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On a problem of H.-H. Kairies concerning Euler's Gamma function

Abstract. The Bohr-Mollerup theorem on the Euler Γ function states: If $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the functional equation $f(x+1) = x\,f(x)$ on $\mathbb{R}_+$, $\log \circ f$ is convex on $(\gamma, +\infty)$ for some $\gamma \geqslant 0$ and $f(1) = 1$ then $f = \Gamma$. We give some partial answers to the question posed by H.-H. Kairies: By what other function can the logarithm be replaced in this statement.

Introduction

Let us introduce the family of functions

$$\mathbf{F} := \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid \forall x \in \mathbb{R}_+ : \;\; f(x+1) = xf(x) \text{ and } f(1) = 1\}\,.$$

Then for every $f \in \mathbf{F}$ and $n \in \mathbb{N}$ we have $f(n) = \Gamma(n) = (n-1)!$ where Γ is the Euler function defined by the formula

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x), \tag{Γ}$$

where

$$\Gamma_n(x) = \frac{n^x n!}{x(x+1)\ldots(x+n)}. \tag{Γ_n}$$

Moreover, $f \in \mathbf{F}$ iff $f(x) = p(x)\Gamma(x)$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is a periodic function of period 1 with $p(1) = 1$.

We begin our considerations with reminding the Bohr-Mollerup Theorem, cf. [1] p. 14.

Theorem (Bohr-Mollerup)
If $f \in \mathbf{F}$ and $\log \circ f$ is convex on $(\gamma, +\infty)$ for some $\gamma \geqslant 0$, then $f = \Gamma$.

H.-H. Kairies proposed (private communication) to investigate the properties of the following set:

$$\mathbf{M} := \{g : \mathbb{R}_+ \to \mathbb{R} \mid (f \in \mathbf{F} \text{ and } g \circ f \text{ is convex on } (\gamma, +\infty) \text{ for some } \gamma \geqslant 0) \Rightarrow f = \Gamma\}.$$

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In this paper we find some elements of the set $\mathbf{M}$ and study its properties.

1. Properties of the set $\mathbf{M}$

We start with the two lemmas which follow directly from the definitions of monotonicity and concavity of the functions involved.

LEMMA 1

If $g : \mathbb{R} \to \mathbb{R}$ is increasing and concave then $g^{-1} : \mathbb{R} \to \mathbb{R}$ is increasing and convex.

LEMMA 2

Let X, Y, Z be some intervals of $\mathbb{R}$. If $f : X \to Y$ is a convex function and $g : Y \to Z$ is increasing and convex then $g \circ f : X \to Z$ is convex.

THEOREM 1

Let $g \in \mathbf{M}$ and let $h : \mathbb{R} \to \mathbb{R}$ be increasing and concave. Then $h \circ g : \mathbb{R}_+ \to \mathbb{R}$ belongs to the set $\mathbf{M}$.

Proof. Let g and h satisfy the assumptions and let $f \in \mathbf{F}$. By Lemma 1 the function h^{-1} is increasing and convex. If $h \circ g \circ f$ is convex then (by Lemma 2) $h^{-1} \circ h \circ g \circ f$ is convex, too. Thus, since $g \in \mathbf{M}$, we have $h \circ g \in \mathbf{M}$.

The above theorem implies

REMARK 1

Let $a > 0$. If $g : \mathbb{R}_+ \to (a, +\infty)$ is increasing and logarithmically convex then $g^{-1} \in \mathbf{M}$.

Proof. By our assumption $\log \circ g$ is a convex function. By Lemma 1 and Lemma 2 the function $g^{-1} \circ \exp = (\log \circ g)^{-1}$ is increasing and concave for $x \in (\log a, +\infty)$. By the Bohr-Mollerup Theorem, $\log \in \mathbf{M}$. Thus, by Theorem 1, $g^{-1} = (\log \circ g)^{-1} \circ \log \in \mathbf{M}$.

In particular, we have

REMARK 2

The function $G = \left(\Gamma\,|_{(2,+\infty)}\right)^{-1}$ is in $\mathbf{M}$.

THEOREM 2

If $g \in \mathbf{M}$, $a > 0$, $b \in \mathbb{R}$, then $a \cdot g + b \in \mathbf{M}$.

Proof. Let $g \in \mathbf{M}$, $a > 0$ and $b \in \mathbb{R}$. Take a function $f \in \mathbf{F}$. If the function $(a \cdot g + b) \circ f$ is convex then so is the function $\frac{1}{a} \cdot [(a \cdot g + b) \circ f] - \frac{b}{a} = g \circ f$. Since we have assumed that $g \in \mathbf{M}$, $f = \Gamma$ and $a \cdot g + b \in \mathbf{M}$.

Besides $f = \Gamma$ there are other convex functions belonging to $\mathbf{F}$, e.g., the functions $f_c : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f_c(x) = \Gamma(x) \exp\left[c \sin(2\pi x)\right].$$

They are convex for sufficiently small $c > 0$ on $(0, +\infty)$ (see [2]). Thus we obtain the following remarks:

REMARK 3
The function $\mathrm{id}_{\mathbb{R}_+}$ does not belong to $\mathbf{M}$.

REMARK 4
If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and convex function then $h \notin \mathbf{M}$.

Proof. Let h be a function satisfying the assumptions and let $f \in \mathbf{F}$. By Lemma 2 if f is convex then so is $h \circ f$. Because Γ is not the only convex element of $\mathbf{F}$, we have $h \notin \mathbf{M}$.

2. Special elements of the set $\mathbf{M}$

THEOREM 3
Let us assume that $h : \mathbb{R}_+ \to \mathbb{R}$ *and*

$$\lim_{x \to +\infty} h(x) = m \in \mathbb{R}. \tag{1}$$

Then $\log + h$ *is an element of the set* $\mathbf{M}$.

Proof. We put $g = \log + h$, and take a function $f \in \mathbf{F}$. Moreover we fix an $n \in \mathbb{N}$ and $x \in (0, 1]$. If $g \circ f$ is convex then the following inequalities hold true:

$$\begin{aligned} g \circ f(n) - g \circ f(n-1) &\leqslant \frac{g \circ f(n+x) - g \circ f(n)}{x} \\ &\leqslant g \circ f(n+1) - g \circ f(n). \end{aligned} \tag{2}$$

Using $f(x+1) = x f(x)$, we have

$$\begin{aligned} x\left(g_{n-1} - g_{n-2}\right) &\leqslant g\left[x(x+1)\ldots(x+n-1) f(x)\right] - g_{n-1} \\ &\leqslant x\left(g_n - g_{n-1}\right) \end{aligned} \tag{3}$$

where we have put, for short,

$$\begin{aligned} g_n &:= g(n!), \\ h_n &:= h(n!). \end{aligned}$$

Since $g = \log + h$, we get

$$x\left[\log(n-1)+h_{n-1}-h_{n-2}\right] \leqslant \log\left[x(x+1)\dots(x+n-1)f(x)\right] \\ + h\left[x(x+1)\dots(x+n-1)f(x)\right] \\ - \log(n-1)! - h_{n-1} \\ \leqslant x\left[\log n + h_n - h_{n-1}\right].$$

Since the exponential function is increasing, we obtain

$$(n-1)^x\left[\frac{\exp h_{n-1}}{\exp h_{n-2}}\right]^x \\ \leqslant \frac{x(x+1)\dots(x+n-1)f(x)}{(n-1)!}\cdot\frac{\exp\circ h\left[x(x+1)\dots(x+n-1)f(x)\right]}{\exp h_{n-1}} \\ \leqslant n^x\left[\frac{\exp h_n}{\exp h_{n-1}}\right]^x.$$

Hence

$$(n-1)^x(n-1)!\,\frac{(\exp h_{n-1})^{x+1}}{(\exp h_{n-2})^x} \\ \leqslant x(x+1)\dots(x+n-1)f(x)\cdot\exp\circ h\left[x(x+1)\dots(x+n-1)f(x)\right] \\ \leqslant n^x(n-1)!\,\frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}}.$$

In turn,

$$n^x n!\,\frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} \\ \leqslant x(x+1)\dots(x+n)f(x)\cdot\exp\circ h\left[x(x+1)\dots(x+n)f(x)\right] \\ \leqslant n^x n!\,\frac{x+n}{n}\cdot\frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}}\cdot\frac{\exp\circ h\left[x(x+1)\dots(x+n)f(x)\right]}{\exp\circ h\left[x(x+1)\dots(x+n-1)f(x)\right]}$$

and

$$\Gamma_n(x)\cdot\frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} \\ \leqslant f(x)\exp\circ h\left[x(x+1)\dots(x+n)f(x)\right] \\ \leqslant \frac{x+n}{n}\cdot\Gamma_n(x)\cdot\frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}}\cdot\frac{\exp\circ h\left[x(x+1)\dots(x+n)f(x)\right]}{\exp\circ h\left[x(x+1)\dots(x+n-1)f(x)\right]}.$$

Notice that by the relations resulting from (1) ($\lim_{n\to\infty} h_n = m$)

$$\lim_{n\to\infty}\exp\circ h\left[x(x+1)\dots(x+n)f(x)\right] = e^m$$

$$\lim_{n\to\infty}\frac{(\exp h_n)^{x+1}}{(\exp h_{n-1})^x} = e^m$$

$$\lim_{n\to\infty}\frac{(\exp h_n)^x}{(\exp h_{n-1})^{x-1}}\cdot\frac{\exp\circ h\,[x(x+1)\dots(x+n)f(x)]}{\exp\circ h\,[x(x+1)\dots(x+n-1)f(x)]}=e^m$$

and by (Γ) we obtain

$$\Gamma(x)e^m \leqslant f(x)e^m \leqslant \Gamma(x)e^m.$$

Thus $f(x)=\Gamma(x)$ for $x\in(0,1]$. We shall show that $f(x)=\Gamma(x)$ for each real positive x.

Let $x\in\mathbb{R}_+$. We proceed by induction. There exists a $k\in\mathbb{N}$ such that $x\in(k-1,k]$. If $k=1$ then we have already proved that $f(x)=\Gamma(x)$. Let us assume that $f(x)=\Gamma(x)$ for $x\in(k-1,k]$. Take $x\in(k,k+1]$ and $y=x-1$. Since $y\in(k-1,k]$, by the inductive assumption we have $f(y)=\Gamma(y)$. By the functional equation for f we have $f(y+1)=yf(y)$. Thus $f(y+1)=y\Gamma(y)=\Gamma(y+1)$ hence $f(x)=\Gamma(x)$ for $x\in(k,k+1]$. Therefore $f(x)=\Gamma(x)$ for $x\in\mathbb{R}_+$.

Remark 5

The function $g=\log+\arctan$ belongs to $\mathbf{M}$.

Remark 6

Let $a,b>0$. Then $\log\circ\left(a\,\mathrm{id}_{\mathbb{R}_+}+b\right)\in\mathbf{M}$.

Proof. Take $h=\log\circ\left(a+\frac{b}{\mathrm{id}_{\mathbb{R}_+}}\right)$, so that $\log\circ\left(a\,\mathrm{id}_{\mathbb{R}_+}+b\right)=\log+h$ and $\lim_{x\to+\infty}h(x)=a$. Thus, by Theorem 3, $\log\circ\left(a\,\mathrm{id}_{\mathbb{R}_+}+b\right)\in\mathbf{M}$.

Theorem 4

Let $m,a>0$ and let $h:\mathbb{R}_+\to\mathbb{R}_+$ be an increasing function such that $h(x)=m-\frac{a}{x}+R(x)$, where $R(x)=o\left(\frac{1}{x}\right)$, $x\to+\infty$. Then $h\cdot\log\in\mathbf{M}$.

Proof. We put $g=h\cdot\log$, and we take a function $f\in\mathbf{F}$. Moreover we fix an $n\in\mathbb{N}$ and $x\in(0,1)$.

If $g\circ f$ is convex then inequalities (2) and (3) (as in the proof of Theorem 3) hold true. Since $g=h\cdot\log$, we get

$$\begin{aligned}
&x\,\{h_{n-1}\log[(n-1)!]-h_{n-2}\log[(n-2)!]\}\\
&\quad\leqslant h\circ f(n+x)\,\log[x(x+1)\dots(x+n-1)]\\
&\qquad-h_{n-1}\log[(n-1)!]\\
&\quad\leqslant x\,\{h_n\log(n!)-h_{n-1}\log[(n-1)!]\}.
\end{aligned}\tag{4}$$

By properties of the logarithmic function we have

$$\begin{aligned}
\log\left[\frac{(n-1)!^{h_{n-1}}}{(n-2)!^{h_{n-2}}}\right]^x &\leqslant \log\frac{[x(x+1)\dots(x+n-1)f(x)]^{h\circ f(x+n)}}{(n-1)!^{h_{n-1}}}\\
&\leqslant \log\left[\frac{n!^{h_n}}{(n-1)!^{h_{n-1}}}\right]^x.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{\left[(n-1)!^{h_{n-1}}\right]^{x+1}}{\left[(n-2)!^{h_{n-2}}\right]^{x}} &\leqslant [x(x+1)\dots(x+n-1)f(x)]^{h\circ f(x+n)} \\ &\leqslant \frac{\left[n!^{h_n}\right]^{x}}{\left[(n-1)!^{h_{n-1}}\right]^{x-1}}\end{aligned}$$

and next

$$\begin{aligned}\left[\frac{(n-1)!^{(x+1)h_{n-1}}}{(n-2)!^{xh_{n-2}}}\right]^{\frac{1}{h\circ f(x+n)}} &\leqslant x(x+1)\dots(x+n-1)f(x) \\ &\leqslant \left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}} .\end{aligned}$$

It follows easily that

$$\begin{aligned}\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} &\leqslant x(x+1)\dots(x+n)f(x) \\ &\leqslant (x+n)\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}} .\end{aligned}$$

So we have

$$\begin{aligned}&\frac{\Gamma_n(x)}{n!n^x}\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} \\ &\qquad\leqslant f(x) \leqslant \frac{\Gamma_n(x)(x+n)}{n!n^x}\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}}\end{aligned} \tag{5}$$

Let us put

$$l_n = \frac{1}{n!n^x}\left[\frac{n!^{(x+1)h_n}}{(n-1)!^{xh_{n-1}}}\right]^{\frac{1}{h\circ f(x+n+1)}} \tag{6}$$

and

$$r_n = \frac{(x+n)}{n!n^x}\left[\frac{n!^{xh_n}}{(n-1)!^{(x-1)h_{n-1}}}\right]^{\frac{1}{h\circ f(x+n)}} . \tag{7}$$

We notice that

$$l_n = [a_n\, b_n\, c_n]^{\frac{1}{h\circ f(n+1+x)}} \tag{8}$$

and

$$r_n = [a_n\, b_{n-1}\, d_n]^{\frac{1}{h\circ f(n+x)}} \cdot \frac{x+n}{n} \tag{9}$$

where

$$\log a_n = x(h_n - h_{n-1})\log[(n-1)!]\,, \tag{10}$$

$$\log b_n = [h_n - h\circ f(n+1+x)]\log(n!)\,, \tag{11}$$

$$\log c_n = x[h_n - h \circ f(n+1+x)] \log n \tag{12}$$

and

$$\log d_n = x[h_n - h \circ f(n+x)] \log n. \tag{13}$$

We shall prove that

$$\lim_{n\to\infty} l_n = 1 \quad \text{and} \quad \lim_{n\to\infty} r_n = 1.$$

By (10) it is obvious that

$$\frac{\log a_{n+1}}{\log a_n} = A_n \cdot \frac{\log(n!)}{\log[(n-1)!]}, \tag{14}$$

where

$$A_n = \frac{h_{n+1} - h_n}{h_n - h_{n-1}}.$$

By the assumptions of the theorem we have

$$A_n = \frac{m - \frac{a}{(n+1)!} + R[(n+1)!] - m + \frac{a}{n!} - R(n!)}{m - \frac{a}{n!} + R(n!) - m + \frac{a}{(n-1)!} - R[(n-1)!]}.$$

Thus

$$A_n = \frac{(n-1)!}{n!} \cdot \frac{a - \frac{a}{n+1} + n!R[(n+1)!] - n!R(n!)}{a - \frac{a}{n} + (n-1)!R(n!) - (n-1)!R[(n-1)!]}.$$

Because $R(x) = o(\frac{1}{x})$, $x \to +\infty$, we have

$$\lim_{x\to+\infty} xR(x) = 0. \tag{15}$$

Consequently $\lim_{n\to\infty} A_n = 0$ and by (14) $\lim_{n\to\infty} \log a_n = 0$, which gives

$$\lim_{n\to\infty} a_n = 1. \tag{16}$$

Similarly, by (11),

$$\frac{\log b_{n+1}}{\log b_n} = B_n \cdot \frac{\log[(n+1)!]}{\log(n!)}, \tag{17}$$

where

$$B_n = \frac{h_{n+1} - h \circ f(n+2+x)}{h_n - h \circ f(n+1+x)}$$

and by (1) we have

$$B_n = \frac{m - \frac{a}{(n+1)!} + R[(n+1)!] - m + \frac{a}{f(n+2+x)} - R[f(n+2+x)]}{m - \frac{a}{n!} + R(n!) - m + \frac{a}{f(n+1+x)} - R[f(n+1+x)]}$$

and further

$$B_n = \frac{n!}{(n+1)!} \times \frac{\frac{a(n+1)!}{f(n+2+x)} - a + (n+1)!R\,[(n+1)!] - (n+1)!R\,[f(n+2+x)]}{\frac{an!}{f(n+1+x)} - a + n!R(n!) - n!R\,[f(n+1+x)]}. \tag{18}$$

Because $f(n+1+x) = x(x+1)\dots(x+n)\,f(x)$, we have (by (Γ_n))

$$\frac{n!}{f(n+1+x)} = \Gamma_n(x)\cdot\frac{1}{f(x)\,n^x}$$

and

$$\lim_{n\to\infty}\frac{n!}{f(n+1+x)} = 0.$$

Note that

$$n!R\,[f(n+1+x)] = f(n+1+x)R\,[f(n+1+x)]\,\frac{n!}{f(n+1+x)}$$

so that by (15) we have

$$\lim_{n\to\infty} n!\,R\,[f(n+1+x)] = 0.$$

Thus (18) yields $\lim_{n\to\infty} B_n = 0$ whence $\lim_{n\to\infty}\log b_n = 0$ (by (17)), and finally

$$\lim_{n\to\infty} b_n = 1. \tag{19}$$

Similarly (using (12)) we can prove that

$$\lim_{n\to\infty} c_n = 1. \tag{20}$$

Finally, by (13) it follows that

$$\frac{\log d_{n+1}}{\log d_n} = D_n\frac{\log(n+1)}{\log n},$$

where

$$D_n = \frac{h_{n+1} - h\circ f(n+1+x)}{h_n - h\circ f(n+x)}.$$

We can observe that

$$D_n = \frac{f(n+x)}{f(n+1+x)}\cdot\frac{S_{n+1}}{S_n},$$

with

$$S_n := a - f(n+x)\left[\frac{a}{n!} - R[f(n+x)] + R(n!)\right]. \tag{21}$$

But

$$\frac{f(n+x)}{n!} = \frac{1}{\Gamma_n(x)}\cdot\frac{n^x f(x)}{x+n} = \frac{f(x)}{\Gamma_n(x)}\cdot\frac{1}{\left(\frac{x}{n}+1\right)n^{1-x}}$$

and because of $x \in (0,\, 1)$ we have

$$\lim_{n\to\infty} \frac{f(n+x)}{n!} = 0.$$

The identity

$$f(n+x)\, R(n!) = n!\, R(n!) \cdot \frac{f(n+x)}{n!}$$

together with (15) imply

$$\lim_{n\to\infty} f(n+x)\, R(n!) = 0.$$

So $\lim_{n\to\infty} D_n = 0$ because of (15) and (21), whence $\lim_{n\to\infty} \log d_n = 0$ and finally

$$\lim_{n\to\infty} d_n = 1. \tag{22}$$

Thus by relations (8), (9), (16), (19), (20) and (22) we obtain

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n = 1. \tag{23}$$

This implies that $f(x) = \Gamma(x)$ for $x \in (0,\, 1]$ (as $f(1) = 1 = \Gamma(1)$).

Applying the same inductive argument as in the proof of Theorem 3 we find that $f(x) = \Gamma(x)$ for $x \in \mathbb{R}_+$, and the proof is completed.

The starting point of the proofs of Theorem 3 and of Theorem 4 is analogous to that in Artin's proof [1] of the Bohr-Mollerup Theorem.

We notice that in a vicinity of $+\infty$ the function arctan is represented by

$$\arctan \mathrm{x} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots\,.$$

Thus we have the following

REMARK 7

The function arctan · log is in **M**.

3. Special convex compositions with Γ

It is known that $g \circ \Gamma$ is convex on $\mathbb{R}_+$ for $g = \log$. We want to present other functions g with this property.

THEOREM 5

Let the functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$ be defined by $g_1 := \log + \arctan$ and $g_2 := \log \circ \big(\mathrm{id}_{\mathbb{R}_+} + a\big)$, where $a > 0$. Then there is a $\gamma > 0$, such that $g_1 \circ \Gamma$ and $g_2 \circ \Gamma$ are convex on $(\gamma, +\infty)$.

Proof. 1°. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be given by $\psi = (\log \circ \Gamma)'$, and let $g_1 = \log + h$ (where $h = \arctan$). We notice by (Γ) that the function $\psi = \frac{\Gamma'}{\Gamma}$ is represented by the formula

$$\psi(x)=\log x-\sum_{n=0}^{\infty}\left[\frac{1}{x+n}-\log\left(1+\frac{1}{x+n}\right)\right],$$

so (by the inequality $\log(1+x)\leqslant x$) we have

$$\psi(x)\leqslant\log x. \tag{24}$$

Moreover, the derivative of ψ is given by

$$\psi'(x)=\frac{\Gamma''(x)\Gamma(x)-[\Gamma'(x)]^2}{[\Gamma(x)]^2}=\frac{1}{x}+\frac{1}{2x^2}+\int_0^{+\infty}\frac{4tx}{(t^2+x^2)^2\,(e^{2\pi t}-1)}\,dt \tag{25}$$

(see [4] p. 250-251). Thus $\psi'=(\log\circ\Gamma)''\geqslant 0$ on $(0,+\infty)$. By the definition of g_1 we see that

$$(g_1\circ\Gamma)''=\frac{\Gamma''\Gamma-(\Gamma')^2}{\Gamma^2}+(h''\circ\Gamma)\,(\Gamma')^2+(h'\circ\Gamma)\Gamma''.$$

By properties of h we can see that

$$\forall x\in\mathbb{R}_+:\ h'\circ\Gamma(x)\cdot\Gamma''(x)\geqslant 0,$$

and

$$h''\circ\Gamma=\frac{-2\Gamma}{(1+\Gamma^2)^2}.$$

So we obtain:

$$\begin{aligned}h''\circ\Gamma(x)\cdot[\Gamma'(x)]^2&\geqslant\frac{-2\Gamma'(x)}{[\Gamma(x)]^3}\\&=-2\left[\frac{\Gamma'(x)}{\Gamma(x)}\right]^2\cdot\frac{1}{\Gamma(x)}=\frac{-2\,[\psi(x)]^2}{\Gamma(x)}.\end{aligned}$$

Thus by conditions (24) and (25) we have:

$$(g_1\circ\Gamma)''(x)\geqslant\frac{1}{x}+\frac{1}{2x^2}-2\frac{\log^2 x}{\Gamma(x)}\geqslant 0$$

for sufficiently large x, say for $x\geqslant\gamma$. Hence $g_1\circ\Gamma$ is convex on $(\gamma,+\infty)$.

2°. Now let $a>0$ and let $g_2=\log\circ\left(\mathrm{id}_{\mathbb{R}_+}+a\right)$. The function $(g_2\circ\Gamma)''$ is given by the formula

$$(g_2\circ\Gamma)''(x)=\frac{\Gamma''(x)\Gamma(x)-[\Gamma'(x)]^2+a\Gamma''(x)}{[\Gamma(x)+a]^2}.$$

Because $\log\circ\Gamma$ and Γ are convex and twice differentiable, we have

$$\forall x\in\mathbb{R}_+:\ \Gamma''(x)\Gamma(x)-[\Gamma'(x)]^2\geqslant 0 \text{ and } \Gamma''(x)\geqslant 0$$

Hence $g_2\circ\log$ is convex on $\mathbb{R}_+$.

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