \text{for } x \in G. \tag{6}$$

If g is bounded, then using (6), we have

$$\begin{aligned} |f(x)| &= |f(x)-g(x)f(0)+g(x)f(0)| \\ &\leqslant \frac{\varepsilon}{2}+|g(x)f(0)|, \end{aligned}$$

which shows that f is also bounded. On the other hand if f is bounded, choose y_0 such that $f(y_0) \neq 0$ and then use (3)$'$,

$$|g(x)| - \left|\frac{f(x+y_0)+f(x-y_0)}{2f(y_0)}\right| \leqslant \left|\frac{f(x+y_0)+f(x-y_0)}{2f(y_0)} - g(x)\right| \leqslant \frac{\varepsilon}{2|f(y_0)|}$$

to get that g is also bounded on G.

It follows easily now that if f (or g) is unbounded, then so is g (or f). Let f and g be unbounded. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in G such that $g(x_n) \neq 0$, $|g(x_n)| \to \infty$ as $n \to \infty$ and $f(y_n) \neq 0$, $\lim_n |f(y_n)| = \infty$.

First we will show that g indeed satisfies (C).

From (3)′ with $y = y_n$ we obtain

$$\left| \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} - g(x) \right| \leqslant \frac{\varepsilon}{2|f(y_n)|},$$

that is,

$$\lim_n \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} = g(x). \tag{7}$$

Using (3)′ again and (K) we have

$$\begin{aligned} &|f(x+(y+y_n)) + f(x-(y+y_n)) - 2g(x)f(y+y_n) \\ &\quad + f(x+(y-y_n)) + f(x-(y-y_n)) - 2g(x)f(y-y_n)| \leqslant 2\varepsilon \end{aligned}$$

so that

$$\begin{aligned} &\left| \frac{f((x+y)+y_n) + f((x+y)-y_n)}{2f(y_n)} + \frac{f((x-y)+y_n) + f((x-y)-y_n)}{2f(y_n)} \right. \\ &\quad \left. -2g(x)\frac{f(y+y_n) + f(y-y_n)}{2f(y_n)} \right| \leqslant \frac{\varepsilon}{|f(y_n)|} \quad \text{for } x, y \in G, \end{aligned}$$

which with the use of (7) implies that

$$|g(x+y) + g(x-y) - 2g(x)g(y)| \leqslant 0,$$

that is g is a solution of (C).

As before applying (3)′ twice and the (K) condition, first we have

$$\lim_n \frac{f(x_n+y) + f(x_n-y)}{2g(x_n)} = f(y), \tag{8}$$

and then

$$\begin{aligned} &|f((x_n+x)+y) + f((x_n+x)-y) - 2g(x_n+x)f(y) \\ &\quad + f((x_n-x)+y) + f((x_n-x)-y) - 2g(x_n-x)f(y)| \leqslant 2\varepsilon \end{aligned}$$

so that

$$\begin{aligned} &\left| \frac{f(x_n+(x+y)) + f(x_n-(x+y))}{2g(x_n)} + \frac{f(x_n+(x-y)) + f(x_n-(x-y))}{2g(x_n)} \right. \\ &\quad \left. - 2 \cdot \frac{g(x_n+x) + g(x_n-x)}{2g(x_n)} f(y) \right| \leqslant \frac{\varepsilon}{|g(x_n)|}. \end{aligned}$$

From (8) and g satisfying (C), it follows that

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leqslant 0,$$

that is, f and g are solutions of (3).

Choose y_0 such that $f(y_0) \neq 0$. Then (3) gives

$$g(x) = \frac{f(x+y_0) + f(x-y_0)}{2f(y_0)}$$

so that g also satisfies the condition (K). Since g satisfies (K), from [6] we see that there exists a homomorphism $m : G \to \mathbb{C}^*$ satisfying the second part of (5).

Finally, applying (3)′, (7) and (K), we get

$$\begin{aligned}|f((x_n+y)+x) + f((x_n+y)-x) - 2g(x_n+y)f(x) + f((x_n-y)+x)\\ + f((x_n-y)-x) - 2g(x_n-y)f(x)| \leqslant 2\varepsilon\end{aligned}$$

and that

$$\begin{aligned}\Bigg|\frac{f(x_n+(x+y)) + f(x_n-(x+y))}{2g(x_n)} + \frac{f(x_n+(x-y)) + f(x_n-(x-y))}{2g(x_n)}\\ - 2f(x)\cdot\frac{g(x_n+y)+g(x_n-y)}{2g(x_n)}\Bigg| \leqslant \frac{\varepsilon}{|g(x_n)|}\end{aligned}$$

resulting to (4). From (3) and (4), it is easy to see that $f(x) = bg(x)$, for some constant b.

This proves the theorem.

We now consider a slight variation of (3)′.

Corollary 2

Let $\varepsilon \geqslant 0$. Let $f_n : G \to \mathbb{C}$ (where G is a group) be a sequence of functions converging uniformly to f on G. Suppose $f, g, f_n : G \to \mathbb{C}$ be such that

$$|f(x+y) + f(x-y) - 2g(x)f_n(y)| \leqslant \varepsilon, \quad \text{for } x, y \in G, \qquad (3)''$$

with f satisfying (K). Then either f is bounded or g satisfies (C) and f and g satisfy (3) and (4).

Proof. Since $\{f_n\}$ is uniformly convergent to f, taking the limit with respect to n in (3)″, we obtain (3)′. The result now follows from Theorem 1.

Now we take up the stability of (4). We prove the following theorems

Theorem 3

Let $\varepsilon \geqslant 0$ and G be a group. Suppose $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x+y) + f(x-y) - 2f(x)g(y)| \leqslant \varepsilon, \quad \text{for } x, y \in G \qquad (4)'$$

with f even (that is, $f(-x) = f(x)$) and f satisfies (K). *Then either f and g are bounded or f and g are unbounded and g satisfies* (C) *and f and g are solutions of* (4) *and* (3).

Proof. We consider only nontrivial f, that is, $f \neq 0$, When f is bounded, choose x_0 such that $f(x_0) \neq 0$ and use $(4)'$ to get

$$|g(y)| - \frac{|f(x_0+y)+f(x_0-y)|}{2|f(x_0)|} \leqslant \left|\frac{f(x_0+y)+f(x_0-y)}{2f(x_0)} - g(y)\right| \leqslant \frac{\varepsilon}{2|f(x_0)|},$$

which shows that g is also bounded.

Suppose f is unbounded. Choose $x = 0$ in $(4)'$ to have $|f(y)+f(-y) - 2f(0)g(y)| \leqslant \varepsilon$, that is, $|f(y)-f(0)g(y)| \leqslant \frac{\varepsilon}{2}$ (this is the only place we use that f is even). Since f is unbounded, $f(0) \neq 0$. Hence g is also unbounded.

Let f and so g be unbounded. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in G such that $f(x_n) \neq 0$, $|f(x_n)| \to \infty$, $g(y_n) \neq 0$, $|g(y_n)| \to \infty$.

Applying twice the inequality (4) and using (K) for f twice, first we get

$$\left|\frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} - g(y)\right| \leqslant \frac{\varepsilon}{2|f(x_n)|}$$

that is,

$$\lim_n \frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} = g(y), \quad \text{for } y \in G, \tag{9}$$

and then we obtain

$$\begin{aligned}|f((x_n+x)+y)+f((x_n+x)-y)-2f(x_n+x)g(y)+f((x_n-x)+y)\\ +f((x_n-x)-y)-2f(x_n-x)g(y)| \leqslant 2\varepsilon,\end{aligned}$$

that is,

$$\begin{aligned}\Bigg|\frac{f(x_n+(x+y))+f(x_n-(x+y)}{2f(x_n)} + \frac{f(x_n+(x-y))+f(x_n-(x-y))}{2f(x_n)}\\ - 2g(y)\,\frac{f(x_n+x)+f(x_n-x)}{2f(x_n)}\Bigg| \leqslant \frac{\varepsilon}{|f(x_n)|}\end{aligned}$$

which by (9) leads to $|g(x+y)+g(x-y)-2g(y)g(x)| \leqslant 0$, so that g satisfies (C).

Again applying the inequality $(4)'$ twice and using (K) condition for f twice, first we have

$$\left|\frac{f(x+y_n)+f(x-y_n)}{2g(y_n)} - f(x)\right| \leqslant \frac{\varepsilon}{2|g(y_n)|} \tag{10}$$

and then we get

$$|f(x+(y_n+y))+f(x-(y_n+y))-2f(x)g(y_n+y)+f(x+(y_n-y)) \\ +f(x-(y_n-y)-2f(x)g(y_n-y)| \leqslant 2\varepsilon$$

that is,

$$\left|\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)} \right. \\ \left. -2f(x)\frac{g(y_n+y)+g(y_n-y)}{2g(y_n)}\right| \leqslant \frac{\varepsilon}{|g(y_n)|}$$

which by (10) and (C) yields

$$|f(x+y)+f(x-y)-2f(x)g(y)| \leqslant 0,$$

so that f and g are solutions of (4).

Consider the inequality

$$|f((y_n+x)+y)+f(y_n+x)-y)-2g(y_n+x)f(y)+f(y_n-x)+y) \\ +f((y_n-x)-y)-2g(y_n-x)f(y)| \leqslant 2\varepsilon.$$

As before using (K), (10), evenness of f and (C) and the division by $2g(y_n)$ yields

$$\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)} \\ -2f(y)\frac{g(y_n+x)+g(y_n-x)}{2g(y_n)} \to 0 \quad \text{as } n\to\infty$$

so that f and g are solutions of (3).

This completes the proof of this theorem.

Note that the evenness of f is used to prove that g is unbounded when f is and nowhere else.

COROLLARY 4

Let $\varepsilon \geqslant 0$. Let $f_n : G \to \mathbb{C}$ (where G is a group) be a sequence of functions converging uniformly to f on G. Suppose $f, f_n, g : G \to \mathbb{C}$ be such that

$$|f(x+y)+f(x-y)-2f_n(x)g(y)| \leqslant \varepsilon, \quad \text{for } x,y\in G, n\in\mathbb{N}, \qquad (4)''$$

where f is even and it satisfies (K). *Then either f is bounded or g satisfies* (C) *and f and g are solutions of* (4) *and* (3).

Proof. Since $\{f_n\}$ is uniformly convergent to f, taking the limit with respect to n in $(4)''$, we get $(4)'$. The result now follows from Theorem 3.

3. Stability of (3) and (4) for vector valued functions

In [1] Badora gave a counter-example to illustrate the failure of the superstability of the cosine functional equation (C) in the case of the vector valued mappings. Here consider the following example. Let f and g be unbounded solution of (3) (or (4)) where $f, g : G \to \mathbb{C}$. Define $f_1, g_1 : G \to M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) by

$$f_1(x) = \begin{pmatrix} f(x) & 0 \\ 0 & c_1 \end{pmatrix}, \quad g_1(x) = \begin{pmatrix} g(x) & 0 \\ 0 & c_2 \end{pmatrix}$$

for $x \in G$ where $c_1 \neq 0$, $c_2 \neq 1$. Then

$$\|f_1(x+y) + f_1(x-y) - 2f_1(y)g_1(x)\| = \text{ constant } > 0$$

(or $\|f_1(x+y) + f_1(x-y) - 2f_1(x)g_1(y)\| = \text{ constant } > 0$) for $x, y \in G$. This f_1 and g_1 are neither bounded nor satisfy (C).

Therefore there is a need to consider the vector valued functions separately. We prove the following two theorems in this section. Let G be a group and A be a complex normed algebra with identity.

Theorem 5

Suppose $f, g : G \to A$ satisfy the inequality

$$\|f(x+y) + f(x-y) - 2g(x)f(y)\| \leqslant \varepsilon, \tag{3'''}$$

for $x, y \in G$ with f satisfying (K) and

$$\|f(x) - f(-x)\| \leqslant \eta, \quad \text{for } x \in G, \tag{11}$$

for some $\varepsilon, \eta \geqslant 0$. Suppose there is a $z_0 \in G$ such that $g(z_0)^{-1}$ exists and $\|f(x)g(z_0)\|$ is bounded for $x \in G$. Then there is an $m : G \to A$ such that

$$\|m(x+y) - m(x)m(y)\| \leqslant a_1, \quad \text{for } x, y \in G \tag{12}$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| \leqslant a_2, \quad \text{for } x \in G \tag{13}$$

for some constants a_1 and a_2.

Proof. Let $M := \sup_{x \in G} \|f(x)g(z_0)\|$. Then using (3)''' and (11), we get by using (K)

$$\begin{aligned} \|f(x)g(-z_0)\| &\leqslant \|f(-x)g(z_0)\| + \|f(x)g(-z_0) - f(-x)g(z_0)\| \\ &\leqslant M + \tfrac{1}{2}\|f(z_0 - x) + f(z_0 + x) - 2g(z_0)f(-x) \\ &\quad - (f(-z_0 + x) + f(-z_0 - x) - 2g(-z_0)f(x)) \\ &\quad - (f(z_0 + x) - f(-z_0 - x) + f(z_0 - x) - f(-z_0 + x))\| \\ &\leqslant M + \varepsilon + \eta. \end{aligned}$$

Define a function $h : G \to A$ by the formula

$$h(x) = \frac{1}{2}(f(x) + f(-x)), \quad \text{for } x \in G.$$

Then h is even, that is, $h(-x) = h(x)$,

$$\|h(x) - f(x)\| \leqslant \frac{\eta}{2} \quad \text{for } x \in G, \quad \|h(x)g(z_0)\| \leqslant M. \tag{14}$$

Define a function $m : G \to A$ by

$$m(x) = h(x) + ig(z_0), \quad \text{for } x \in G.$$

Utilizing (14), we get (using first commutativity in A)

$$\begin{aligned}\|m(x+y) - m(x)m(y)\| &= \|h(x+y) + ig(z_0) - h(x)h(y) \\ &\quad + i(h(x) + h(y))g(z_0) + g(z_0)^2\| \\ &\leqslant \|h(x+y)\| + \|h(x)h(y)\| \\ &\quad + \|(h(x) + h(y))g(z_0)\| + \|g(z_0)\| + \|g(z_0)\|^2 \\ &\leqslant \|h(x+y) - f(x+y)\| + \|f(x+y)\| \\ &\quad + \|h(x)h(y)g(z_0)^2 \cdot g(z_0)^{-2}\| + \|h(x)g(z_0)\| \\ &\quad + \|h(y)g(z_0)\| + \|g(z_0)\| + \|g(z_0)\|^2 \\ &\leqslant \frac{\eta}{2} + M\|g(z_0)\|^{-1} + M^2\|g(z_0)\|^{-2} \\ &\quad + 2M + \|g(z_0)\| + \|g(z_0)\|^2 \\ &= a_1\end{aligned}$$

(say) which is (12). Finally by (14), we have

$$\begin{aligned}\left\|f(x) - \frac{1}{2}(m(x) + m(-x))\right\| &= \Big\|f(x) - h(x) + h(x) \\ &\quad - \frac{1}{2}(h(x) + h(-x)) - ig(z_0)\Big\| \\ &\leqslant \frac{\eta}{2} + \|g(z_0)\| = a_2\end{aligned}$$

(say), which is (13). This proves the theorem.

Lastly we prove the following theorem.

THEOREM 6

Let $f, g : G \to A$ satisfy the inequality

$$\|f(x+y) + f(x-y) - 2f(x)g(y)\| \leqslant \varepsilon, \quad x, y \in G, \tag{4$'''$}$$

with f satisfying (K) *and*

$$\|f(x) - f(-x)\| \leqslant \eta, \quad \textit{for } x \in G,$$

for some nonnegative ε *and* η. *Suppose there exists a* $z_0 \in G$ *such that* $g(z_0)^{-1}$ *exists and* $\|f(x)g(z_0)\|$ *is bounded over* G. *Then there exists a mapping* $m : G \to A$ *such that*

$$\|m(x+y) - m(x)m(y)\| \leqslant a_1, \quad \text{for } x, y \in G$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| \leqslant a_2, \quad \text{for } x \in G,$$

for some constants a_1 *and* a_2.

The proof runs parallel to that of Theorem 5.

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