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On the stability of derivations of higher order

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Abstract. Derivations of order n as defined by L. Reich are *additive* and *nonlinear* functions $f : \mathbb{R} \to \mathbb{R}$ with $f(1) = 0$ which satisfy the functional equation $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f = 0$ for all $a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}$, where $\delta_a f(x) := f(ax) - af(x)$. Here we prove several stability results concerning this (and similar) functional equations.

1. In [K], Chap. XIV a *derivation* $f : \mathbb{R} \to \mathbb{R}$ is defined to be an *additive* mapping which additionally satisfies the Leibniz rule

$$f(xy) = xf(y) + yf(x), \quad x, y \in \mathbb{R}.$$

In [R92] the operators δ_a are introduced. For functions $f : \mathbb{R} \to \mathbb{R}$ and reals a we have

$$\delta_a(f)(x) := f(ax) - af(x).$$

In [R98] it was shown that

an additive function $f : \mathbb{R} \to \mathbb{R}$ is a derivation if and only if and

$$f(1) = 0 \quad \textit{and} \quad (\delta_a \circ \delta_b) f = 0, \quad a, b \in \mathbb{R}.$$

Moreover in the same paper (Satz 2) and in [UR] this leads to the following generalization.

Definition 1

A function $f : \mathbb{R} \to \mathbb{R}$ is called a derivation of order n ($\in \mathbb{N}_0$) if and only if f is additive with $f(1) = 0$ and if f satisfies the equation

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x) = 0, \quad a_1, a_2, \ldots, a_{n+1}, x \in \mathbb{R}. \tag{1}$$

Actually the original definition was different. But for our purpose (stability investigations) it is convenient to use the definition above.

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Before we proceed to this topic we prove the following theorem which was mentioned in [R98] (Satz 3) and proved there for $n = 2$.

THEOREM 1

For $a \in \mathbb{R}$ and $n \in \mathbb{N}$ let $\delta_a^{n+1} := \underbrace{\delta_a \circ \delta_a \circ \ldots \circ \delta_a}_{n+1 \text{ times}}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a derivation of order n if and only if f is additive with $f(1) = 0$ and if

$$\delta_a^{n+1} f(x) = 0, \quad a, x \in \mathbb{R}. \tag{2}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be additive. Obviously it is enough to show that (2) implies (1). Moreover δ_a maps additive functions to additive functions. Thus δ_a is an endomorphism of the vector space of all additive functions defined on $\mathbb{R}$ with real values. Since $\delta_a \pm \delta_b = \delta_{a \pm b}$ and $\delta_a \circ \delta_b = \delta_b \circ \delta_a$ the ring generated by the operators δ_a, $a \in \mathbb{R}$, is commutative. It is well-known (see for example [S]) that in any commutative ring we have

$$m! 2^m x_1 \cdot x_2 \cdot \ldots \cdot x_m = \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m \left(\sum_{j=1}^{m} \varepsilon_j x_j \right)^m .$$

Using this for $m = n + 1$, $x_j = \delta_{a_j}$ (and $\cdot = \circ$) we see that the operator $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}}$ is a linear combination of certain operators δ_b^{n+1}. But this gives the desired result.

REMARK 1

Since $\delta_a (\mathrm{id}_{\mathbb{R}}) = 0$ it can easily be seen that the general additive solution of (1) or (2) is given by sums of a derivation of order n and of a linear function. In the following we will call additive solutions of (1) or (2) *generalized* derivations of order n (and for convenience omit the term "generalized").

For future use we formulate the explicit form of $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x)$.

LEMMA 1

For all $a_1, a_2, \ldots, a_{n+1}, x \in \mathbb{R}$ and all $f : \mathbb{R} \to \mathbb{R}$ we have

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x) = \sum_{J \subseteq \{1,2,\ldots,n+1\}} (-1)^{n+1-\#J} \prod_{j \notin J} a_j \cdot f \left(\prod_{j \in J} a_j \cdot x \right) \tag{3}$$

Proof. For real a the operator δ_a may be written as $\delta_a = M_a - \mu_a$, where $M_a f(x) := f(ax)$ and $\mu_a f(x) := a f(x)$. Then $M_a \circ M_b = M_{ab}$, $\mu_a \circ \mu_b = \mu_{ab}$ and $(M_a \circ \mu_b) f(x) = b f(ax) = (\mu_b \circ M_a) f(x)$. Thus all the operators M_a, M_b, μ_c, μ_d commute in pairs and we get

$$\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} = \bigcirc_{j=1}^{n+1} (M_{a_j} - \mu_{a_j})$$
$$= \sum_{J \subseteq \{1,2,\ldots,n+1\}} (-1)^{n+1-\#J} \bigcirc_{j \notin J} \mu_{a_j} \circ \bigcirc_{j \in J} M_{a_j},$$

from which the assertion follows.

We also mention a suitably adapted version of a stability theorem (see [K], Chap. XVII and the references given there in).

THEOREM 2
Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x+y) - f(x) - f(y)| \leqslant \varepsilon$ for all $x, y \in \mathbb{R}$. Then there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $|f - g|$ is bounded (by ε.) Moreover g is given by $g(x) = \lim_{n\to\infty} \frac{f(nx)}{n}$

Surprisingly the same problem for exponential functions has a completely different answer (see e.g. [BLZ]).

THEOREM 3
Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x+y) - f(x)f(y)| \leqslant \varepsilon$ for all $x, y \in \mathbb{R}$. Then either f is bounded or an unbounded exponential function, i.e., an unbounded function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

The phenomenon described here is called "superstability" by some authors.

2. The stability results

The possibility of investigating the stability of derivations of order n depends on the choice of equations to be replaced by suitable inequalities. One result is the following.

THEOREM 4
Let $\varepsilon > 0$, let $b : \mathbb{R}^{n+1} \to \mathbb{R}$ be an arbitrary function, and let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon, \quad x, y \in \mathbb{R} \tag{4}$$

and

$$\left|\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}} f(x)\right| \leqslant b(a_1, a_2, \ldots, a_{n+1}), \quad x, a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}. \tag{5}$$

Then we have:

i) *There is one and only one derivation d of order n such that $f - d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r:\mathbb{R}\to\mathbb{R}$ the function $f:=d+r$ satisfies* (4) *and* (5) *for some suitable number ε and some function $b:\mathbb{R}^{n+1}\to\mathbb{R}$.*

iii) *If b is independent of at least one of its variables and if f satisfies* (4) *and* (5)*, then f is already a derivation of order n itself.*

Proof. Using (4) and Theorem 2 we get a unique additive function d such that $f-d$ is bounded. Moreover we know that $d(x)=\lim_{m\to\infty}\frac{f(mx)}{m}$. For fixed m we put $f_m(x):=\frac{f(mx)}{m}$. Then (5) together with the linearity of the operators δ_a gives

$$\left|\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}f_m(x)\right|\leqslant\frac{b(a_1,a_2,\ldots,a_{n+1})}{m}$$

for all $x,a_1,a_2,\ldots,a_{n+1}\in\mathbb{R}$ and all $m\in\mathbb{N}$.

But for $m\to\infty$ we get that $f_m(x)\to d(x)$ and by (1)

$$\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}f_m(x)\to\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}d(x).$$

Since $\frac{1}{m}b(a_1,a_2,\ldots,a_{n+1})\to 0$ this means that

$$\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}d(x)=0$$

for all $x,a_1,a_2,\ldots,a_{n+1}\in\mathbb{R}$, thus proving the first part of the theorem.

Let r and d be as required in the second part of the theorem, and let R be an upper bound for $|r(x)|$, $x\in\mathbb{R}$. Then

$$\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(d+r)(x)=\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(r)(x)$$

and by (1)

$$\left|\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(r)(x)\right|\leqslant b(a_1,a_2,\ldots,a_{n+1}):=\sum_{J\subseteq\{1,2,\ldots,n+1\}}\left|\prod_{j\notin J}a_j\right|R.$$

Moreover

$$\begin{aligned}|(d+r)(x+y)-(d+r)(x)-(d+r)(y)|&=|r(x+y)-r(x)-r(y)|\\&\leqslant\varepsilon:=3R.\end{aligned}$$

To prove the third part we may observe that by the first part there is a unique derivation d such that $F:=f-d$ is bounded. We have to show that $F=0$. Let us assume that the function b does not depend on, say, the last variable a_{n+1}. Since $\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(F)=\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(f)$ we get

$$\left|\delta_{a_1}\circ\delta_{a_2}\circ\ldots\circ\delta_{a_{n+1}}(F)(x)\right|\leqslant b(a_1,a_2,\ldots,a_n,a_{n+1})=:B(a_1,a_2,\ldots,a_n)$$

Assuming that $a_1,a_2,\ldots,a_{n+1}$ are different from zero and using (1) we get

$$\left| \sum_{J\subseteq\{1,2,\dots,n+1\}} (-1)^{n+1-\#J} \frac{F\left(\prod_{j\in J} a_j \cdot x\right)}{\prod_{j\in J} a_j} \right| \leqslant \frac{B(a_1,a_2,\dots,a_n)}{|a_1a_2\cdots a_{n+1}|}.$$

If we fix $a_1,a_2,\dots,a_n$ and if we let a_{n+1} tend to infinity we get

$$\sum_{J\subseteq\{1,2,\dots,n\}} (-1)^{n+1-\#J} \frac{F\left(\prod_{j\in J} a_j \cdot x\right)}{\prod_{j\in J} a_j} = 0$$

since $\frac{B(a_1,a_2,\dots,a_n)}{|a_1a_2\cdots a_{n+1}|} \to 0$ for $a_{n+1}\to\infty$ and since also $\frac{F(\prod_{j\in J} a_j\cdot x)}{\prod_{j\in J} a_j} \to 0$ if the subset $J\subseteq\{1,2,\dots,n+1\}$ is such that $n+1\in J$.

But the sum above contains the term $\pm F(x)$ (for $J=\emptyset$) and all the other terms tend to zero when all the a_j tend to infinity, which means that $F(x)=0$ (for arbitrary x).

Concerning the characterization of derivations as given by Theorem 1 we have the following result.

THEOREM 5
Let $\varepsilon>0$, let $b:\mathbb{R}\to\mathbb{R}$ be an arbitrary function, and let $f:\mathbb{R}\to\mathbb{R}$ be such that (4) *is satisfied and such that*

$$\left|\delta_a^{n+1} f(x)\right| \leqslant b(a), \quad x,a\in\mathbb{R} \tag{6}$$

holds. Then we have:

i) *There is one and only one derivation d of order n such that $f-d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r:\mathbb{R}\to\mathbb{R}$ the function $f:=d+r$ satisfies* (4) *and* (6) *for some suitable number ε and some function $b:\mathbb{R}\to\mathbb{R}$.*

iii) *If b is constant and if f satisfies* (4) *and* (6), *then f is already a derivation of order n itself.*

Proof. The first part and also the second one can be proved as the corresponding parts of the theorem above. Especially the desired derivation of order n is given by $d(x):=\lim_{m\to\infty}\frac{f(mx)}{m}$. As for the third part we put $F:=f-d$ and observe that F is bounded and satisfies $\delta_a^{n+1}F=\delta_a^{n+1}f$. Moreover

$$\begin{aligned}\delta_a^{n+1}F(x) &= \sum_{J\subseteq\{1,2,\ldots,n+1\}} (-1)^{n+1-\#J}a^{n+1-\#J}F\left(a^{\#J}x\right)\\ &= \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}a^{n+1-j}F\left(a^jx\right).\end{aligned}$$

Thus for $a \neq 0$ (and putting $B := b(a)$)

$$\left|\frac{\delta_a^{n+1}F(x)}{a^{n+1}}\right| = \left|(-1)^{n+1}F(x) + \sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\frac{F\left(a^jx\right)}{a^j}\right| \leqslant \frac{B}{|a|^{n+1}}.$$

This for $a \to \infty$ implies $F(x) = 0$, as desired.

REMARK 2

The last parts of both theorems are remarkable since they show the phenomenon of "strong" superstability: *Every solution of the inequality is also a solution of the equation!*

3. Characterization of derivations of order n by a single equation (and its stability)

Actually the definition of a derivation of order n contains two requirements (additivity and the condition connected with the operators δ_a). For $n = 0$ formally this is nothing but the definition of *linearity* by the two requirements $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$ which are equivalent to the single condition $f(a(x+y)) = af(x) + af(y)$. Generalizing this we have the following theorem.

THEOREM 6

For $n \in \mathbb{N}_0$ and $f : \mathbb{R} \to \mathbb{R}$ the conditions (a) *and* (b) *below are equivalent.*

(a) *f is a derivation of order n.*

(b) $f\left(a^{n+1}(x+y)\right) + \sum_{j=0}^{n}(-1)^{n+1-j}\binom{n+1}{j}a^{n+1-j}\left(f(a^jx) + f(a^jy)\right) = 0$ *for all $x, y, a \in \mathbb{R}$.*

Proof. Obviously the condition given in (b) is nothing but

$$f\left(a^{n+1}(x+y)\right) + \delta_a^{n+1}f(x) + \delta_a^{n+1}f(y) - f(a^{n+1}x) - f(a^{n+1}y) = 0. \qquad (7)$$

If f is a derivation of order n the terms $\delta_a^{n+1}f(x)$ and $\delta_a^{n+1}f(y)$ vanish. Moreover by the additivity of f the three remaining terms on the left-hand side of (7) also disappear.

Conversely, if (7) is satisfied, we may put $x = y = 0$ in this relation to get

$$f(0) + 2(1-a)^{n+1}f(0) - 2f(0) = 0 \quad \text{or} \quad \left(2(1-a)^{n+1} - 1\right)f(0) = 0.$$

Using this with (for example) $a=-1$ we get $(2^{n+2}-1)f(0)=0$, i.e. $f(0)=0$. This gives $\delta_a^{n+1}f(0)=0$. Applying the equation for $y=0$ gives

$$f(a^{n+1}x)+\delta_a^{n+1}f(x)+0-f(a^{n+1}x)-0=0 \quad \text{or} \quad \delta_a^{n+1}f(x)=0.$$

Using this and (7) for $a=1$ once more also gives the additivity of f.

The corresponding stability result is contained in the following theorem.

THEOREM 7

Let $b:\mathbb{R}\to\mathbb{R}$ be an arbitrary function and let $f:\mathbb{R}\to\mathbb{R}$ be given such that

$$\left|f\left(a^{n+1}(x+y)\right)+\delta_a^{n+1}f(x)+\delta_a^{n+1}f(y)-f(a^{n+1}x)-f(a^{n+1}y)\right|\leqslant b(a) \quad (8)$$

for all $x,y,a\in\mathbb{R}$. Then we have:

i) *There is one and only one derivation d of order n such that $f-d$ is bounded.*

ii) *For any derivation d of order n and any bounded function $r:\mathbb{R}\to\mathbb{R}$ the function $f:=d+r$ satisfies* (8) *for some suitable function $b:\mathbb{R}\to\mathbb{R}$.*

iii) *If b is constant and if f satisfies* (8)*, then f is already a derivation of order n itself.*

Proof. Since $\delta_1^{n+1}f(x)=(1-1)^{n+1}f(x)=0$, equation (8) for $a=1$ implies

$$|f(x+y)-f(x)-f(y)|\leqslant b(1)=:\varepsilon.$$

Putting $y=0$ in (8) leads to

$$\left|f(a^{n+1}x)+\delta_a^{n+1}f(x)+\delta_a^{n+1}f(0)-f(a^{n+1}x)-f(0)\right|\leqslant b(a). \quad (9)$$

Moreover

$$\delta_a^{n+1}f(0)-f(0)=\left((1-a)^{n+1}-1\right)f(0)=:c(a).$$

Thus with $b'(a):=b(a)+|c(a)|$ we get

$$\left|\delta_a^{n+1}f(x)\right|\leqslant b'(a), \quad x,a\in\mathbb{R}.$$

Accordingly we may apply Theorem 5 to get the first part of the theorem.

The second part may be proved in a similar way as in previous cases. (If R is an upper bound for $|r|$ we may take $b(a)=3R+2(1+|a|)^{n+1}$.)

With $F=f-d$ we have (again as in previous cases) that F is bounded and that

$$|F(x+y)-F(x)-F(y)|\leqslant b(1)=\varepsilon.$$

Thus (8) with $x=y$ implies

$$\left|2\delta_a^{n+1}f(x)\right|\leqslant 2\varepsilon, \quad x,a\in\mathbb{R}.$$

Now we again apply Theorem 5.

REMARK 3

It is possible to formulate (and prove) similar results with δ_a^{n+1} replaced by the operator $\delta_{a_1} \circ \delta_{a_2} \circ \ldots \circ \delta_{a_{n+1}}$. But we will not do this here.

The results of this paper have been presented earlier for example at a joint Graz-Maribor seminar in 1996 and at the 36-th ISFE in Brno in 1998.

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