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Károly Lajkó

Generalized Hosszú functional equations

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. Pexiderizations of the Hosszú functional equation

$$f(xy) + f(x + y - xy) = f(x) + f(y)$$

are considered on a variety of domains.

1. Introduction

The functional equation

$$f(xy) + f(x + y - xy) = f(x) + f(y) \tag{H}$$

was first considered by M. Hosszú.

The general solution for real functions was given by Blanuša [1] and Daróczy [3]. They proved the following

THEOREM B-D
The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (H) *for all $x, y \in \mathbb{R}$ if and only if*

$$f(x) = A(x) + C, \quad x \in \mathbb{R}, \tag{1}$$

where A is an additive function on $\mathbb{R}^2$ and $C \in \mathbb{R}$ is an arbitrary constant.

Equation (H) was also studied on other structures (see [2], [5], [6], [7], [9], [10], [14], [15], [16]).

In [12] and [13] we studied the functional equation

$$f(xy) + g(x + y - xy) = f(x) + f(y) \tag{GH1}$$

and we proved the following theorems.

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Theorem L1

If the functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation (GH1) *for all $x, y \in \mathbb{R}$ then*

$$f(x) = g(x) = A(x) + C, \quad x \in \mathbb{R},$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $C \in \mathbb{R}$ is an arbitrary constant.

Theorem L2

The functions $f : \mathbb{R}_0 \to \mathbb{R}$ ($\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$) and $g : \mathbb{R} \to \mathbb{R}$ satisfy the equation (GH1) *for all $x, y \in \mathbb{R}_0$ if and only if*

$$f(x) = A_1(x) + A_2(\log|x|) + C, \quad x \in \mathbb{R}_0, \tag{2}$$

$$g(x) = A_1(x) + C, \quad x \in \mathbb{R}, \tag{3}$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions on $\mathbb{R}^2$ and C is a real constant.

In this paper we shall deal with the following problems.

Problem A

Let $f, g : (0,1) \to \mathbb{R}$ be real functions which satisfy the functional equation (GH1) for all $x, y \in (0,1)$. What is the general solution of (GH1) on the interval $(0,1)$?

Problem B

Let $f, h : \mathbb{R}_0$ (or $\mathbb{R}_1 = \mathbb{R} \setminus \{1\}$) $\to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be real functions satisfying the functional equation

$$f(xy) + g(x + y - xy) = h(x) + h(y) \tag{GH2}$$

for all $x, y \in \mathbb{R}_0$ or $x, y \in \mathbb{R}_1$ respectively. Find the general solution of (GH2).

Problem C

Let $f, g, h : (0,1) \to \mathbb{R}$ be real functions satisfying (GH2) for all $x, y \in (0,1)$. Find the general measurable solution of (GH2).

2. Problem A

Here we shall use the following result by Z. Daróczy and L. Losonczi (see [4]).

Theorem D-L

If f is additive on an open connected domain of $\mathbb{R}^2$, then f has one and only one quasi-extension.

Theorem 1

The functions $f, g : (0,1) \to \mathbb{R}$ satisfy the generalized Hosszú equation (GH1) *for all $x, y \in (0,1)$ if and only if*

$$f(x) = A_1(x) + A_2(\log x) + C, \quad x \in (0,1), \tag{4}$$

$$g(x) = A_1(x) + C, \quad x \in (0,1), \tag{5}$$

where A_1 and A_2 are additive functions on $\mathbb{R}^2$ and C is an arbitrary real constant.

Proof. First we follow the idea used in [14] for the proof of Lemma 2.

The function

$$F(x,y) = f(x) + f(y) - f(xy)$$

satisfies the equation

$$F(xy,z) + F(x,y) = F(x,yz) + F(y,z) \tag{6}$$

for all $x, y, z \in (0,1)$. On the other hand we have

$$F(x,y) = g(x + y - xy).$$

Putting this into (6), we obtain the equation

$$g(xy + z - xyz) + g(x + y - xy) = g(x + yz - xyz) + g(y + z - yz) \tag{7}$$

for all $x, y, z \in (0,1)$.

By the substitution

$$\left.\begin{aligned} xy + z - xyz &= t + \frac{1}{2} \\ x + y - xy &= s + \frac{1}{2} \\ y + z - yz &= \frac{1}{2} \end{aligned}\right\} \tag{8}$$

we obtain from (7) the functional equation

$$g\left(t + \frac{1}{2}\right) + g\left(s + \frac{1}{2}\right) = g\left(t + s + \frac{1}{2}\right) + g\left(\frac{1}{2}\right)$$

on the domain (t,s) defined by (8), i.e. on

$$D = \left\{(t,s) \mid -\frac{1}{2} < t < 0,\ \frac{1}{2} + \frac{1}{2t-1} < s < \frac{1}{2} + t\right\}.$$

Thus the function

$$A^* : \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbb{R}, \quad A^*(t) = g\left(t + \frac{1}{2}\right) - g\left(\frac{1}{2}\right) \tag{9}$$

is additive on the open connected domain D. So, by Theorem D-L, A^* has one and only one quasi-extension A_1 with $A^*(x) = A_1(x)$ for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Now, using (9), we have

$$g(x) = A_1\left(x - \frac{1}{2}\right) + g\left(\frac{1}{2}\right), \quad x \in (0,1).$$

This implies (5) with an arbitrary real constant $C = g\left(\frac{1}{2}\right) - A_1\left(\frac{1}{2}\right)$.

Substituting (5) in (GH1), we get that the function φ defined by

$$\varphi(x) = f(x) - A_1(x) - C, \quad x \in (0,1) \tag{10}$$

satisfies the functional equation

$$\varphi(xy) = \varphi(x) + \varphi(y), \quad x, y \in (0,1).$$

On setting

$$x = e^{-t},\ y = e^{-s} \quad (t, s > 0), \quad B(t) = \varphi(e^{-t}), \tag{11}$$

this is transformed into

$$B(t+s) = B(t) + B(s), \quad t, s > 0.$$

So, using again Theorem D-L, B has one and only one quasi-extension A with $B(t) = A(t)$ for all $t \in \mathbb{R}_+$. This, together with (11) implies

$$\varphi(x) = A_2(\log x), \quad x \in (0,1), \tag{12}$$

where $A_2 = -A$ is an additive function on $\mathbb{R}^2$.

Finally, from (10) and (12), (4) follows for the function f.

It is easy to see that (4) and (5) indeed satisfy (GH1).

3. Problem B

THEOREM 2

The functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation (GH2) *for all $x, y \in \mathbb{R}$ if and only if*

$$f(x) = A(x) + C_2, \quad x \in \mathbb{R}, \tag{13}$$

$$g(x) = A(x) + C_3, \quad x \in \mathbb{R}, \tag{14}$$

$$h(x) = A(x) + C_1, \quad x \in \mathbb{R}, \tag{15}$$

where A is an additive function on $\mathbb{R}^2$ and $C_i \in \mathbb{R}$ $(i = 1,2,3)$ are arbitrary constants with $2C_1 = C_2 + C_3$.

Proof. Putting into (GH2) $y = 0$ or $y = 1$, one gets

$$g(x) = h(x) + h(0) - f(0), \quad x \in \mathbb{R} \tag{16}$$

and

$$f(x) = h(x) + h(1) - g(1), \quad x \in \mathbb{R} \tag{17}$$

respectively. Substituting these into (GH2) we have

$$h(xy) + h(x + y - xy) = h(x) + h(y), \quad x, y \in \mathbb{R}.$$

This is the Hosszú functional equation. So, by Theorem B-D h is of the form (15). Taking (16), (17) and (15) into consideration also, we have proved (13) and (14).

One can easily to see that (13), (14) and (15) satisfy (GH2) if $2C_1 = C_2 + C_3$.

THEOREM 3

The functions $f, h : \mathbb{R}_0 \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in \mathbb{R}_0$ *if and only if*

$$f(x) = A_1(x) + A_2(\log|x|) + C_3, \quad x \in \mathbb{R}_0, \tag{18}$$

$$g(x) = A_1(x) + C_2, \quad x \in \mathbb{R}, \tag{19}$$

$$h(x) = A_1(x) + A_2(\log|x|) + C_1, \quad x \in \mathbb{R}_0, \tag{20}$$

where A_1, A_2 *are additive functions on* $\mathbb{R}^2$ *and* $C_i \in \mathbb{R}$ $(i = 1, 2, 3)$ *are arbitrary constants with* $2C_1 = C_2 + C_3$.

Proof. Setting $y = 1$ in (GH2) we obtain the identity

$$f(x) = h(x) + h(1) - g(1), \quad x \in \mathbb{R}_0 \tag{21}$$

and putting this into (GH2) we get

$$h(xy) + g(x + y - xy) + h(1) - g(1) = h(x) + h(y),$$

where $x, y \in \mathbb{R}_0$. This is an instance of the generalized Hosszú equation (GH1). Thus, by Theorem L1, h and g are of the forms (20) and (19) respectively, where $C_2 = C_1 + g(1) - h(1)$ is arbitrary constant. Finally, by (19), (20) and (21), we get (18) for f.

An easy calculation shows that the functions (18), (19) and (20) satisfy (GH2) if $2C_1 = C_2 + C_3$.

THEOREM 4

The functions $f : \mathbb{R} \to \mathbb{R}$ *and* $g, h : \mathbb{R}_1 \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in \mathbb{R}_1$ *if and only if*

$$f(x) = A_1(x) + C_3, \quad x \in \mathbb{R}, \tag{22}$$

$$g(x) = A_1(x) + A_2(\log|1 - x|) + C_2, \quad x \in \mathbb{R}_1, \tag{23}$$

$$h(x) = A_1(x) + A_2(\log|1 - x|) + C_1, \quad x \in \mathbb{R}_1, \tag{24}$$

where A_1 *and* A_2 *are additive functions on* $\mathbb{R}^2$ *and* $C_i \in \mathbb{R}$ $(i = 1, 2, 3)$ *are arbitrary constants with* $2C_1 = C_2 + C_3$.

Proof. Letting $y = 0$ in (GH2), we obtain

$$g(x) = h(x) + h(0) - f(0), \quad x \in \mathbb{R}_1. \tag{25}$$

Using (25) in (GH2), we get

$$f(xy) + h(x + y - xy) + h(0) - f(0) = h(x) + h(y), \quad x, y \in \mathbb{R}.$$

Replacing here x, y by $1 - x$, $1 - y$ respectively, we have

$$f(1 - (x + y - xy)) + h(0) - f(0) + h(1 - xy) = h(1 - x) + h(1 - y)$$

for all $x, y \in \mathbb{R}_0$, which implies that the functions f^* and h^* defined by

$$\begin{aligned} f^*(x) &= f(1-x) + h(0) - f(0), \quad x \in \mathbb{R}, \\ h^*(x) &= h(1-x), \quad x \in \mathbb{R}_0 \end{aligned} \tag{26}$$

satisfy the functional equation

$$f^*(x+y-xy) + h^*(xy) = h^*(x) + h^*(y), \quad x, y \in \mathbb{R}_0.$$

This is (GH1). Now, using Theorem L2, we have

$$h^*(x) = A_1^*(x) + A_2(\log|x|) + C_1^*, \quad x \in \mathbb{R}_0, \tag{27}$$

$$f^*(x) = A_1^*(x) + C_3^*, \quad x \in \mathbb{R}. \tag{28}$$

From (26), (27) and (28), we obtain (22) and (24) with $A_1 = -A_1^*$, $C_1 = A_1^*(1) + C_1^*$ and $C_3 = A_1^*(1) + C_3^* + f(0) - h(0)$. Finally (24) and (25) imply (23).

The functions (22), (23) and (24) indeed satisfy (GH2).

4. Problem C

We need the following result of A. Járai ([11] Theorem 2.7.2).

THEOREM J

Let $\mathcal{T}$ be a locally compact metric space, let Z_0 be a metric space, and let Z_i $(i = 1, 2, \ldots, n)$ be separable metric spaces. Suppose that D is an open subset of $\mathcal{T} \times \mathbb{R}^k$ and $X_i \subset \mathbb{R}^k$ for $i = 1, 2, \ldots, n$. Let $f_0 : \mathcal{T} \to Z_0$, $f_i : X_i \to Z_i$, $g_i : D \to X_i$, $H : D \times Z_1 \times Z_2 \times \ldots \times Z_n \to Z_0$ be functions. Suppose, that the following conditions hold:

(1) *For every $(t, y) \in D$*

$$f_0(t) = H(t, y, f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))).$$

(2) *f_i is Lebesgue measurable over X_i for $i = 1, 2, \ldots, n$.*

(3) *H is continuous on compact sets.*

(4) *For $i = 1, 2, \ldots, n$, g_i is continuous, and for every fixed $t \in \mathcal{T}$ the mapping $y \to g_i(t, y)$ is differentiable with the derivative $D_2 g_i(t, y)$ and with the Jacobian $J_2 g_i(t, y)$; moreover, the mapping $(t, y) \to D_2 g_i(t, y)$ is continuous on D and for every $t \in \mathcal{T}$ there exists a $(t, y) \in D$ so that*

$$J_2 g_i(t, y) \neq 0 \quad \textit{for } i = 1, 2, \ldots, n.$$

Then f_0 is continuous on $\mathcal{T}$.

LEMMA 1

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ *satisfy the functional equation* (GH2) *for all* $x, y \in (0,1)$ *then the functions* $f, g, h : (0,1) \to \mathbb{R}$ *are continuous.*

Proof. First we prove the continuity of f. From (GH2), with $t = xy$, we obtain

$$f(t) = h\left(\frac{t}{y}\right) + h(y) - g\left(1 - \frac{t}{y} - y + t\right), \quad (0 < t < y < 1). \tag{29}$$

Let $\mathcal{T} = (0,1)$, $n = 3$, $Z_0 = Z_1 = Z_2 = Z_3 = \mathbb{R}$, $X_1, X_2, X_3 = (0,1)$, $D = \{(t,y) \in \mathbb{R}^2 \mid 0 < t < y < 1\}$. Define the functions g_i on D by $g_1(t,y) = \frac{t}{y}$, $g_2(t,y) = y$, $g_3(t,y) = 1 - \frac{t}{y} - y + t$ and let $H(t,y,z_1,z_2,z_3) = z_1 + z_2 - z_3$. It follows from (29) that the functions f_i $(i = 0,1,2,3)$ given by

$$f_0 = f, \quad f_1 = f_2 = h, \quad f_3 = g$$

satisfy the functional equation occurring in (1) of Theorem J for all $(t,y) \in D$ and f_i $(i = 0,1,2,3)$ is measurable by the conditions of our lemma. H is clearly continuous and condition (4) of Theorem J holds too, since calculating $D_2 g_i$ one can see that for every $t \in \mathcal{T} = (0,1)$

$$D_2 g_i(t,y) \neq 0 \quad \text{for } i = 1,2,3 \text{ if } y \neq \sqrt{t}.$$

Thus, by Theorem J, $f = f_0$ is continuous on $(0,1)$.

The continuity of g can be proved by making the substitutions $x \to 1 - x$, $y \to 1 - y$ in (GH2) and repeating the above argument.

Substituting $y = \frac{1}{2}$ in (GH2) and solving the equation obtained for h we get

$$h(x) = g\left(\frac{x+1}{2}\right) + f\left(\frac{1}{2}x\right) - h\left(\frac{1}{2}\right), \quad x \in (0,1), \tag{30}$$

whence, by the continuity of f, g it follows that h is continuous as well.

LEMMA 2

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ *satisfy the functional equation* (GH2) *then they are differentiable infinitely many times on* $(0,1)$.

Proof. Write (GH2) in the form (29) and let $[\alpha, \beta] \subset (0,1)$ be arbitrary and choose the interval $[\lambda, \mu]$ such that $\sqrt{\beta} < \lambda < \mu < 1$ (then $[\alpha, \beta] \times [\lambda, \mu] \subset D = \{(t,y) \mid 0 < t < y < 1\}$ holds). Integrating (29) with respect to y on $[\lambda, \mu]$ we obtain

$$(\mu - \lambda) f(t) = \int_\lambda^\mu h\left(\frac{t}{y}\right) dy + \int_\lambda^\mu h(y)\, dy - \int_\lambda^\mu g\left(1 - \frac{t}{y} - y + t\right) dy.$$

We use the substitutions $g_1(t,y) = \frac{t}{y} = u$ and $g_3(t,y) = 1 - \frac{t}{y} - y + t = u$ in the first and third integral respectively. It is easy to check that these equations can uniquely be solved for y if $t \in [\alpha, \beta]$. In the case of $\frac{t}{y} = u$ this is clear. In the case of $1 - \frac{t}{y} - y + t = u$ this uniqueness is ensured by the assumption $\sqrt{\beta} < \lambda$, namely, by this condition, the derivative of the function $y \to g_3(t,y)$:

$$D_2 g_3(t,y) = \frac{t}{y^2} - 1$$

is negative on $[\alpha,\beta] \times [\lambda,\mu]$ hence our function is strictly decreasing. The solutions

$$y = \frac{t}{u} = \gamma_1(t,u) \quad \text{and} \quad y = \frac{1+t-u+\sqrt{(1+t-u)^2-4t}}{2} = \gamma_2(t,u)$$

are infinitely many times differentiable functions of t and u. Performing the substitutions we have for $t \in [\alpha,\beta]$

$$f(t) = \frac{1}{\mu-\lambda}\left[\int_{\frac{t}{\lambda}}^{\frac{t}{\mu}} h(u) D_2\gamma_1(t,u)\,du - \int_{1-\frac{t}{\lambda}-\lambda+t}^{1-\frac{t}{\mu}-\mu+t} g(u) D_2\gamma_2(t,u)\,du + C\right],$$

where $C = \int_\lambda^\mu h(y)\,dy$. The functions h, g are at least continuous. Hence, by repeated application of the theorem concerning the differentiation of parametric integrals (see e.g. Dieudonné [8]), the sum on the right hand side is differentiable infinitely many times on $[\alpha,\beta]$. Since $[\alpha,\beta]$ is an arbitrary subinterval of $(0,1)$, we have that f is differentiable infinitely many times on $(0,1)$. The differentiability of g can be obtained similarly. Finally, from (30), we can deduce that h is also differentiable infinitely many times on $(0,1)$.

LEMMA 3

If the functions $f, g, h : (0,1) \to \mathbb{R}$ satisfy the functional equation (GH2) *and they are twice differentiable in* $(0,1)$*, then there exist constants $\gamma, C_i, \delta_i \in \mathbb{R}$* $(i = 1,2)$ *such that*

$$f(x) = C_1 \ln x + \gamma x + \delta_1, \quad x \in (0,1), \tag{31}$$

$$g(x) = C_2 \ln(1-x) + \gamma x + \delta_2, \quad x \in (0,1), \tag{32}$$

$$h(x) = C_1 \ln x + C_2 \ln(1-x) + \gamma x + \frac{\delta_1+\delta_2}{2}, \quad x \in (0,1). \tag{33}$$

Proof. Differentiating (GH2) with respect to x, then the resulting equation with respect to y, we have

$$f'(xy) + xyf''(xy) - g'(x+y-xy) + (1-x)(1-y)g''(x+y-xy) = 0, \quad x,y \in (0,1).$$

This can hold if and only if

$$tf''(t) + f'(t) = \gamma = (s-1)g''(s) + g'(s), \quad t, s \in (0,1)$$

for some constant γ.

The general solutions of the differential equations

$$tf''(t) + f'(t) = \gamma, \quad t \in (0,1)$$

and

$$(s-1)g''(s) + g'(s) = \gamma, \quad s \in (0,1)$$

have the following forms

$$f(t) = C_1 \ln t + \gamma t + \delta_1, \quad t \in (0,1),$$
$$g(s) = C_2 \ln(1-s) + \gamma s + \delta_2, \quad s \in (0,1).$$

Then, from (30), (31) and (32), we get (33) for h.

Thus we have proved our lemma.

We may sum up the results of Lemmas 1, 2 and 3 in the following theorem.

THEOREM 5

If the measurable functions $f, g, h : (0,1) \to \mathbb{R}$ satisfy the functional equation (GH2) *for all $x, y \in (0,1)$, then there exist constants $\gamma, C_i, \delta_i \in \mathbb{R}$ $(i = 1, 2)$ such that f, g, h have the forms* (31), (32) *and* (33) *respectively.*

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Institute of Mathematics and Informatics
Lajos Kossuth University
H-4010 Debrecen
P.O. Box 12
Hungary
E-mail: lajko@math.klte.hu

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