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Dynamical interpretation of Schröder's equation. Its consequences

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. This text deals with the domain of existence of the solution of Schröder's equation, related to a two-dimensional real iteration process, defined by functions which do not satisfy the Cauchy-Riemann conditions. Its purpose is limited to the identification of the difficulties generated by the determination of this domain. When the Cauchy-Riemann conditions are verified the answer to this problem was given by Fatou at the beginning of the 20th century. The qualitative theory of dynamical systems permits to identity the difficulties which may be met, from the notion of immediate basin of an attractor (stable fixed point in our case), and the singular set generated by the iteration associated with Schröder's equation.

1. Introduction

An evident link exists between *autonomous recurrences* (equivalent denominations depending on the mathematical field: *iterations, maps*), and some functional equations like those of Schröder, or Böttcher, or Abel, or Perron-Frobenius in some special case, or the equation of automorphic functions. This paper is essentially devoted to Schröder's functional equation. It is well known that the first set of studies on this equation appeared from the end of the 19th century in n-dimensional problems. The fundamental contributions are those of Grevy, Leau, Koenigs, Lattés whose papers concern a "*local*" study of the solution, i.e. its determination inside a sufficiently small neighborhood of a fixed point, or a cycle. A first "*global*" study is due to Julia (1918) [1], and Fatou (1919) [2] [3]. It concerns one-dimensional "rational" iterations with a complex variable, i.e. equivalently, two-dimensional iterations defined by functions with real variables satisfying the *Cauchy-Riemann conditions*. In particular Fatou's results suppose the existence of a stable fixed point with a non-zero multiplier, the boundary of its basin (domain of convergence toward

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this point) being what is called now a "*Julia set*". In this case it was stated that the fundamental solution of the Schröder's equation is a holomorphic function inside the ***immediate basin*** of the fixed point, the basin boundary belonging to the set of the ***essential singular points*** of this solution. As far as I know, this question has remained unexplored after these results for Schröder's equations related to classes of two-dimensional real iterations which do not satisfy the Cauchy-Riemann conditions. In this last more general case, it is a question of knowing if the notion of immediate basin can play the same role. In fact for noninvertible maps, and maps with canceling denominators, it appears that it is prudent to consider only a part of the immediate basin as domain of existence of the solution of the Schröder's equation associated with a stable fixed point of the iteration.

This paper does not pretend to deal with a close mathematical presentation of an extension of the Fatou's results in the case of *two-dimensional iterations with real variables*. Such a presentation would imply very long developments related to the convergence of series expansions, or infinite products, with the inherent difficulties induced by the boundaries of the domains of convergence. The aim is more modest. Indeed this text only tries to show how the dynamical approach permits to outline an extension of the Fatou's results. For the mathematicians specialists of functional equations this might give some first indications about the "landscape" of this question and its difficulties, from a point of view external to their field of study. For such a limited purpose, in the framework of the *qualitative theory of Dynamics*, it is sufficient to expose with commentaries a summarized presentation of certain results, obtained since some 30 years, on the basins structure generated by two-dimensional iterations with real variables. About the qualitative methods, it is well known that the solutions of equations of nonlinear dynamic systems are in general not classical transcendental functions of the Mathematical Analysis, which are very complex. So analytical methods generally failing, the "qualitative strategy" is of the same type as the one used for the characterization of a function of the complex variable by its singularities: zeros, poles, essential singularities. Here, for *two-dimensional maps with real variables* (topic of the paper) the complex transcendental functions are defined by the ***singularities*** of continuous (or discrete) dynamic systems such as:

— stationary states which are equilibrium points (fixed points), or periodical solutions (cycles); which can be stable, or unstable;
— trajectories (invariant curves) passing through *saddle* singularities of two dimensional systems;
— stable and unstable manifold for a dimension greater than two;
— boundary, or separatrix, of the influence domain of a stable (attractive) stationary state, called domain of attraction, or *basin*;
— *homoclinic*, or *heteroclinic singularities*;

— or more complex singularities of *fractal*, or nonfractal type.

The qualitative methods consist in the identification of two spaces associated with the map (iteration, recurrence relationship). The first space, called ***phase space*** (defined by the map variables), is related to the nature of the above singularities. The second space, called ***parameter space***, characterizes the singularities evolution when the system parameters vary, or in presence of a continuous structure modification of the system (definition of a ***function space***), by identification of the ***bifurcation sets***, loci of points boundary between two different qualitative changes. In the dynamics framework an *iteration* (equivalent denominations: recurrence relationship, map) is considered as a mathematical model of a discrete dynamical system. Since 1960, the important development of the computer means has given a large extension to the numerical approach of the problems of dynamic systems. Such an approach constitutes a powerful tool, when it is associated with the qualitative, or analytical, methods. In particular such a "mixed" approach has permitted to understand the complex structure of basins, and their bifurcations, that is the change of their qualitative properties in presence of parameter variations, cf. [7].

The paper is limited to ***two-dimensional Schröder's equations with real variables*** considered in the framework of the qualitative approach. This implies to define different classes of problems associated with basin boundaries (singular sets) of different nature. So problems involving ***invertible iterations***, ***noninvertible ones***, ***iterations defined by functions with a vanishing denominator*** must be differentiated.

The first part is a reminder of the Julia-Fatou's results. It is followed by the presentation of the matter related to two-dimensional maps not satisfying the Cauchy-Riemann conditions. The considered maps are firstly invertible, then noninvertible, without vanishing denominators in these two cases. The case of a vanishing denominator is dealt with in the last part.

2. Reminder of the Julia-Fatou's results

Let

$$z' = R(z) \tag{1}$$

be a one-dimensional iteration (or map, or recurrence relationship, or substitution), $R(z)$ being a rational function of the complex variable z, supposed not being of "fundamental circle" type. For simplicity sake it is assumed that the map has a unique attracting stable fixed point O, $S = R'(O)$ is its multiplier, $|S| < 1$, $S \neq 0$.

Let E be the set of all the unstable cycles generated by the iteration. Julia and Fatou [1]-[3] proved that the derived set E' of E contains E and is perfect. They showed that ***la structure de*** E' ***est la même dans toutes ses parties***, which means that the E' structure is self-similar, called *fractal* from

1976. The set E' can be either continuous, or discontinuous, it constitutes the set of essential singularities for any function limit of functions, extracted from an iterated sequence. It is also the set, the iterates of which do not form a *normal sequence* in the Montel sense. E' is the basin boundary of O, i.e., the boundary of the open domain of convergence toward O. It contains the whole set of the increasing rank preimages of the points of E. When it is a continuous set, the basin is generally disconnected, and made up of infinitely many disjoint parts. Then the part D_0 containing O is called the *immediate basin*, it includes a *critical point* (image of the point at which $\frac{dR}{dz} = 0$) of $R(z)$, and may be multiply connected with infinitely many holes.

The Fatou's contribution to the Schröder equation,

$$\gamma[R(z)] = S\gamma(z), \tag{2}$$

constitutes a particular case of more general functional equations considered in Chap. 7 of [3]. The main result states:

> The fundamental solution of the functional Schröder equation is a holomorphic function inside the immediate basin D_0 of O. Inside this domain it has infinitely many zeros having as limit points all the points of the immediate basin boundary ∂D_0. In the neighborhood of each of these boundary points, the function is completely indeterminated and takes all the values except infinity. Then the points of the immediate basin boundary are essential singular points of $\gamma(z)$.

The domain of existence of the function $\gamma(z)$ coincides with the connected domain of convergence (containing O) of the infinite product which permits to define $\gamma(z)$. The total basin of O may be disconnected. Then it is made up of the immediate basin D_0 and infinitely many domains which are the "arborescent" infinite sequences of its increasing rank preimages. Let D_1 be a rank-one preimage of D_0, different from D_0. So inside D_1 a function $\gamma_1(z)$ is defined. The variable z being in D_1, $R(z)$ is in D_0 and one has:

$$S\gamma_1(z) = \gamma[R(z)].$$

When z is inside D_0, the function $\gamma(z)$ satisfies (2). A generalization of a process of analytic continuation would give $\gamma_1(z)$ as the continuation of $\gamma(z)$ inside D_1. But in general D_1 and D_0 have no common points. So it would be necessary to find some lines out of the total basin, having contacts with the boundary, and leading along such lines to a uniform convergence of the expressions defining $\gamma(z)$. Until now it seems that this process has not been realized. So the functional equation defines infinitely many analytical functions having different bounded domains of existence. In the framework of the qualitative theory of dynamical systems, the solution $\gamma(z)$ of the Schröder equation is defined by the singular set made up of the zeros of $\gamma(z)$, which are the successive preimages of O in infinite number inside D_0, and the points of the boundary $\partial D_0 \subseteq E'$ of the immediate basin of O.

It is worth noting that the one-dimensional map (1) of the complex variable $z = x + jy$, $j^2 = -1$, is equivalent to the two-dimensional map with real variables:

$$x' = f(x, y), \quad y' = g(x, y), \tag{3}$$

the functions $f(x, y)$, $g(x, y)$ satisfying the conditions of Cauchy-Riemann:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}. \tag{4}$$

An example illustrating the properties of the complex set ∂D_0 is given by the map:

$$z' = \frac{3z - z^3}{2};$$
$$f(x, y) = \frac{3x}{2} - \frac{x^3 - 3xy^2}{2}, \quad g(x, y) = \frac{3y}{2} + \frac{y^3 - 3x^2y}{2}. \tag{5}$$

The origin is an unstable fixed point. This map has two stable fixed points ($x = \pm 1$, $y = 0$). The set E' made up of double points is everywhere dense. It is formed by the union of infinitely many closed simple Jordan curves, every points of one of these curves being the limit points of similar curves out of the one considered, their sizes tending toward zero, cf. [1]. The whole fractal set E' is symmetric with respect to the two axes. Figure 1 (see p. 74) represents the basin of each of the two stable fixed points from two different grey shades, and an enlargement of a basins part. The domain of existence of the solution of the Schröder equation related to one of the stable fixed point is its immediate basin.

3. Two-dimensional maps not satisfying the Cauchy-Riemann conditions

3.1. Difficulties generated by the problem

Consider the two-dimensional map (3) with real variables, the functions $f(x, y)$, $g(x, y)$ being analytic, and not satisfying conditions (4). Denote this map by T, and put $X = [x, y]$. The map (3) can be written in the form $X' = TX$. Let $O(0; 0)$ be a stable fixed point of T, i.e., with multipliers $0 <| S_i |< 1$, $i = 1, 2$. Consider the corresponding Schröder's equation:

$$\gamma_i(x', y') = S_i \gamma_i(x, y)], \quad \text{or} \quad \Gamma(TX) = S\Gamma(X), \tag{6}$$

with $\Gamma = [\gamma_1, \gamma_2]$, $S = [S_1, S_2]$. In the case of a stable cycle of period (or order) k, i.e., made up of k consequent points verifying: $T^k(X) = X$, $T^m(X) \neq X$, $0 < m < k$ ($k = 1$ gives a fixed point), the conclusions will be the same by considering T^k in (6).

An outline of extension of the Fatou's results would be given remarking that, if X varies in the whole immediate basin D_0 of the fixed point $O(0; 0)$, it would seem reasonable to conjecture that the infinite products which define

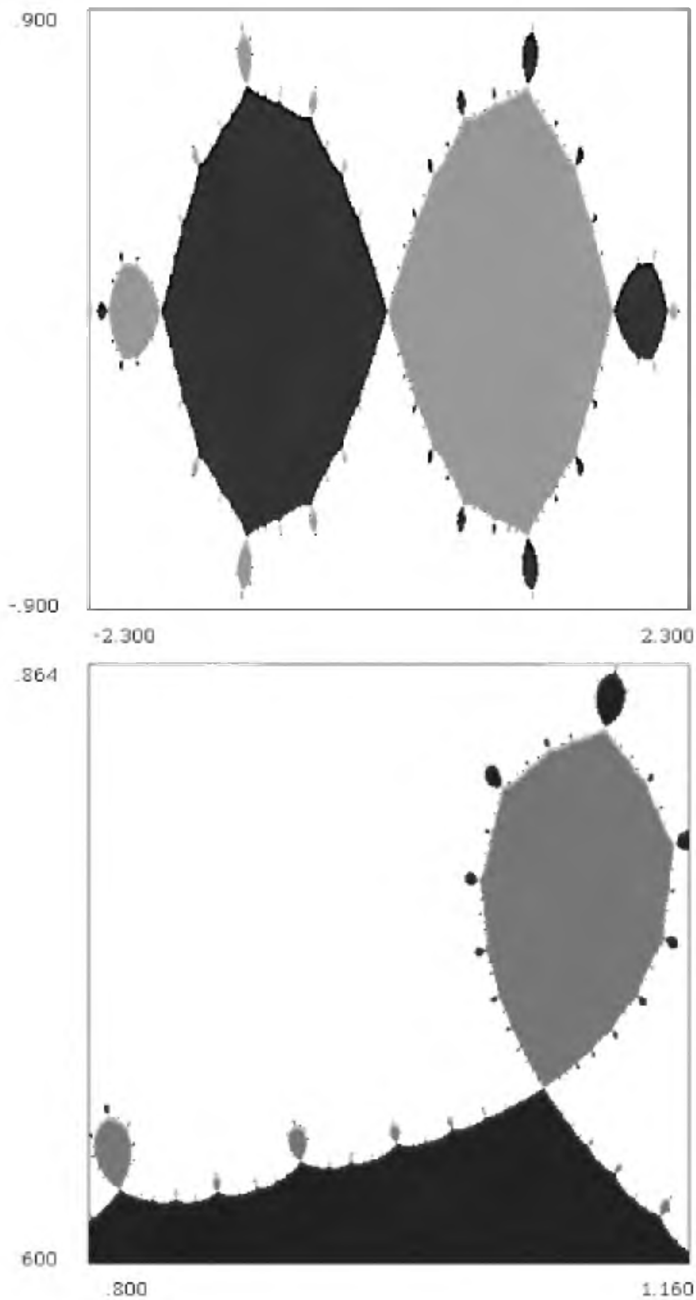

Fig. 1. Map (5). Basins of the fixed points ($x = +1$; $y = 0$; clear grey) and ($x = -1$; $y = 0$; dark gray)

$\Gamma(X)$ are uniformly convergent inside all close domains fully interior to D_0. Then $\Gamma(X)$ would be analytic inside D_0 and would satisfy (6) for each of its points. In this case the boundary ∂D_0 of the immediate basin would belong to the singular set related to $\Gamma(X)$. Therefore, in the framework of a qualitative approach, the problem boils down to study the structure of ∂D_0 and its qualitative modifications in presence of parameters variations. I think that such a conjecture is not true for all the iteration (or map) forms. It depends on the nature of the map, which in particular implies the consideration of the following classes of problems:

(a) T is a *diffeomorphism* defined by functions without canceling denominator,

(b) T is a *noninvertible map* defined by functions without vanishing denominator,

(c) For each of the two last cases T is defined by *functions with vanishing denominator.*

For the two-dimensional maps considered now it is important to note that ∂D_0 loses the properties of the perfect set E' mentioned in Sec. 2. Generally the new situations also present difficulties explained as follows. In the dynamics approach the knowledge of cells, giving the same qualitative behavior of solutions in the parameter space, is of prime importance for the analysis and the synthesis of continuous, or discrete mathematical models. On the boundary (*bifurcation set*) of a cell, a dynamic system is *structurally unstable.* In order to identify the difficulties, it is necessary to remind that the study of ordinary differential equations can be made via a Poincaré section leading to a map, the effective dimension of which is smaller. So a three-dimensional *flow* (vector field, or autonomous ordinary differential equation) leads to the formulation of a two-dimensional invertible map. In 1966 Smale showed that *for n-dimensional vector fields, $n > 2$, structurally stable systems are generally not dense in the function space*, which does not occur for $n = 2$. This means that p-dimensional maps, $p \geqslant 2$, have the same properties. So it appears that, with an increase of the problem dimension, one has an increase of complexity of the parameter (or function) space. This complexity appears for flows from the case $n = 3$, or for maps from the dimension $p = 2$. It results that the boundaries of the cells defined in the phase space (basins), as well as in the parameter space, have in general a complex structure which may be fractal (self-similarity properties) for n-dimensional vector fields, $n > 2$, and for p-dimensional maps with $p \geqslant 2$.

In 1979 Newhouse stated that in any neighborhood of a C^r-smooth ($r \geqslant 2$) dynamical system, in the space of dynamical systems, there exist regions for which systems with homoclinic tangencies (then with structurally unstable, or nonrough, homoclinic orbits) are dense. Domains having this property are called *Newhouse regions.* This result, as completed in [4], asserts that systems

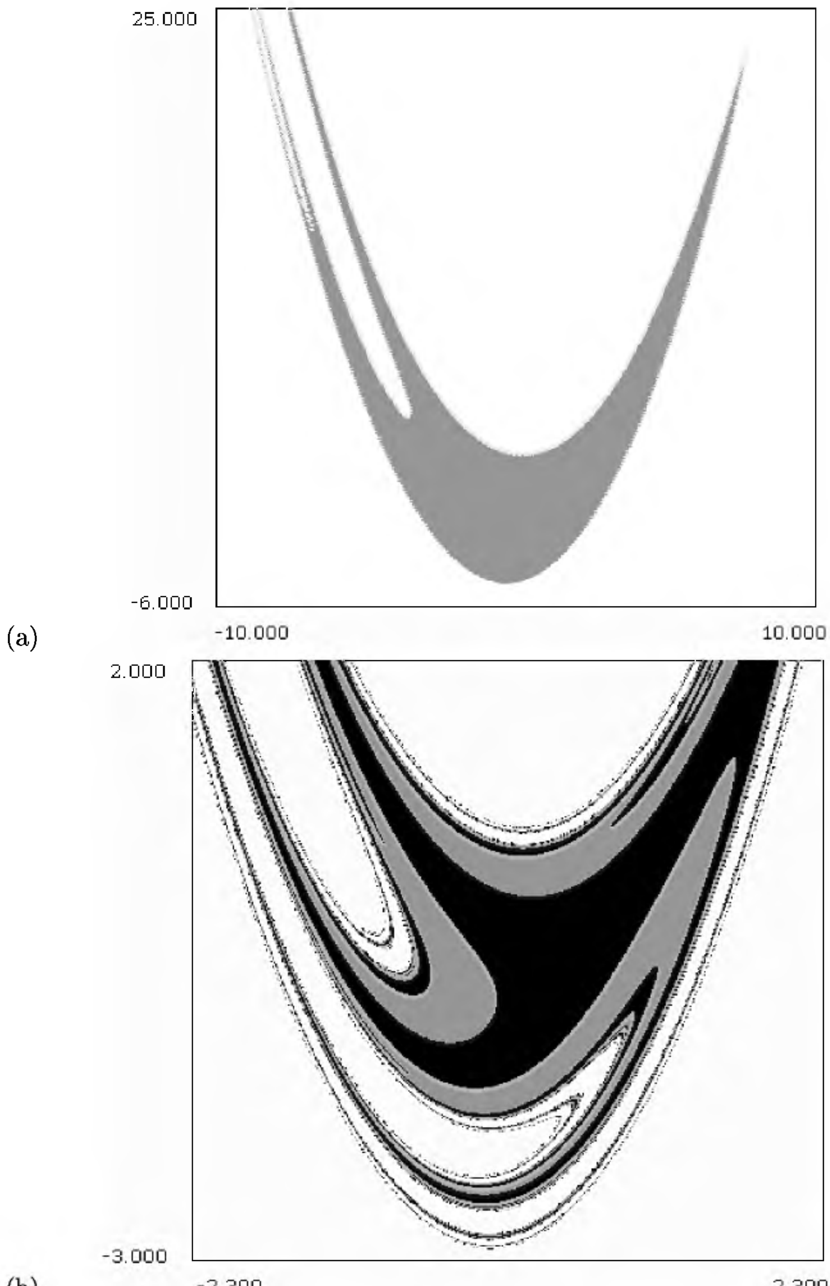

Fig. 2. Map (7). Basin of the fixed point q_2 (grey), and basin of the period 3 cycle (black). (a) $a = 0.4$, $b = 0.6$. (b) $a = 0.92$, $b = 0.7$.

with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has the important consequence concerning the dynamical properties:

Systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible.

More particularly, many of the attractors obtained numerically contain a "large" hyperbolic subset in presence of a finite, or an infinite number of stable periodic solutions. Generally such stable solutions have large periods, and narrow "oscillating" tangled basins, which are very difficult to display numerically. So it is only possible to consider some of the characteristic properties of the system, their interest depending on the nature of the problem nature, cf. [5]. The general problem of defining globally, and not locally, the solution of the functional equation (6) suffers from such limitations, due to the complex structure of a basin boundary. In the case of two-dimensional non-invertible maps this complexity increases, due to the introduction of another type of singularity: the *critical curve*, locus of points having two coincident rank-one preimages.

3.2. Diffeomorphisms without canceling denominator

Such maps, being invertible, have the important property: *the basin D of an attractor is always simply connected*, that is the immediate basin coincides with the total basin and contains no hole. In the simplest case, i.e., in absence of homoclinic and heteroclinic points, in general the boundary ∂D of D belongs to the *stable manifold* of some saddle cycles of period k ($k = 1$ corresponding to a fixed point). Locally this stable manifold can be defined by the series expansion given by S. Lattés. Its global determination is obtained by using the terms of the series until a given rank as a "germ" in a numerical method, for constructing ∂D, which belongs to the singular set of the solution of (6). Another numerical method consists in a scanning of the phase plane (x, y), which checks the convergence of the iterated sequence generated by each pixel of this plane as an initial condition. For each of these two methods it is possible to control the precision of the result.

Figure 2 (see p. 76) shows a type of basin (with fractal properties) obtained from the map:

$$x' = 1 - ax^2 + y, \quad y' = bx \tag{7}$$

which has two fixed points, $q_1(x_1, bx_1)$, and $q_2(x_2, bx_2)$, where

$$x_1 := \frac{1}{2a}\left(b - 1 - \sqrt{\Delta}\right), \quad x_2 := \frac{1}{2a}\left(b - 1 + \sqrt{\Delta}\right); \quad \Delta := (1-b)^2 + 4a.$$

For $a = 0.4$, $b = -0.6$, q_1 is a saddle point, and q_2 is asymptotically stable. The basin D of q_2 is given by the grey marked region of Fig. 2a, the white one being the domain of divergence. It is bounded by the stable manifold of the saddle q_1, $W^S(q_1) \equiv \partial D$, a branch going to infinity. Such a parameter value of the plane (a, b) belongs to a region of the parameter plane, called *Morse-Smale region*, for which a unique attractor exists, with absence of homoclinic points, cf. [6]. It results a simple structure of the basin boundary, and then a "simple" singular set related to the solution of the Schröder equation (6).

Fig. 2b corresponds to $a = 0.92$, $b = -0.7$, a parameter point out of the Morse-Smale region, leading to the presence of homoclinic and heteroclinic points. The grey part is the basin of the stable fixed point q_2, the dark one is the basin of a stable period three cycle, the white region gives rise to divergence. The basins of q_2 and that of the stable period three cycle are separated by the stable manifold of its "satellite" period three saddle (i.e., the two period three cycles come from the same fold bifurcation). The two basins present infinitely many more and more narrow oscillating parts, tangled with the domain of divergence. A section of such regions by a line gives a *Cantor set*. The stable manifold of the saddle q_1 is a line of accumulations of the above oscillations. For this situation it is worth noting that the parameter point is in a Newhouse region. Therefore a numerically obtained image as Fig. 2b cannot make appear other eventual stable states having large periods, and very narrow "oscillating" tangled basins. This situation increases the complexity of the true "mathematical" structure of the basin of q_2, with its consequences on the structure of the singular set of the solution of the Schröder equation (6). Nevertheless the total basin being simply connected, an extension of results of Sec. 2 might present no difficulty in principle. In such a case the only "practical" difficulty lies in the fact that the domain of existence of the solution of (6) has a very complex structure. We shall say that this domain permits to define the "global" solution of Schröder's equation (6).

3.3. Non-invertible maps without canceling denominator

3.3.1. Difficulties generated by "global" solution of Schröder's equation in the simplest case

This section essentially concerns a family of two-dimensional smooth non-invertible maps, $X \to T(X)$, $X = [x, y]$, such that the critical curve LC is made up of only one branch separating the plane $\mathbb{R}^2$ in two open regions Z_0 and Z_2, the points of which have respectively 0 and 2 preimages (or antecedents or backward iterates) of rank one. The two real preimages of a point X belonging to Z_2 are given by the two inverses $T_1^{-1}(X)$, $T_2^{-1}(X)$ of T. Such noninvertible maps (which are the simplest ones) are called of (Z_0-Z_2) *type* (cf. [7]). Their study is indispensable before considering more complex types, which locally may have the (Z_0-Z_2) properties, plus others induced by more than two first rank preimages in certain regions of $\mathbb{R}^2$. The curve LC is the locus of points having two coincident rank-one preimages, located on a curve LC_{-1}, with $LC = T[LC_{-1}]$. If the map T is smooth, LC_{-1} is contained in the set on which the Jacobian J of T vanishes.

Denote by R_1, R_2 the two open regions such that $LC_{-1} = \overline{R_1} \cap \overline{R_2}$, and for every $X \in Z_2$, let $T_1^{-1}(X) \in R_1$, $T_2^{-1}(X) \in R_2$ be the two first rank preimages of X. If $X \in LC$ then $T_1^{-1}(X) = T_2^{-1}(X) \in LC_{-1}$.

It is recalled that a closed and invariant set A is called an *attracting set* if some neighborhood U of A exists such that $T(U) \subset U$, and $T^n(X) \to A$ as $n \to \infty$, $\forall X \in U$. An attracting set A may contain one or several *attractors* coexisting with sets of repulsive points (*strange repulsors*) giving rise to either *chaotic transients* towards these attractors, or *fuzzy boundaries* of their basin, cf. [6], [7]. The set $D = \bigcup_{n \geqslant 0} T^{-n}(U)$ is the *total basin* (or simply: basin of attraction, or influence domain) of A. That is D is the open set of points X whose forward trajectories (set of images of X with increasing rank) converge towards A. D is invariant under backward iteration T^{-1} of T, but not necessarily invariant by T:

$$T^{-1}(D) = D, \qquad T(D) \subseteq D. \tag{8}$$

In (8), the strict inclusion holds iff D contains points of Z_0, i.e., points without preimages. The relations in (8) hold also for the closure of D. The boundary (or frontier) of D is denoted by ∂D. The boundary ∂D is defined by the geometrical equality $\partial D = \overline{D} \cap \overline{C'(D)}$ where $C'(D)$ denotes the complementary set of D. This boundary satisfies:

$$T^{-1}(\partial D) = \partial D, \qquad T(\partial D) \subseteq \partial D \tag{9}$$

We remark that $T^{-1}(D) = D$ implies that D must contain the set of preimages of any of its cycles, that is ∂D must contain the stable set W^S of any cycle of T belonging to ∂D, while $T(\partial D) \subseteq \partial D$ means that the images of any of its points belongs to $\partial D \cap Z_2$. It is worth noting that, for unstable node and focus cycles, the stable set W^S is made up of the set of increasing rank preimages of cycle points (such a set does not exist in the case of an invertible map). For a saddle cycle W^S is made up of the local stable set W_l^S, associated with the determination of the inverse map which let invariant this cycle, and its increasing rank preimages.

Properties (8) and (9) with the strict inclusion are illustrated by the following example, cf. [7]:

$$x' = y, \qquad y' = 0.8x + 0.02y + x^2 + y^2, \tag{10}$$

leading to a simply connected basin of the stable fixed point $O(0;0)$. The curve of coincident first rank preimages LC_{-1} is $x = -0.4$, and the critical curve LC is the parabola $y = x^2 + 0.02x - 0.16$. The region R_1 is defined by $x > -0.4$, and R_2 by $x < -0.4$. The fixed point $O(0;0)$, $O \in R_1$, is a stable node. A second fixed point $P(0.09;0.09)$, $P \in R_1$, is a saddle with multipliers of opposite signs. Figure 3 (see p. 80) represents the boundary ∂D of the simply connected basin D of O. This boundary consists of the stable invariant set W^S of P. The determination of T^{-1} which let O and P invariant leads to T_1^{-1}. The set W_l^S is the open segment $]B, B_{-1}[$, where B_{-1} belongs to LC_{-1}, $B = T(B_{-1})$, and $W^S = W_l^S \cup T^{-1}(W_l^S) \cup T^{-2}(W_l^S)$. P has only one first rank preimage P_{-1} different from P. The two first rank preimages of B_{-1} are noted B^1_{-2} and B^2_{-2} in Fig. 3. Finally, $T^{-1}(C) = C_{-1} \in LC_{-1}$. The basin of O is simply connected, and satisfies: $T(D) = D \cap \overline{Z_2} \subset D$.

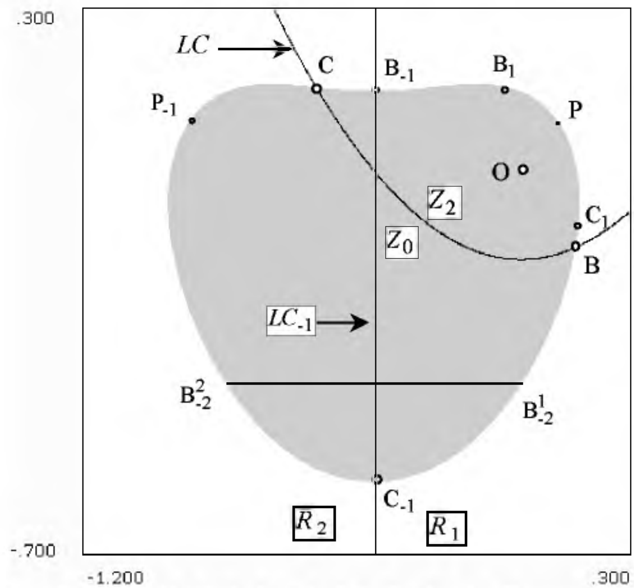

Fig. 3. Map (10). Basin (grey) of the fixed point O.

More generally a basin D may be *connected*, or *disconnected*. A connected basin may be *simply connected*, or *multiply connected* (which means connected with holes). A disconnected basin consists of a finite or infinite number of connected components (which may be simply or multiply connected). The properties and bifurcations related to these different situations will be considered in the next section. If A is a connected attractor (particular example: A is a fixed point), the *immediate basin* D_0 of A, is defined as the widest connected component of D containing A.

Let us return to the example of the map (10), where the two fixed points O and P are located in the region R_1. Consider the Schröder equation related to the stable fixed point O and note that the following properties:

$$T(D \cap Z_0) = D \cap Z_2, \quad T(D \cap R_2) = D \cap R_1 \cap Z_2$$

are satisfied. Using the Fatou's arguments, the extension of Sec. 2 results might present no essential difficulty in the region $D \cap R_1 \cap Z_2$, where the inverse of T which let O invariant is T_1^{-1}. Then one can conjecture that this region is at least a sufficient domain of existence of the solution of (6). It is not the case in the complementary region inside the basin D. Indeed a process of analytic continuation is not evident in this last region.

3.3.2. Problems generated by "global" solution of Schröder's equation in more general cases

The basin boundaries, belonging to the singular set of the solution of Schröder's equation, can have very complex structure with fractal properties described in [7]. From parameter variations they can undergo qualitative changes, related to ***bifurcations*** resulting from the contact of the basin boundary with a critical curve, or one of its image of a certain rank. The general case induces more complex situations with respect to the (10) one. This is due in particular to a large variety of qualitative modifications, with different types of fractalization, undergone by an immediate basin, and so by the domain of existence of the solution of the corresponding Schröder's equation.

The following example dealing with the map T:

$$x' = y, \quad y' = \left(\frac{y}{5} - \frac{x}{2}\right)\left(x^2 + y^2 - 6\frac{x}{5} - \lambda y + \frac{1}{2}\right) + x, \tag{11}$$

illustrates such modifications, when λ varies in the interval $1.05 < \lambda < 4.8$. For the case $\lambda = 4.1$ see Fig. 4 below.

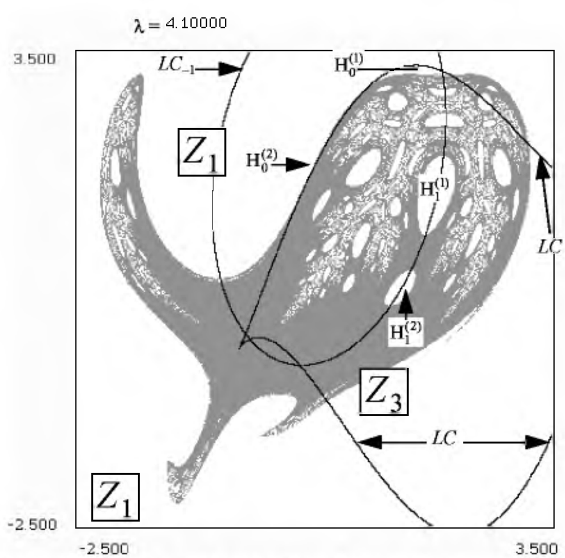

Fig. 4. Map (11), $\lambda = 4.1$. Basin (grey) of the fixed point O.

With the map T the fixed point $O(0;0)$ is always a stable node whatever be the parameter λ. The "global" solution of Schröder's equation, that is the domain of existence of the solution of (6), is considered for this fixed point O. It can be deduced from the detailed study of the basin modifications related to O described in [7] (cf. pages 439-446). Figure 4 gives an idea of the complexity of the O basin (grey marked) for $\lambda = 4.1$. This basin is multiply connected with a fractal structure.

3.4. Maps defined by functions with canceling denominator

Such maps T, invertible or non-invertible, introduce new types of singular sets, which has consequences on the determination of the domain of existence of the solution of the Schröder's equation (6). The first singularity concerns the set of nondefinition δ_s, locus of points in which at least one denominator vanishes, and the set of its successive preimages. The map is well defined provided that the initial condition belongs to the set $\hat{E}$ given by:

$$\hat{E} = \mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty} T^{-n}(\delta_s).$$

Indeed the points of the singular set δ_s, as well as all their preimages of any rank, which constitute a set of zero Lebesgue measure, must be excluded from the set of initial conditions in order to generate well defined sequences by iteration of T, so that $T : \hat{E} \to \hat{E}$.

Many other types of basin bifurcations, generated by two-dimensional non-invertible maps, and so many other qualitative changes of the existence domain of the solution of (6) are possible. Some of them are described in [7].

The presence of δ_s is followed by two other singular sets: *focal points* and *prefocal curve.* Roughly speaking a prefocal curve is a set of points for which at least one inverse exists, which maps (or "focalizes") the whole set into a single point, called focal point, which belongs to δ_s . More details on the consequences of such singularities on the structure of a basin, and on its bifurcations are given in [8]. It is evident that the domain of existence of the solutions of Schröder equation (6) must not contain a prefocal curve. Then when a prefocal curve cuts the immediate basin of a stable fixed point, separating this basin into two regions, the one which does not contain the fixed point must be excluded from the domain of existence of the solution of the Schröder equation (6). Indeed a process of analytic continuation might fail on the prefocal curve.

4. Conclusion

This text has the limited purpose to identify the difficulties generated by the determination of the domain of existence of the solution of a Schröder's equation, related to a two-dimensional real iteration (map) process, defined by functions which do not satisfy the Cauchy-Riemann conditions. The qualitative theory of dynamical systems permits to outline an answer from the notion of immediate basin of the stable fixed point of the considered iteration, and the singular sets generated by this iteration. It is worth noting that the dynamical approach permits to deal with another application of Schröder's equation. It concerns a method of construction of a class of iterations (recurrences, maps) giving rise to chaotic behaviors, which can be described from elementary functions. This process, taken on the pages 33-45 of [7] from results published in 1982, also leads to the definition of multi-dimensional function having properties similar to those of the *Chebyshev's polynomials.*

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