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A Wiener Tauberian Theorem on discrete abelian torsion groups

Dedicated to Professor Zenon Moszner
on his 70th birthday

Abstract. One version of the classical Wiener Tauberian Theorem states that if G is a locally compact abelian group then any nonzero closed translation invariant subspace of $L^\infty(G)$ contains a character. In other words, spectral analysis holds for $L^\infty(G)$. In this paper we prove a similar theorem: if G is a discrete abelian torsion group then spectral analysis holds for $C(G)$, the space of all complex valued functions on G.

1. Introduction

A possible formulation of one version of the classical Wiener Tauberian Theorem on locally compact abelian groups is the following: if G is a locally compact abelian group then any nonzero closed translation invariant subspace of $L^\infty(G)$ contains a character. Similar problem can be formulated concerning $C(G)$ instead of $L^\infty(G)$ but in that case "characters" should be replaced by "generalized characters": does any nonzero closed translation invariant subspace of $C(G)$ contain a generalized character? The answer is positive in some special cases but the general problem is far from being solved. The problem is still open even in the case where G is discrete. In this paper we give a positive answer to the above question if G is a discrete abelian torsion group.

2. Spectral: analysis and synthesis on discrete abelian groups

In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is an abelian group then $C(G)$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the topology of pointwise convergence. The dual of $C(G)$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G.

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Homomorphisms of G into the additive group of complex numbers, resp. into the multiplicative group of nonzero complex numbers are called *additive*, resp. *exponential functions.* Bounded exponential functions are exactly the characters of G, hence exponential functions are sometimes called generalized characters. Products of additive functions are called *monomials*, products of monomials and exponential functions are called *exponential monomials.*

The basic question of spectral analysis on $C(G)$ can be formulated as follows: does any nonzero closed translation invariant subspace of $C(G)$ contain an exponential function? If so, then we say that *spectral analysis holds for* $C(G)$. For instance, if G is finite then by the Wiener Tauberian Theorem the answer is "yes". Another basic problem concerns spectral synthesis on $C(G)$: given a nonzero closed translation invariant subspace of $C(G)$, do the linear combinations of exponential monomials in this subspace form a dense subset? If so, then we say that *spectral synthesis holds for* $C(G)$. This is the case, for instance, if G is finitely generated, due to the following theorem.

THEOREM 1 (M. Lefranc [2])

If G is a finitely generated discrete abelian group then spectral synthesis holds for $C(G)$.

3. Exponential monomials on abelian torsion groups

Let G be an abelian group. We say that G is a *torsion group* if every element of G has finite order. In other words, for every x in G there exists a positive integer n with $nx = 0$. Hence G is not a torsion group if and only if there exists an element of G which generates a subgroup isomorphic to $\mathbb{Z}$, the additive group of integers.

In what follows we need the following lemma (see [1]).

LEMMA 2

Let G be an abelian group, $H \subseteq G$ a subgroup and let D be a divisible abelian group. If $\varphi : H \to D$ is a homomorphism, then there exists a homomorphism $\Phi : G \to D$ which extends φ, that is, $\Phi(x) = \varphi(x)$ for all x in H.

THEOREM 3

Let G be an abelian group. Then G is a torsion group if and only if every nonzero exponential monomial on G is a character.

Proof. Suppose that G is a torsion group and let $a : G \to C$ be an additive function, and $m : G \to \mathbb{C}$ an exponential function. For every x in G there exists a positive integer n with $nx = 0$ and hence $0 = a(nx) = na(x)$, which implies $a(x) = 0$. This means that every additive function on G is zero. Further $1 = m(nx) = m(x)^n$, which implies $|m(x)| = 1$. This means that every exponential function on G is a character. Now we conclude that if G is a torsion group then every nonzero exponential monomial on G is a character.

Assume now that G is not a torsion group, that is, there exists an x_0 in G such that the cyclic group generated by x_0 is isomorphic to $\mathbb{Z}$. Let $\alpha \neq 0$ be a complex number and we define $\varphi(nx_0) = n\alpha$ for any integer n. Then φ is a homomorphism of the subgroup generated by x_0 into the additive group of complex numbers. As this latter group is divisible, by Lemma 2. this homomorphism can be extended to a homomorphism $a : G \to \mathbb{C}$ of G into the additive group of complex numbers. By $a(x_0) = \varphi(x_0) = \alpha \neq 0$ we have that a is a nonzero additive function, that is, a nonzero exponential monomial on G, which is obviously not a character. The theorem is proved.

4. A Wiener Tauberian Theorem on abelian torsion groups

In this paragraph we show that if G is a discrete abelian torsion group, then any nonzero closed translation invariant subspace of $C(G)$ contains a character. The proof heavily depends on Theorem 1.

THEOREM 4

Let G be an abelian torsion group. Then any nonzero closed translation invariant subspace of $C(G)$ contains a character.

Proof. Let $V \subseteq C(G)$ be any nonzero closed translation invariant subspace. Then by the Hahn-Banach theorem V is equal to the annihilator of its annihilator, that is, there exists a set $\Lambda \subseteq \mathcal{M}_c(G)$ of finitely supported complex measures on G such that V is exactly the set of all functions in $C(G)$ which are annihilated by all members of Λ:

$$V = V(\Lambda) = \{f \mid f \in C(G),\ \lambda(f) = 0 \quad \text{for all } \lambda \in \Lambda\}.$$

We show that for any finite $\Gamma \subseteq \Lambda$, its annihilator, $V(\Gamma)$, contains a character. Indeed, let F_Γ denote the subgroup generated by the supports of the measures belonging to Γ. Then F_Γ is a finitely generated torsion group. The measures belonging to Γ can be considered as measures on F_Γ and the annihilator of Γ in $C(F_\Gamma)$ will be denoted by $V(\Gamma)_{F_\Gamma}$. This is a closed translation invariant subspace of $C(F_\Gamma)$. It is also nonzero. Indeed, if f belongs to V then its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$. If, in addition, we have $f(x_0) \neq 0$ and y_0 is in F_Γ, then the translate of f by $x_0 - y_0$ belongs to V, its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$ and at y_0 it takes the value $f(x_0) \neq 0$. Hence $V(\Gamma)_{F_\Gamma}$ is a nonzero closed translation invariant subspace of $C(F_\Gamma)$. As F_Γ is finitely generated, by Theorem 1. spectral synthesis holds for $C(F_\Gamma)$, and, in particular $V(\Gamma)_{F_\Gamma}$ contains nonzero exponential monomials. As F_Γ is a torsion group, any nonzero exponential monomial on F_Γ is a character. That means, $V(\Gamma)_{F_\Gamma}$

contains a character of F_Γ. By Lemma 2. any character of F_Γ can be extended to a character of G, and obviously any such extension belongs to $V(\Gamma)$.

We have proved that for any finite $\Gamma \subseteq \Lambda$ the annihilator $V(\Gamma)$ contains a character. Let $char(V)$ denote the set of all characters contained in V. Obviously $char(V)$ is a compact subset of $\hat{G}$, the dual of G, because $char(V)$ is closed and $\hat{G}$ is compact. On the other hand, the system of sets $char(V(\Gamma))$, where $\Gamma \subseteq \Lambda$ is finite, is a centered system of nonempty compact sets:

$$char(V(\Gamma_1 \cup \Gamma_2)) \subseteq char(V(\Gamma_1)) \cap char(V(\Gamma_2)).$$

We infer that the intersection of this system is nonempty, and obviously

$$\emptyset \neq \bigcap_{\Gamma \subseteq \Lambda \text{ finite}} char(V(\Gamma)) \subseteq char(V).$$

That means, $char(V)$ is nonempty, and the theorem is proved.

References

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