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Functional and differential equations

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. Functional equations play an important role in the theory of differential equations. Euler functional equation for homogeneous functions, Abel and Schröder functional equations and their systems, iteration groups of functions are essential tools for studying transformations and asymptotic properties of their solutions. And conversely, differential equations give answer to some problems in the theory of functional equations, e.g., decomposition of functions.

I. Introduction

Let us start with a historical remark. Floquet theory deals with linear differential systems

$$Y' = P(x)Y, \tag{1}$$

where $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuos periodic matrix,

$$P \in C^0(\mathbb{R}), \quad P(x+1) = P(x),$$

and $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a matrix solution of the system (1).

It is known, e.g., R. Bellman [2], that the solution Y is of the form

$$Y(x) = Q(x) \cdot e^{Bx},$$

where $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $Q \in C^1(\mathbb{R})$, is a periodic matrix, $Q(x+1) = Q(x)$, and B is a constant $n \times n$ matrix with generally complex elements.

The proof of this result is essentially based on the fact that together with a solution $x \mapsto Y(x)$ of the system (1) the function $x \mapsto Y(x+1)$ is also a solution. Since $Y(x)\cdot C$, C being a regular constant matrix, is a general solution of equation (1), there exists a constant regular matrix C_0 such that

$$Y(x+1) = Y(x) \cdot C_0, \quad \det C_0 \neq 0, \quad x \in \mathbb{R}. \tag{2}$$

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However, this is a matrix functional equation. If we suppose its solution Y of the form

$$Y(x) = Q(x) \cdot e^{Bx}, \tag{3}$$

where $Q(x+1) = Q(x)$ is a periodic matrix, then

$$Y(x+1) = Q(x+1) \cdot e^{Bx} \cdot e^{B} = Q(x) \cdot e^{Bx} \cdot e^{B} = Y(x) \cdot e^{B}.$$

If now $e^B = C_0$ ($B = \ln C_0$, $\det C_0 \neq 0$) then we see that any solution Y of the functional equation (2) must have the form (3). And this is the essence of the Floquet theory.

After this historical remembrance of an application of functional equations, let us continue with more recent results when functional equations play an important role in the study of differential equations. In general we may observe that functional equations occur when solutions of differential equations are considered in different points, e.g., in consecutive zeros, with delayed or advanced arguments, or when transformations of differential equations are considered.

II. Abel functional equation and linear differential equations

Consider the second order equation in the Jacobi form

$$y'' = p(x)y, \quad p \in C_0(I),\ I = (a,b) \subset \mathbb{R}, \tag{p}$$

$-\infty \leqslant a < b \leqslant \infty$. Suppose that this equation (p) is oscillatory for $x \to b$, i.e. each solution y of (p) has infinitely many zeros when x approaches the right end of the interval of definition.

In accordance with O. Borůvka [3], introduce the following notions.

DEFINITION 1

A *phase* α of equation (p) having two linearly independent solutions y_1, y_2 is defined as a continuous function on I satisfying the relation

$$\tan \alpha(x) = \frac{y_1(x)}{y_2(x)}$$

for all x where $y_2(x) \neq 0$.

PROPERTY 1

Phase α being continuous on the whole interval I, is also in $C^3(I)$ and $\alpha'(x) \neq 0$ on I.

PROPERTY 2

If α is a phase of equation (p) then its general solution is

$$y(x) = y(x; c_1, c_2) = c_1\, |\alpha'(x)|^{-\frac{1}{2}} \sin(\alpha(x) + c_2).$$

DEFINITION 2

Let $x_0 \in I$ be arbitrary, and y be a nontrivial solution of equation (p) vanishing at x_0, $y(x_0) = 0$. Denote by x_1 the first zero to the right of x_0 of this solution y. Define the *dispersion* of the equation (p) as the function

$$\varphi : I \to I, \quad \varphi(x_0) = x_1 \quad \text{for each } x_0 \in I.$$

The dispersion φ is well-defined since all solutions of equation (p) having a zero in x_0 have x_1 as its first zero to the right of x_0. Moreover, all such x_1 exist because equation (p) oscillates when $x \to b$.

O. Borůvka has proved

PROPOSITION 1

The dispersion φ and the phase α of an equation (p) satisfy the Abel equation

$$\alpha(\varphi(x)) = \alpha(x) + \pi \operatorname{sign} \alpha'. \tag{4}$$

Hence

$$\varphi : I \to I, \quad \varphi(x) > x, \quad \varphi \in C^3(I) \quad \text{and} \quad \varphi'(x) > 0.$$

Using these properties we proved [9] for differential equations (p) the following result ($\varphi^{[i]}$ denoting the i-th iterate of φ).

PROPERTY 3

If the dispersion φ satisfies one of the conditions

a) $\varphi - \mathrm{id}_I$ *is a nondecreasing function, or*

b) $\varphi - \mathrm{id}_I$ *is a nonincreasing function, or*

c) $\varphi - \mathrm{id}_I = \delta = \text{const.} > 0$,

then one of the three cases hold, respectively:

a′) *the maxima of absolute values of each solution of (p) on consecutive intervals $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, $i = 0, 1, 2, \ldots$, form a nondecreasing sequence,*

b′) *those maxima form a nonincreasing sequence,*

c′) *each solution of (p) is periodic or half-periodic with the period δ.*

Roughly speaking, if the distances between consecutive zeros of solutions are increasing, or decreasing or are equal, then their maxima are increasing, or decreasing, or equal (solutions are half-periodic).

By using this Abel equation (4) and results of B. Choczewski [5], M. Kuczma [7] and E. Barvínek [1], the second order equations with prescribed properties were constructed [12].

Recently the notion of dispersion was extended to some linear differential equations of an arbitrary order. The same effect concerning relations between distances of consecutive zeros of solutions and their asymptotic behaviour was proved in [14]. Also a construction of all n-th order linear differential equations with prescribed asymptotic properties was presented there.

III. Systems of Abel and Schröder functional equations, iteration groups of functions

Consider a generall nonlinear functional differential equation,

$$F(x, y(x), \dots, y^{(n)}(x), y(\xi_1(x)), \dots, y^{(n)}(\xi_1(x)),\\ \dots, y(\xi_k(x)), \dots, y^{(n)}(\xi_k(x))) = 0,$$

and the substitution $x = h(t)$, $z(t) = y(h(t))$ converting the above equation into

$$G(t, z(t), \dots, z^{(n)}(t), z(\eta_1(t)), \dots, z^{(n)}(\eta_1(t)),\\ \dots, z(\eta_k(t)), \dots, z^{(n)}(\eta_k(t))) = 0.$$

Then $y \circ \xi_i(x) = y \circ \xi_i \circ h(t) = (y \circ h) \circ (h^{-1} \circ \xi \circ h(t)) = z(\eta_i(t))$, i.e., $h^{-1}(\xi_i(h(t)) = \eta_i(t)$, or

$$h \circ \eta_i(t) = \xi_i \circ h(t), \quad i = 1, \dots, k,$$

expressing the fact that deviating arguments ξ_i and η_i are conjugate functions.

If we consider a possibility of a special choice of *canonical* deviations $\xi_i(x) = x + c_i$, $c_i =$ const., see [10], then we come to a problem of a common solution h of a *system of Abel functional equations* for prescribed η_i:

$$h(\eta_i(t)) = h(t) + c_i, \quad i = 1, \dots, k.$$

If $k = 1$, i.e. when we have a single Abel equation, there were lot of results in the literature, see e.g., [7]. For $k > 1$ there has recently been investigated these problems in Brno, Katowice and Kraków. We discovered several sufficient conditions for the existence of a solution of a system of Abel equations [10]. Then a systematic research was done by M.C. Zdun [16].

For linear functional differential equations we may take even more general transformations of Kummer's type $z(t) = f(t)y(h(t))$ which enable us to impose one more condition on coefficients because of a rather arbitrary function f in the transformation.

In the simplest case of linear functional differential equations of the first order with one delay

$$y'(x) + a(x)y(x) + b(x)y(\xi(x)) = 0$$

we may consider their *canonical form* as

$$z'(t) + c(t)z(t-1) = 0.$$

For another choice of special deviations, e.g., of the form $\xi_i(x) = c_i x$ we get a system of Schröder functional equations,

$$h(\eta_i(t)) = c_i h(t), \quad i = 1, \dots, k.$$

In general, zeros of solutions are preserved and they may be studied on canonical forms only. Since the factor f in the transformation can be explicitly evaluated from coefficients, asymptotic properties of solutions of equations, their boundedness, classes L^p, convergency to zero, or the rate of growth, can be obtained from these properties of canonical equations.

For some cases we have also a *criterion of equivalence,* see [13].

Iteration groups of continuous functions were studied by many authors in connection with flows, dynamical systems, fractional iterates, etc. At the beginning of the eighties the study of solutions of a system of Abel equations, or equivalently, embedding of a finite number of functions into an iteration group as its elements, was initiated by investigating functional differential equations.

IV. Euler functional equation for homogeneous functions

Consider a linear differential equation of the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \quad \text{on } I, \tag{P}$$

I being an open interval of the reals, p_i are real-valued continuous functions defined on I for $i = 0, 1, \dots, n-1$, i.e. $p_i \in C^0(I)$, $p_i : I \to \mathbb{R}$.

Take functions $f : J \to \mathbb{R}$ and $h : J \to I$ such that

$$f \in C^n(J), \quad f(t) \neq 0 \quad \text{for each } t \in J,$$

and

$$h \in C^n(J), \quad h'(t) \neq 0 \quad \text{for each } t \in J, \text{ and } h(J) = I.$$

For each solution y of equation (P) the function z defined as

$$z : J \to \mathbb{R}, \quad z(t) := f(t)\, y(h(t)), \quad t \in J, \tag{f,h}$$

satisfies again a differential equation of the same form

$$z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \quad \text{on } J. \tag{Q}$$

Since h is a C^n-diffeomorphism of J onto I, solutions y are transformed into solutions z on their whole intervals of definition. This is why we also speak about a *global* transformation of equation (P) into equation (Q).

Let $\mathbf{y}(x) = (y_1(x), \dots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation (P) considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \dots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \dots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation (Q), then the global transformation (f, h) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x))$$

or, for an arbitrary regular constant $n \times n$ matrix A,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x)),$$

expressing only the fact that another n-tuple of linearly independent solutions of the *same* equation (Q) is taken.

To emphasize this situation, let us denote by $(P_{\mathbf{y}})$ and $(Q_{\mathbf{z}})$ the equations (P) and (Q), respectively. Capital P refers to the coefficients p_i of the equation $(P_{\mathbf{y}})$, subscript $\mathbf{y}$ expresses a particular choice of an n-tuple of linearly independent solutions. Similarly for $(Q_{\mathbf{z}})$.

Denote by $W[\mathbf{y}](x)$ the Wronski determinant of $\mathbf{y}$, i.e.

$$\det(\mathbf{y}(x), \mathbf{y}'(x), \ldots, \mathbf{y}^{(n-1)}(x)).$$

The coefficient p_{n-1} in $(P_{\mathbf{y}})$ is given by

$$p_{n-1}(x) = -(\ln |W[\mathbf{y}](x)|)'.$$

We have $p_{n-1} \equiv 0$ exactly when $W[\mathbf{y}](x) = \mathit{const.} \neq 0$. Since

$$W[f \cdot \mathbf{y}(h)](t) = (f(t))^n \, (h'(t))^{\frac{n(n-1)}{2}} \, W[\mathbf{y}](h(t)),$$

for the coefficient q_{n-1} in $(Q_{\mathbf{z}})$ we have

$$q_{n-1}(t) = -n\frac{f'(t)}{f(t)} - \frac{n(n-1)}{2}\frac{h''(t)}{h'(t)} + p_{n-1}(h(t))\, h'(t). \tag{5}$$

Namely, if $p_{n-1} \equiv 0$ then $q_{n-1} \equiv 0$ occurs exactly when

$$f(t) = c\,|h'(t)|^{\frac{1-n}{2}}, \quad c = \text{const.} \neq 0. \tag{6}$$

Since the factor f belongs to $C^n(J)$, we have $h \in C^{n+1}(J)$.

For the criterion of equivalence of linear differential equations it was essential to find covariant functors from the second order equations (p) to the n-th order equations with the vanishing coefficient of $y^{(n-1)}$. The condition on the commutativity of the diagram of transformations leads to the relation

$$F\left(|h(t)|^{-\frac{1}{2}} u_1(h(t)), |h(t)|^{-\frac{1}{2}} u_2(h(t))\right) = |h(t)|^{\frac{(1-n)}{2}} F(u_1(h(t)), u_2(h(t))) \tag{7}$$

for linearly independent solutions u_1, u_2 of equation (p). Set $a = h(t)^{-\frac{1}{2}}$, $r = u_1(h(t))$ and $s = u_2(h(t))$, then from (7) we get

$$F(ar, as) = a^{n-1}\, F(s, r),$$

the Euler functional equation. Under the additional condition that each second order equation with analytic coefficients should be mapped on its whole interval of definition into an n-th order equation again with analytic coefficients, the only possible solutions are linear combinations with constant coefficients of

$$r^{n-1}, r^{n-2}s, \ldots, s^{n-1}.$$

It means that the n-th order linear differential equation to which the equation (p) with a couple u_1, u_2 is covariantly mapped is uniquely determined by its n-tuple of linearly independent solutions

$$u_1^{n-1}, u_1^{n-2}u_2, \ldots, u_2^{n-1}.$$

These special n-th order linear differential equations are called iterative equations and serve for effective criterion of equivalence of linear differential equations of an arbitrary order in general case, see [12].

V. Decomposition of functions

Here is a brief comment to results connected with decompositions of functions h into finite sums of the form

$$h(x,y) = \sum_{k=1}^{n} f_k(x) \cdot g_k(y). \tag{8}$$

For sufficiently smooth h, determinants of the form

$$\det \begin{pmatrix} h & h_y & \ldots & h_{y^n} \\ h_x & h_{xy} & \ldots & h_{xy^n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^n} & h_{x^n y} & \ldots & h_{x^n y^n} \end{pmatrix}$$

are involved in expressing a sufficient and necessary condition for such a decomposition. The correct formulation of the condition was first given in [11]. Functions f_k, g_k in the decomposition (8) and the number n as the minimal number possible for such a decomposition was determined there by using solutions of certain linear ordinary differential equations.

A sufficient and necessary condition for not sufficiently smooth functions h defined on arbitrary (even discrete) sets without any regularity conditions was also formulated in [11] by introducing there a new, special matrices

$$\begin{pmatrix} h(x_1,y_1) & h(x_1,y_2) & \ldots & h(x_1,y_n) \\ h(x_2,y_1) & h(x_2,y_2) & \ldots & h(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ h(x_n,y_1) & h(x_n,y_2) & \ldots & h(x_n,y_n) \end{pmatrix}.$$

Several authors, M. Čadek, H. Gauchman, Z. Moszner, Th.M. Rassias, L.A. Rubel, J. Šimša, in [4], [6], [8], [15] and others dealt with problems concerning decompositions of functions of several variables and similar questions.

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