Annales Academiae Paedagogicae Cracoviensis

Folia 4 Studia Mathematica I (2001)

Agostino Prástaro

Themistocles M. Rassias

Results on the J. d'Alembert equation

Dedicated to Professor Zenon Moszner
on his 70-th birthday

Abstract. In this paper the authors provide an account of some of their recent results concerning the J. D'Alembert equation especially in a suitable category of noncommutative manifolds.

Introduction

Questions of representation of functions in several variables by means of functions of a smaller number of variables have captured the interest of mathematicians for centuries (see [14]). One of these questions is closely connected with the thirteenth problem of D. Hilbert (1862-1943) and concerns the solvability of algebraic equations (see [5]). Let us mention the surprising result of A.N. Kolmogorov here (see [6]):

> Each continuous function h on the unit n-dimensional cube can be represented in the form
>
> $$h(x^1, x^2, \dots, x^n) = \sum_{1 \le i \le 2n+1} \phi_i \Big(\sum_{1 \le j \le n} \alpha_{ij}(x^j) \Big)$$
>
> with some continuous functions ϕ_i and α_{ij}. Moreover, the inner functions α_{ij} can be chosen in advance, i.e., independently of the function h.

Functions of certain special forms have been investigated by several authors, including J. d'Alembert (1717-1783), who as early as 1747 proved that each sufficiently smooth scalar function h of the form $h(x, y) = f(x).g(y)$ has to satisfy the following partial differential equation

$$\frac{\partial^2 \log h}{\partial x \partial y} = 0 \tag{A}$$

AMS (2000) Subject Classification: 58698, 58618.

Work partially supported by grants MURST "Geometry of PDEs and Applications" and GNFM/INDAM.

(see [2]). This equation can be also expressed in the following "Wronskian form":

$$\det W_{2+1}(h) \equiv \begin{vmatrix} h & h_y \\ h_x & h_{xy} \end{vmatrix} = 0.$$

A generalization to a finite sum of products of functions in single variables of the form

$$h(x,y) = \sum_{1\le i\le n} f_i(x).g_i(y) \tag{P}$$

has been considered since the beginning of the twentieth century. This forms *the fundamental problem* in the subject. The functions of the above tensor product play a significant role in many areas of both pure and applied mathematics. In the year 1904 in the section *Arithmetics and Algebra* at the Third International Congress of Mathematicians in Heidelberg, Cyparissos Stéphanos announced the following result ([16]):

> Functions of the type (P) form the space of all solutions of the partial differential equation with the "Wronskian" of order $(n+1)$: $\det W_{n+1}(h) = 0$.

However, no proof of the above result was given and no smoothness condition on the given function h was mentioned. In fact, Th.M. Rassias gave in [13] a counterexample to Stéphanos statement. It was F. Neuman ([7]) who proved the basic theorem involving the equation $\det W_{n+1}(h) = 0$ for functions of class C^n.

The problem of representing a function f in several (more than two) variables by:

$$h(x^1,x^2,\dots,x^k) = \sum_{1\le i\le n} f_{i1}(x^1)\cdot f_{i2}(x^2)\cdot\cdots\cdot f_{ik}(x^k), \tag{Q}$$

was proposed by Th.M. Rassias in [13]. H. Gauchman and L.A. Rubel [4] obtained some new results and extensions on finite sums expansions of the form (P), especially for real analytic functions. The first existence theorem on the decomposition (Q) was proved by F. Neuman [7]. Later M. Čadek and J. Šimša [1] found an effective criterion for the existence of the decomposition (Q) by making use of a system of functional equations, which does not require any assumption on the function h. Furthermore, they outlined a way to find systems of partial differential equations whose solutions space form the family of all sufficiently smooth functions h of type (Q). J. Šimsa [15], among other things, has introduced some new functional equations for functions of the form (P) using the so called Casorati determinant.

By using a geometric framework for partial differential equations A. Prástaro and Th.M. Rassias [11] proved that the set of solutions of the J. d'Alembert equation (A) is larger than the set of smooth functions h of two variables x, y of the form (P). This agrees with the above mentioned counterexample by Th.M.

Rassias. The book by Th.M. Rassias and J. Šimša [14] discusses the work of both past and mainly current research in the subject. Then, A. Prástaro and Th.M. Rassias [10] extended their results on the d'Alembert equation to functions of more than two variables by considering the generalized d'Alembert equation

$$\frac{\partial^n \log h}{\partial x_1 \partial x_2 \cdots \partial x_n} = 0,$$

in which $h = h(x^1, x^2, \ldots, x^n)$ is a scalar unknown function, smoothly depending on the variables $x^1, \ldots, x^n$. Recently A. Prástaro has given a general method to calculate integral and quantum (co)bordism groups in PDEs [8]. This method has proved to be very useful in order to show existence of global solutions, their topological structure and tunneling effects in PDE's, i.e., existence of solutions that change their sectional topology. Furthermore, A. Prástaro and Th.M. Rassias in [12] have extended such results also to generalized d'Alembert equations built in the category of quantum manifolds. These objects are noncommutative manifolds introduced by A. Prástaro who has also formulated a general geometric theory of quantum PDEs [8,9]. By utilizing such a theory we proved the existence of quantum tunneling effects for solutions of noncommutative d'Alembert equations.

In this paper we provide an account of some recent results in the subject. (For more details see the original papers [8-12].)

1. The commutative generalized d'Alembert equation

The **n-d'Alembert equation**:

$$\frac{\partial^n log f}{\partial x_1 \cdots \partial x_n} = 0, \qquad (d'A)_n$$

is an n-th order partial differential relation on the fiber bundle $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, i.e., it defines a subset $Z_n \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$. Let $\{x^\alpha, u, u_\alpha, u_{\alpha\beta}, \ldots, u_{\alpha_1 \cdots \alpha_n}\}$ be a coordinate system on the jet space $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ adapted to the fiber structures $\pi_n : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n, \bar{\pi}_{n,0} : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$. Then Z_n can be defined as the following subset:

$$\begin{aligned} Z_n \equiv \{ & D^n f(x^1, \ldots, x^n) \in J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \mid \\ & f(x^1, \ldots, x^n) = f_1(x^2, \ldots, x^n) \cdots f_n(x^1, \ldots, x^{n-1})\}. \end{aligned}$$

Furthermore, Z_n can be locally characterized as

$$Z_n = F^{-1}(0), \quad F : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R},$$

where the value of F is a sum of terms of the type

$$F[s; r|\alpha, \beta_1\beta_2, \cdots, \gamma_1 \ldots \gamma_q] \equiv s u^r u_\alpha u_{\beta_1\beta_2} \cdots u_{\gamma_1 \ldots \gamma_q},$$

with $\alpha \neq \beta_1 \neq \beta_2 \neq \cdots \neq \gamma_1 \neq \ldots \neq \gamma_q \leq n, s \in \mathbb{Z}, r \in \mathbb{N} \cup \{0\}$. Furthermore, the term in F containing $u_{1\ldots n}$ is just $u_{1\ldots n}u^{n-1}$. For example,

$$F = u_{xy}u - u_xu_y \quad \text{for } n = 2;$$
$$F = u_{xyz}u^2 - u_{xy}u_zu - u_{xz}u_yu + u_xu_yu_z \quad \text{for } n = 3.$$

Note that F has not locally constant rank on all Z_n, so Z_n is not a submanifold of $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$. Furthermore, on the open subset $C_n \equiv u^{-1}(\mathbb{R} \setminus 0) \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$, one recognizes that F has locally constant rank 1. Hence $Z_n \cap C_n$ is a subbundle of $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n$, of dimension $n + \frac{(2n)!}{(n!)^2} - 1$. In the following, for abuse of notation, we shall denote by $(d'A)_n$ either Z_n or $Z_n \cap C_n$. The fundamental geometric structure of $(d'A)_n$ is given by the following:

THEOREM 1.1

1) *The n-d'Alembert equation $(d'A)_n \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ is an n-th order PDE, formally integrable on the trivial fiber bundle $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$.*

2) *The characteristic distribution of $(d'A)_n$ is zero.*

REMARK 1.1

Note that, even if the characteristic distribution of $(d'A)_n$ is zero, we can built regular solutions by means of characteristic method if one considers the infinitesimal symmetry of $(d'A)_n$ (for $n = 2$ it is generated by the following vector fields ζ on $\pi : W(\equiv)\mathbb{R}^3 \to \mathbb{R}^2$:

$$\zeta = f(u)\partial x + g(y, u)\partial y + [s(y) + r(x)]\, u\partial u, \tag{$\bullet$}$$

where f, s and r are arbitrary functions of a single variable and g is an arbitrary function of two variables).

In fact we have the following:

THEOREM 1.2

Let $\psi : P \to (d'A)$ be the mapping that characterizes a 1-dimensional regular integral manifold $N \subset (d'A)$ such that the second holonomic prolongation $\zeta^{(2)}$ of a vector field ζ, as given in ($\bullet$), for suitable functions f, g, r and s, satisfies the following conditions:

(i) transversality condition*: $\psi^*(\zeta^{(2)} \rfloor \eta) \neq 0$;*

(ii) initial conditions*: $\psi^*\mathcal{I} = 0$, $\psi^*(\zeta^{(2)} \rfloor \mathcal{I} = 0$,*

where $\mathcal{I}$ is the Pfaffian ideal defining the contact structure of $(d'A)$ (see equation (1.3) below), and η is a differential 2-form defining the horizontalization for N. Then, if ϕ is the flow associated to $\zeta^{(2)}$: $\partial\phi = \zeta^{(2)}$, one has that $V \equiv \bigcup_{s \in J} \phi_s(N)$ is a regular 2-dimensional integral manifold of $(d'A)$, where J is a suitable neighborhood of $0 \in \mathbb{R}$.

Proof. The conditions for $\zeta^{(2)}$ to be a symmetry for $\mathcal{I}$ and transversal to N imply that $\phi_s(N) \equiv N_s$ are 1-dimensional regular integral manifolds of $(d'A)$, for s in a suitable neighborhood J of $0 \in \mathbb{R}$. Furthermore, the conditions (i) and (ii) assure that the 2-dimensional manifold $V \equiv \bigcup_{s\in J} N_s$ is integral also for $(d'A)$.

REMARK 1.2

Another way to built solutions by means of the characteristic method is just to recognize characteristic strips in $(d'A)_n$, cf. [8]. In the following lemma we explicitly give the characteristic strips for the case $n = 2$.

LEMMA 1.1

The equation

$$uu_{xy} - u_x u_y = 0 \qquad (d'A)$$

admits the following two 1*-dimensional characteristic strips:*

$$\begin{aligned} v_1 &\equiv X^x\left(\partial x + u_x \partial u + u_{xx}\partial u^x + u_{xy}\partial u^y + u_{xxx}\partial u^{xx} + u_{xxy}\partial u^{xy} + u_{xyy}\partial u^{yy}\right) \\ v_2 &\equiv X^y\left(\partial y + u_y \partial u + u_{yx}\partial u^x + u_{yy}\partial u^y + u_{yxx}\partial u^{xx} + u_{xyy}\partial u^{xy} + u_{yyy}\partial u^{yy}\right) \end{aligned} \qquad (1.1)$$

where X^x and X^y are arbitrary numerical functions on $J\mathcal{D}^2(\mathbb{R}^2, \mathbb{R})$.

Now, we are ready to prove the first main theorem.

THEOREM 1.3

The set $Sol(d'A)_n$ of all solutions of the n-d'Alembert equation: $(d'A)_n$, considered in domains contained in $\mathbb{R}^n$, is larger than the set of all functions f that can be represented in the form

$$\begin{aligned} &f(x^1, \dots, x^n) \\ &\quad = f_1(x^2, x^3, \dots, x^n) f_2(x^1, x^3, \dots, x^n) \cdots f_n(x^1, x^2, \dots, x^{n-1}). \end{aligned} \qquad (1.2)$$

Proof. The Cartan distribution $\mathbb{E}_n \subset T(d'A)_n$ of $(d'A)_n$ that characterizes the solutions of $(d'A)_n$ is the annihilator of the Pfaffian ideal $\mathcal{I}_n$ generated by the following differential 1-forms on $J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$:

$$\omega_\alpha \equiv \begin{cases} \omega_0 & \equiv dF = (\partial x_\alpha.F)dx^\alpha + (\partial u.F)du + (\partial u^\alpha.F)du_\alpha \\ & \quad + \cdots + (\partial u^{\alpha_1\cdots\alpha_n}.F)du_{\alpha_1\cdots\alpha_n} \\ \omega_1 & \equiv du - u_\alpha dx^\alpha \\ \omega_{2\alpha} & \equiv du_\alpha - u_{\alpha\beta}dx^\beta \\ \cdots & \\ \omega_{k\alpha_1\cdots\alpha_{n-1}} & \equiv du_{\alpha_1\cdots\alpha_{n-1}} - u_{\alpha_1\cdots\alpha_{n-1}\beta}dx^\beta \end{cases} \qquad (1.3)$$

with the function F that defines $(d'A)_n$. One has a canonical embedding $((d'A)_{n-1})_{+1} \to (d'A)_n$. Let us consider, now, a vector field $\zeta : J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \to TJ\mathcal{D}^n(\mathbb{R}^n, \mathbb{R})$ of the following type:

$$\zeta \equiv \partial x_n + u_n \partial u + u_{n\alpha}\partial^\alpha + \cdots + u_{n\alpha_1\ldots\alpha_n}\partial u^{\alpha_1\ldots\alpha_n} \tag{1.4}$$

such that $u_{n\alpha_1\ldots\alpha_n}$ are functions on $J\mathcal{D}^{n+1}(\mathbb{R}^n, \mathbb{R})$ satisfying the equations which define the first prolongations of $(d'A)_n$, $\{F = 0\}$:

$$\left\{\begin{array}{l} F_\alpha \equiv (\partial x_\alpha.F) + (\partial u.F)u_\alpha \\ \qquad + \cdots + (\partial u^{\alpha_1\ldots\alpha_n}.F)u_{\alpha\alpha_1\ldots\alpha_n} \\ \quad = 0, \quad 1 \leq \alpha \leq n \\ \;F = 0 \end{array}\right. . \tag{$((d'A)_n)_{+1}$}$$

Then ζ is necessarily transversal to

$$((d'A)_{n-1})_{+1} = J\mathcal{D}((d'A)_{n-1}) \bigcap J\mathcal{D}^n(\mathbb{R}^{n-1}, \mathbb{R})$$

and it generates a characteristic strip for $(d'A)_n$. Therefore, if N is an $(n-1)$-dimensional integral manifold contained in $((d'A)_{n-1})_{+1}$, a vector field ζ, as defined in (1.4), generates from N an n-dimensional integral manifold V contained in $(d'A)_n$. As N is not, in general, a regular solution of the equation $(d'A)_{n-1}$, then the so generated integral manifold V, solution of $(d'A)_n$, cannot be represented as the graph of some n-derivative of function $f : \mathbb{R}^n \to \mathbb{R}$. Hence, in particular, V cannot be represented as the image of the n-derivative of a function $f(x^1, \ldots, x^n)$, of the type (1.2).

We shall prove, now, that in $\mathcal{S}ol(d'A)$ there are solutions that change their sectional topologies. We shall use some recent results obtained by A. Prástaro about tunneling effects and quantum and integral (co)bordism in PDE's [8]. In the following we shall consider the n-d'Alembert equation given as a submanifold $(d'A)_n$ of the jet space $J^n_n(\mathbb{R}^{n+1})$ by means of the embedding $(d'A)_n \hookrightarrow J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \hookrightarrow J^n_n(\mathbb{R}^{n+1})$, where $J^n_n(\mathbb{R}^{n+1}) \equiv \{[N]^n_a\}$ with $[N]^n_a$ the set of n-dimensional submanifolds of $\mathbb{R}^{n+1}$ that have with the n-dimensional submanifold $N \subset \mathbb{R}^{n+1}$ a contact of order n at an $a \in N$. In the following table we report the explicitly calculated expressions of the integral bordism groups $\Omega^{(d'A)_n}_{n-1}$ of $(d'A)_n$, for $n \in \{2, 3, 4, 5\}$.

$\Omega_1^{(d'A)_2} = 0$	$\Omega_2^{(d'A)_3} = \mathbf{Z}_2$	$\Omega_3^{(d'A)_4} = 0$	$\Omega_4^{(d'A)_5} = \mathbf{Z}_2 \oplus \mathbf{Z}_2$

Tab. 1.1 Integral bordism groups of (d'A)$_n$ for $2 \leq n \leq 5$

Now, by means of these integral bordism groups, we see that there are solutions of $(d'A)_n$ that change their sectional topology. In fact, for example, if $n = 2$ or $n = 4$ one has: $\Omega_1^{(d'A)_2} = \Omega_3^{(d'A)_4} = 0$. Thus, in the case $n = 2$, any compact closed admissible integral 1-dimensional manifold N of $(d'A)$ is a disjoint union of copies of S^1: $N = S^1\dot{\cup}\dots_p\dots\dot{\cup}S^1$. Hence, we can always find a connected 2-dimensional integral manifold V, contained into $(d'A)$, such that $\partial V = N$. In other words, if $N_0 = S^1\dot{\cup}\dots_r\dots\dot{\cup}S^1$ and $N_1 = S^1\dot{\cup}\dots_s\dots\dot{\cup}S^1$ are two compact closed admissible integral 1-dimensional manifolds of $(d'A)$, we can always find a 2-dimensional integral manifold $V \subset (d'A)$ such that $\partial V = N_0\dot{\cup}N_1$. Of course, if $r \neq s$ one has a tunnel effect, i.e., a change in the sectional topology of V, passing from N_0 to N_1. Similar considerations hold for $n = 4$. Furthermore, if $n = 3$ one has: $\Omega_2^{(d'A)_3} = \mathbb{Z}_2$. In this case we have two types of compact closed admissible integral 2-dimensional manifolds. But the above considerations can be extended to each of these types of integral manifolds.

We now state our second main theorem.

THEOREM 1.4

In the set of solutions $Sol(d'A)_n$ of the n-d'Alembert equation, $(d'A)_n \subset J\mathcal{D}^n(\mathbb{R}^n, \mathbb{R}) \subset J^n_n(\mathbb{R}^{n+1})$, there are also some manifolds enjoying a change of sectional topology (tunneling effect).

2. The quantum generalized d'Alembert equation

In order to give a geometrical model for quantum physics, A. Prástaro has introduced in [8,9] a new category of noncommutative manifolds (*quantum manifolds*) built by means of a suitable structured noncommutative Frèchet algebra, (*quantum algebra*). An example for such an algebra can be the C^*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ of continuous linear operators on a Hilbert space $\mathcal{H}$ corresponding to the canonically quantized observables of a classical system.

The aim of this section is to consider the extension of the generalized D'Alembert equation $(d'A)_m$ to this new noncommutative framework given by A. Prástaro and Th.M. Rassias in [11]. Let us recall some fundamental definitions and results on quantum manifolds as given by A. Prástaro.

A *quantum algebra* is a triplet (A, ϵ, c), where:

(i) A is a metrizable, complete, Hausdorff, locally convex topological $\mathbf{K}$-vector space, that is also an associative $\mathbf{K}$-algebra with unit;

(ii) $\epsilon : \mathbf{K} \to A_0 \subset A$ is a $\mathbf{K}$-algebra homomorphism, where A_0 is the centre of A;

(iii) $c : A \to \mathbf{K}$ is a $\mathbf{K}$-linear morphism, with $c(e) = 1$, $e =$unit of A.

A *quantum vector space* of dimension $(m_1, \dots, m_s) \in \mathbf{N}^s$, built on the quantum algebra $A \equiv A_1 \times \dots \times A_s$, is a locally convex topological $\mathbf{K}$-vector space E isomorphic to $A_1^{m_1} \times \dots A_s^{m_s}$.

A *quantum manifold* of dimension $(m_1, \dots, m_s)$ over a quantum algebra $A \equiv A_1 \times \dots \times A_s$ of class Q^k_w, $0 \leq k \leq \infty, \omega$, is a locally convex manifold M modelled on E and with a Q^k_w-atlas of local coordinate mappings, where Q^k_w means class C^k_w (weak differentiability, e.g., H.H. Keller [3]), and derivatives A_0-linear. So for each open coordinate set $U \subset M$ we have a set of $m_1 + \cdots + m_s$ coordinate functions $x^A : U \to A$, (*quantum coordinates*). The *tangent space* T_pM at $p \in M$, is the vector space built in the usual way (cf. [9]). Then, derived tangent spaces associated to a quantum manifold M can be naturale defined.

A *quantum PDE* (QPDE) of order k on the fibre bundle $\pi : W \to M$, defined in the category of quantum manifolds, is a subfibrebundle $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ of the jet-quantum derivative space $J\hat{\mathcal{D}}^k(W)$ over M. $J\hat{\mathcal{D}}^k(W)$ is, in the category of quantum manifolds, the counterpart of the jet-derivative space for usual manifolds.

For more details see [9, 10], where there is also formulated a geometric theory for quantum PDEs that generalizes the theory of PDEs for usual manifolds.

In order to state existence of local solutions of QPDEs the following two theorems are very useful. (For the terminology used see [9, 10].)

THEOREM 2.1 (A. PRÁSTARO [9])

1) (δ-POINCARÉ LEMMA FOR QUANTUM PDEs). *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular QPDE. If A_0 is a Noetherian $\mathbf{K}$-algebra, then $\hat{E}_k$ is a δ-regular QPDE.*

2) (CRITERION OF FORMAL QUANTUM INTEGRABILITY). *Let $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$ be a quantum regular, δ-regular QPDE. Then if $\dot{g}_{k+r+1}$ is a bundle of A_0-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq n$, then $\hat{E}_k$ is formally quantumintegrable.*

A *solution* of $\hat{E}_k$ that satisfies the initial condition $q \in \hat{E}_k$ is an m-dimensional quantum manifold $N \subset \hat{E}_k$ such that $q \in N$ and N can be represented in a neighborhood of any of its points $q' \in N$, except for a nowhere dense subset $\Sigma(N) \subset N$ of dimension $\leq m-1$, as image of the k-derivative $D^k s$ of some Q^k_w-section s of $\pi : W \to M$. We call $\Sigma(N)$ the set of *singular points* (of Thom-Bordman type) of N. If $\Sigma(N) \neq \emptyset$ we say that N is a *regular solution* of $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W)$.

Let us denote by $\hat{J}^k_m(W)$ the k-jet of m-dimensional quantum manifolds (over A) contained into W. One has the natural embeddings $\hat{E}_k \subset J\hat{\mathcal{D}}^k(W) \subset \hat{J}^k_m(W)$. Then, with respect to the embedding $\hat{E}_k \subset \hat{J}^k_m(W)$ we can consider solutions of $\hat{E}_k$ as m-dimensional (over A) quantum manifolds $V \subset \hat{E}_k$ such that V can be representable in the neighborhood of any of its points $q' \in V$; except for a nowhere dense subset $\Sigma(V) \subset V$, of dimension $\leq m-1$; as $N^{(k)}$ — the k-quantum prolongation of an m-dimensional (over A) quantum manifold $N \subset W$.

In the case that $\Sigma(V) = \emptyset$, we say that V is a *regular solution* of $\hat{E}_k \subset \hat{J}^k_m(W)$. Of course, solutions V of $\hat{E}_k \subset \hat{J}^k_m(W)$, even regular ones in general are not diffeomorphic to their projections $\pi_k(V) \subset M$, hence they are not representable by means of sections of $\pi : W \to M$. Therefore, the above two theorems allow us to obtain existence theorems of local solutions.

Now, in order to study the structure of global solutions it is necessary to consider the integral bordism groups of QPDEs. In [9] A. Prástaro has extended to QPDEs his previous results on the determination of integral bordism groups of PDEs [8]. Let us denote by $\Omega_p^{\hat{E}_k}$, $0 \leq p \leq m-1$, the integral bordism groups of a QPDE $\hat{E}_k \subset \hat{J}^k_m(W)$ for closed integral quantum submanifolds of dimension p and class Q^∞_w, over a quantum algebra A of $\hat{E}_k$. The structure of smooth global solutions of $\hat{E}_k$ are described by the integral bordism group $\Omega_{m-1}^{\hat{E}_\infty}$ corresponding to the quantum prolongation $\hat{E}_\infty$ of $\hat{E}_k$.

Let us, pass to the study of a noncommutative case. For, now, set $\mathbf{K} = \mathbb{R}$ and let A be a quantum algebra such that A_0 is Noetherian. Let us consider the following trivial fiber bundle: $\pi : W \equiv A^{m+1} \to A^m$, with quantum coordinates $(x^{A_1}, \dots, x^{A_m}, u) \mapsto (x^{A_1}, \dots, x^{A_m})$. Then the **noncommutative generalized m-d'Alembert equation**, $m \in \mathbb{N}$, $2 \leq m < \infty$, is the QPDE $\widehat{(d'A)}_m \subset J\hat{\mathcal{D}}^m(W) \subset \hat{J}^m_m(W)$ defined by means of the following Q^∞_w-function:

$$F : J\hat{\mathcal{D}}^m(A^m; A) \to \overset{m}{\hat{A}}$$
$$\equiv Hom_{A_0}(A \otimes_{A_0} \cdots_m \cdots \otimes_{A_0} A; A) \equiv (\dot{T}^m_0 A) O A \equiv (\dot{T}^m_0 A)^+,$$

where F is the sum of formally the same terms with the commutative case. Of course more care must be taken on their meaning. For details see [9] and [11]. The quantum jet-derivative space $J\hat{\mathcal{D}}^m(W) \subset \hat{J}^m_m(W)$ is a quantum manifold of dimension $(m+1, m, m^2, \dots, m^m)$ over the quantum algebra $C \equiv A \times \overset{1}{\hat{A}} \times \dots \times \overset{m}{\hat{A}}$, i.e., $J\hat{\mathcal{D}}^m(W)$ is modelled on $A^{m+1} \times (\overset{1}{\hat{A}})^m \times \dots \times (\overset{m}{\hat{A}})^{m^m}$. Moreover $J\hat{\mathcal{D}}^m(W)$ is an open quantum submanifold of $\hat{J}^m_m(W)$, and $\widehat{(d'A)}_m$ is a quantum regular QPDE as the mappings $(\widehat{(d'A)}_m)_{+r} \to (\widehat{(d'A)}_m)_{+(r-1)}$, $r \geq 1$, are surjective. Hence, taking into account that $\widehat{(d'A)}_m$ is also δ-regular, it follows that $\widehat{(d'A)}_m$ is formally quantum integrable. Then, since in the open subset $Z_m \equiv u^{-1}(0) \subset J\hat{\mathcal{D}}^m(W)$ the QPDE $\widehat{(d'A)}_m$ is quantum analytic, in a suitable neighborhood U of any point $q \in Z_m \cap \widehat{(d'A)}_m$ one is able to build a quantum analytic solution that is diffeomorphic to $\pi_m(U) \subset M \equiv A^m$. Therefore we have the following:

THEOREM 2.2

The noncommutative generalized m-d'Alembert equation $\widehat{(d'A)}_m$ is a formally quantum integrable QPDE. For any point $q \in Z_m \cap \widehat{(d'A)}_m$ passes a quantum analytic solution V that is diffeomorphic to $\pi_m(V) \subset M \equiv A^m$.

In order to state existence theorems of global solutions for $\widehat{(d'A)}_m$ it is necessary to calculate the integral bordism groups $\Omega_p^{\widehat{(d'A)}_m}$, $0 \leq p \leq m-1$. From the above theorem, and since W is p-connected, $p \in \{0, \ldots, m-1\}$, one has the following isomorphism: $\Omega_p^{\widehat{(d'A)}_m} \cong A \otimes_{\mathbf{K}} H_p(W; \mathbf{K})$, $\quad 0 \leq p \leq m-1$. For a proof see [9]. On the other hand $H_0(W; \mathbf{K}) \cong \mathbf{K}$, and $H_p(W; \mathbf{K}) = 0$, for $1 \leq p \leq m-1$. Therefore, one obtains: $\Omega_0^{\widehat{(d'A)}_m} \cong A$, $\Omega_p^{\widehat{(d'A)}_m} \cong 0$, for $1 \leq p \leq m-1$. Hence, in particular, the following result holds:

THEOREM 2.3

Any admissible integral closed quantum manifold $N \subset \widehat{(d'A)}_m$, of dimension $m-1$ over A, bounds an integral quantum manifold of dimension m over A that is a solution of $\widehat{(d'A)}_m$. Moreover, for two admissible integral closed quantum manifolds $N_0, N_1 \subset \widehat{(d'A)}_m$, of dimension $m-1$ over A, there exists a solution V of $\widehat{(d'A)}_m$ such that $\partial V = N_0 \dot{\cup} N_1$.

In particular if N_0 and N_1 are homotopically different and V is connected, then V is a solution with change of sectional topology. Thus, we get the following:

COROLLARY 2.1

*In the set $Sol(\widehat{(d'A)}_m)$ of solutions of the quantum generalized m-d'Alembert equation, there are solutions with change of sectional topology (*quantum tunnel effect*). Such solutions, in general, cannot be represented as mappings $f : A^m \to A$.*

Conclusions

The geometric theory of PDEs introduced by A. Prástaro in [8, 9] is a handable framework where problems in the theory of partial differential equations find their natural solutions. In fact, the J. d'Alembert equation is one such application.

References

[1] M. Čadek, J. Šimša, *Decomposable functions of several variables*, Aequationes Math. **40** (1990), 8-25; *Decomposition of smooth functions of two multidimensional variables*, Czechoslovak Math. J. **41(116)** (1991), 342-358.

[2] J. d'Alembert, *Recherches sur la courbe que forme une corde tendue mise en vibration I-II*, Hist. Acad. Berlin, (1747), 214-249.

[3] H.H. Keller, *Differential Calculus in Locally Convex Spaces*, Lect. Notes in Mathematics **417**, Springer-Verlag, Berlin – New York, 1974.

[4] H. Gauchman, A.L. Rubel, *Sums of products of functions of x times functions of y*, Linear Algebra Appl. **125** (1989), 19-63.

[5] D. Hilbert, *Mathematical problems* (Lecture delivered before the International Congress of Mathematicians at Paris in 1900), Bull. Amer. Math. Soc. **8** (1902), 437-479.

[6] A.N. Kolmogorov, *On the representation of continuous functions of several variables in the form of a superposition of continuous functions of one variable and addition* (in Russian), Dokl. Akad. Nauk. SSSR. **114** (1957), 953-956, (Amer. Math. Soc. Transl. **28** (1963), 55-59).

[7] F. Neuman, *Functions of two variables and matrices involving factorizations*, C. R. Math. Rep. Acad. Sci. Canada **3**(1981), 7-11; *Factorizations of matrices and functions of two variables*, Czechoslovak Math. J. **32(107)** (1982), 582-588; *Finite sums of products of functions in single variables*, Linear Algebra Appl. **134** (1990), 153-164.

[8] A. Prástaro, *Cobordism groups in PDEs and quantum PDEs*, Rep. Math. Phys. **38(4)** (1996), 443-455; *Geometry of PDEs and Mechanics*, World Scientific, Singapore, 1996; *Quantum and integral (co)bordism groups in partial differential equations*, Acta Appl. Math. **51** (1998), 243-302; *Quantum and integral bordism groups in the Navier-Stokes equation*, New Developments in Differential Geometry, Budapest 1996, J.Szenthe (ed.), Kluwer Academic Publishers, Dordrecht 1999, 343-359; *(Co)bordism groups in PDEs*, Acta Appl. Math. **59(2)** (1999), 111-201.

[9] A. Prástaro, *(Co)bordism groups in quantum PDEs*, Acta Appl. Math. **64(2/3)** (2000), 111-217.

[10] A. Prástaro, Th.M. Rassias, *A geometric approach to an equation of J. d'Alembert*, Proc. Amer. Math. Soc. **123(5)** (1975), 1597-1606; *On the set of solutions of the generalized d'Alembert equation*, C. R. Acad. Sci. Paris Sér. Math. **328** (1999), 389-394.

[11] A. Prástaro, Th.M. Rassias, *A geometric approach of the generalized d'Alembert equation*, J. Comp. Appl. Math. **113**(1-2) (1999), 93-122.

[12] A. Prástaro, Th.M. Rassias, *A geometric approach to a noncommutative generalized d'Alembert equation*, C. R. Acad. Sci. Paris Sér. Math. **330** (2000), 545-550.

[13] Th.M. Rassias, *A criterion for a function to be represented as a sum of products of factors*, Bull. Inst. Math. Acad. Sinica **14** (1986), 377-382; *Problem P286*, Aequationes Math. **38** (1989), 111-112; *Remark 20*, Aequationes Math. **56** (1998), 311.

[14] Th.M. Rassias, J. Šimša, *Finite Sums Decompositions in Mathematical Analysis*, J.Wiley & Sons, Chichester, 1995.

[15] J. Šimša, *A note on certain functional determinants*, Aequationes Math. **44** (1992), 35-41.

[16] C.M. Stéphanos, *Sur une categorie d'équations fonctionelles*, Rend. Circ. Mat. Palermo **18** (1904), 360-362; *Sur une categorie d'équations fonctionelles*, in: Math. Kongress, Heidelberg, 1904, (1905), 200-201.

Agostino Prástaro
Dipartimento di Metodi e Modelli Matematici
Università degli Studi di Roma "La Sapienza"
Via A. Scarpa
10-00176 Roma
Italy
E-mail: Prastaro@dmmm.uniroma1.it

Themistocles. M. Rassias
Department of Mathematics
National Technical University
Zografou Campus
Athens 15780
Greece
E-mail: trassias@math.ntua.gr

Manuscript received: November 24, 1999 and in final form: April 10, 2001