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Report of Meeting

7th International Conference on Functional Equations and Inequalities, Zlockie, September 12-18, 1999

The Seventh International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University in Kraków, was held from September 12 to September 18, 1999, in the hotel "Geovita" at Złockie. The preceding ICFEI took place at: Sielpia (1984), Szczawnica (1987), Koninki (1991), Krynica (1993) and Muszyna-Złockie (1995 and 1997). The substantial support of the Polish State Committee for Scientific Research (KBN) and of the Foundation for Advancement of Science "Kasa im. Józefa Mianowskiego" is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Eugeniusz Wachnicki, the Dean of the Faculty of Mathematics, Physics and Technics of the Pedagogical University in Kraków. He conveyed participants' best greetings and congratulations to Professor János Aczél, whom the University of Miskolc conferred on September 11, 1999 the degree of Doctor Honoris Causa. It was the fourth of Prof. Aczél's honorary doctorates, after those granted by the Universities of Karlsruhe, Graz and Katowice.

There were 73 participants who came from: Austria (1), Canada (3), Czech Republic (2), Germany (3), Hungary (8), Italy (2), Japan (1), Russia (1), Sweden (1), Ukraine (3), The U.S.A. (1), Venezuela (1), Yugoslavia (1); and from Poland: Gdańsk (1), Gliwice (1), Katowice (16), Kielce (1), Kraków (19), Rzeszów (5), Zielona Góra (2).

During 19 sessions 63 talks were delivered mainly on: functional equations in several variables, their stability, and applications; iteration theory (also for multifunctions) and dynamical systems; functional and general inequalities.

The organizing Committee was chaired by Professors Dobiesław Brydak and Bogdan Choczewski. Dr. Jacek Chmieliński acted as a scientific secretary. Miss Ewa Dudek, Mrs. Anna Grabiec, Miss Janina Wiercioch and Mr. Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference office at Złockie.

At Wednesday, September 15, there was a half-day excursion to the Pieniny region. A group of participants took part in the scenic rafting race on the Dunajec river through a gorge in the Pieniny mountains. Other group visited the castle Dunajec at Niedzica (constructed about 1310 and owned by Hungarian noble families till 1943) and the world famous wooden 15th century church at Dębno (with its polychromy which have kept its fastness for 500 years up to the present).

When closing the Conference, Professor D. Brydak first asked to commemorate two our colleagues, who attended former meetings and passed away in the last two years.

Professor György Targonski died at the age of 69, on January 10, 1998, in Munich. The European Conference on Iteration Theory held in September, 1999, also in "Geovita", was dedicated to his memory.

Mr. Martin Grinč, Slovakian citizen, who studied at the Silesian University in Katowice, died of cancer at the age of 28, on January 18, 1999, at Stará L'ubovňa, a week after submitting his Ph. D. thesis.

Expressing cordial thanks to the participants, and especially to Professor János Aczél, Prof. Brydak pointed out that the present meeting was the best also in numbers: of participants (73), talks (63) and contributions (14) presented on problems-and-remarks parts of many sessions. He extended best thanks to the members of the office staff at Złockie for their effective and dedicated work and assistance, and to the managers of the hotel "Geovita" for their hospitality and quality of services.

The 8th ICFEI was announced to be held in 2001, most probably in September, at the same place.

The abstracts of talks are printed in the alphabetical order, and the contributions to the problems-and-remark sessions — in the order of presentation. They were completed by Dr. J. Chmieliński and prepared for printing by him and Mr. W. Wilk.

Bogdan Choczewski

Abstracts of Talks

János Aczél *The strictly monotonic solutions of a functional equation arising from coordination of two ways to measure utility*

Joint work with Gy. Maksa and Zs. Páles.

Gyula Maksa, Duncan Luce and I dealt in 1996 with the pair of functional equations

$$H(x,y)z = H[xz, yP(x,z)],$$

$$G[H(x,y)] = G(x)G(y),$$

(x and y are in $[0,1[$, z in $[0,1]$) originating from a problem of utility theory described in the title of this talk. That problem makes it natural to assume that G is strictly monotonic and maps $[0,1[$ onto $]0,1]$ (this determines H from the second equation) and P maps $[0,1[\times[0,1]$ into $[0,1]$. From these conditions we proved that

$$P(x,z) = g(x)/g(xz),$$

where g is continuous, strictly decreasing and maps $]0,1[$ into the set of positive reals, while $P(0,z) = z$ and $P(x,0) = 0$. This reduces the first of the above equations to

$$G[yg(x)/g(xz)] = G[H(x,y)z]/G(xz),$$

with H defined, as before, by the second equation. We in 1996 and several others since then, however, succeeded to advance further to the complete solution of the problem only under differentiability conditions. Eventually I came to the paradox idea that the limit equation (as y tends to 1; continuity has already been established)

$$G[g(x)/g(xz)] = G(z)/G(xz)$$

may be easier to solve. We succeeded to do this with help of an idea (by now method) of Páles which derives the Jensen inequality from this equation. That is what this talk is about.

Roman Badora *On approximate additive derivations*

The aim of the talk is to present a stability theorem for additive derivations.

Karol Baron *On a linear functional equation in a complex domain*

Studies of the problem how the brain works have led Thomas L. Saaty (University of Pittsburgh) to the functional equation

$$f(a_1 z_1, \dots, a_N z_N) = bf(z_1, \dots, z_N),$$

where $a_1, \dots, a_N$ and b are given complex numbers. It is the purpose of the talk to present a result on its solutions $f : (\mathbb{C}\setminus\{0\})^N \to \mathbb{C}$ which are continuous on polycircles about the origin.

Lech Bartłomiejczyk *Solutions with big graph of iterative functional equations of the first order*

We obtain a result on the existence of a solution with big graph of functional equations of the form

$$g(x, \varphi(x), \varphi(f(x))) = 0$$

and we show that it is easily applicable to some particularly important equations, both linear and nonlinear, as, e.g., those of Abel, Böttcher and Schröder. The graph of such a solution has some strange properties: it is dense and connected, has full outer measure and is topologically big.

Bogdan Batko *On the stability of an alternative Cauchy equation*

The talk is based on the results obtained jointly with Jacek Tabor.

Let G be a commutative semigroup and let $f : G \to \mathbb{R}$. We deal with the stability (in the Hyers-Ulam sense) of the functional equation

$$|f(x+y)| = |f(x) + f(y)| \quad \text{for } x, y \in G$$

and its generalizations. We obtain the following results.

THEOREM 1

Let $V \subset G$ be such that for every $x \in G \setminus \{0\}$ there exists an $n \in \mathbb{N}$ with $kx \notin V$ for $k \geqslant n$. Suppose that $f : G \to \mathbb{R}$ satisfies for some $\delta > 0$ the inequality

$$\big|\,|f(x+y)| - |f(x) + f(y)|\,\big| \leqslant \delta \quad \textit{for } (x, y) \in G \times G \setminus V \times V.$$

We prove that there exists a unique additive function $\gamma : G \to \mathbb{R}$ such that

$$|f(x) - \gamma(x)| \leqslant 3\delta \quad \textit{for } x \in G.$$

The constant of the approximation is, in general, the best possible one.

We also show that an analogon of this result for functions $f : G \to \mathbb{R}^2$ does not hold.

THEOREM 2

Let L be a complete Archimedean Riesz Space. Suppose that $F : G \to L$ satisfies for some $e \in L_+$ the inequality

$$\big|\,|F(x+y)| - |F(x) + F(y)|\,\big| \leqslant e \qquad \textit{for } x, y \in G.$$

Then there exists a unique additive mapping $A : G \to L$ such that

$$|F(x) - A(x)| \leqslant e \qquad \textit{for } x \in G.$$

As the method of the proof we use the Johnson-Kist Representation Theorem.

Zoltán Boros *Stability of the Cauchy equation in ordered fields*

Let R be an ordered field and $I \subset R$ be an interval. We give sufficient conditions for R and I so that the following statement hold: if $(X, +)$ is a commutative semigroup and $f : X \to R$ such that

$$f(x) + f(y) - f(x+y) \in I \quad \text{for every } x, y \in X,$$

then there exists an additive function $g : X \to R$ such that $f(x) - g(x) \in I^*$ for every $x \in X$, where $I^* = I$ (if I denotes the set of infinitesimal or finite elements) or I^* is infinitesimally close to I (if I is of the form $[-\delta, \delta]$).

Janusz Brzdęk *On the isosceles orthogonally exponential mappings*

Let X be a real normed space with $\dim X > 1$ and K be a field. We have the following theorem.

THEOREM 1
Suppose $f : X \to K$ satisfies

$$f(x+y) = f(x)f(y) \quad \textit{whenever } \|x+y\| = \|x-y\|. \tag{1}$$

Then $f(X \setminus \{0\}) = \{0\}$ or $0 \notin f(X)$.

Theorem 1 yields the subsequent two corollaries.

COROLLARY 1
Suppose X is not an inner product space and $\dim X > 2$. *Then every solution $f : X \to K$ of* (1) *is exponential, i.e.*

$$f(x+y) = f(x)f(y) \quad \textit{for every } x, y \in X. \tag{2}$$

COROLLARY 2
Let X be as in Corollary 1, $(S, \cdot)$ be a commutative semigroup with the neutral element e, and $f : X \to S$ be a solution of (1). *Suppose that $f(x_0)$ is invertible (in S) for some $x_0 \in X \setminus \{0\}$. Then* (2) *holds.*

Jacek Chmieliński *Almost approximately inner product preserving mappings*

Motivated by previous papers dealing with mappings preserving the inner product almost everywhere well as by stability results for the orthogonality equation we investigate a combination of these two problems. We show that a mapping that preserves inner product approximately and up to a negligible set of arguments has to be almost everywhere close to an exact solution of the orthogonality equation.

Jacek Chudziak *Continuous solutions of the generalized Gołąb-Schinzel equation*

We consider the functional equation

$$f(g(x)\phi(f(y)) + h(y)\varphi(f(x))) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{1}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$ and $\phi, \varphi : f(\mathbb{R}) \to \mathbb{R}$ are unknown functions such that

(i) f and ϕ are continuous;

(ii) $f(0) = 1$;

(iii) g, h are bijections with $g(0) = h(0) = 0$.

The equation (1) is a generalization of the well known Gołąb-Schinzel equation.

Krzysztof Ciepliński *On non-singular iteration groups on the unit circle*

Let S^1 be the unit circle with positive orientation and V be a vector space over $\mathbb{Q}$ such that $\dim V \geqslant 1$.

We consider non-singular iteration groups $\{F^v,\ v \in V\}$ of homeomorphisms of S^1, that is iteration groups possesing at least one element without periodic point. We present some results on such groups with no further assumptions as well as we give the general form of some particular non-singular iteration groups.

Stefan Czerwik *Stability of the quadratic functional equation in L^p spaces*

Joint work with Krzysztof Dłutek.

Let (X, v) be an abelian complete measurable group with $v(X) = \infty$ and let E be a metric abelian group. A function $f : X \to E$ is called quadratic iff it satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in X. \tag{1}$$

We define the quadratic difference Qf by

$$Qf(x,y) := 2f(x) + 2f(y) - f(x+y) - f(x-y). \tag{2}$$

By L_p^+, $p > 0$ we denote some generalization of the space L^p. The following results can be proved.

Lemma 1

Let E be a space without elements of order two. If $f(x) = q(x) + c$, $x \in X$, where q is quadratic and $c \in E$ is fixed and $f \in L_p^+(x, E)$ for a certain $p > 0$, then

$$q = 0 \quad \text{and} \quad c = 0.$$

Lemma 2

Let G, H be abelian groups. Then for every $f : G \to H$, the quadratic difference Qf satisfies the functional equation:

$$\begin{aligned} &Qf(x, u+v) + Qf(x, u-v) + 2Qf(u,v) \\ &\quad = Qf(x+u, v) + Qf(x-u, v) + 2Qf(x,u). \end{aligned} \tag{3}$$

Theorem 1

Let E be uniquely divisible by two. Let $f : X \to E$ be such that $Qf(x,y) \overset{v\times v}{=} 0$. Then there exists a quadratic function $q : X \to E$ such that

$$f(x) \overset{v}{=} q(x), \quad x \in X. \tag{4}$$

THEOREM 2

Let E be a space without elements of order two. If $f : X \to E$ is such that $Qf \in L_p^+(X \times X, E)$, then

$$Qf(x,y) \overset{v\times v}{=} 0. \tag{5}$$

Zoltán Daróczy *Characterization of Matkowski pairs*

This work is joint with Gy. Maksa and Zs. Páles.

Let $I \subset \mathbb{R}$ be an open interval and let $CM(I)$ denote the class of all continuous and strictly monotonic real-valued functions defined on I. If $\phi \in CM(I)$, then we define

$$A_\phi(x,y) = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)$$

for all $x, y \in I$. A pair of functions $(\phi, \psi) \in CM(I)^2$ is called a Matkowski pair if the functional equation

$$A_\phi(x,y) + A_\psi(x,y) = x + y$$

holds for all $x, y \in I$. We characterize Matkowski pairs in the following two cases:

(i) there exists a nonvoid open interval $K \subset I$ such that either ϕ or ψ is continuously differentiable on K;

(ii) there exists a nonvoid open interval $K \subset I$ such that A_ϕ and A_ψ are strictly comparable in K.

Thomas M.K. Davison *D'Alembert's functional equation and the Chebyshev polynomials*

The functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{d'Alembert}$$

is studied where the domain of f is the additive group of the integers, and the codomain of f is an arbitrary commutative ring R. We show there is a function

$$T : \mathbb{Z} \to \mathbb{Z}[X] \quad \text{denote} \quad n \mapsto T_n$$

such that if $f : \mathbb{Z} \to R$ satisfies d'Alembert and $f(0) = 1$ then, for all $n \in \mathbb{Z}$

$$f(n) = T_n(f(1)).$$

The sequence T_n of polynomials is identified with the sequence of Chebyshev polynomials using Kannappan's fundamental result

$$f(x) = \frac{e(x) + e(-x)}{2}$$

where $e(x+y)=e(x)e(y)$ and $e(0)=1$.

In our case

$$e(n) := \begin{bmatrix} X & 1 \\ X^2-1 & X \end{bmatrix}^n \qquad n \in Z.$$

Certain consequences of our result are discussed.

[1] P. Kannappan, *The functional equation $f(xy)+f(xy^{-1})=2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc. **19** (1968), 69-74.

Joachim Domsta *Regularly varying solutions of Schröder's and related linear equations*

This is a presentation of an extension of former results by B. Choczewski, M. Kuczma, E. Seneta, A. Smajdor and other authors on regularly varying at 0 solutions of the linear equations

$$\Psi(f(x)) = g(x)\cdot\Psi(x), \quad x \in \mathbb{R}_+ := (0,\infty)\,. \tag{$S_{f,g}$}$$

DEFINITION

We say, that $h:\mathbb{R}_+\to\mathbb{R}_+$ *is almost constant at 0*, if

(C_0) h is continuous and $h(0^+)\in\mathbb{R}_+$;

(C_1) at least one of the following conditions is fulfilled,

(a) h is of bounded variation locally at 0;

(b) $h(x)-h(0^+)=O(|\log x|^{-1-\nu})$, as $x\to 0^+$, where $\nu>0$.

The solutions are analysed with the use of the following assumptions and notation,

(H_1) f is a nondecreasing continuous selfmapping of $\mathbb{R}_+$, $0<f(x)<x$, for $x\in\mathbb{R}_+$ and $f'_+(0) := \lim_{x\to 0^+}\frac{f(x)}{x}\in(0,1)$;

(H_2) g is almost constant at 0;

$$\gamma_n(x|y) := \frac{G_n(y)}{G_n(x)}\,,$$

where $G_n(x) := \prod_{k=0}^{n-1} g_0(f^k(x))$, $g_0 := \frac{g}{g(0^+)}$, $x\in\mathbb{R}_+$, $n\in\mathbb{N}$.

LEMMA

For every $y\in\mathbb{R}_+$, the limit $\gamma(x\,|\,y) := \lim_{n\to\infty}\gamma_n(x\,|\,y)$, $x\in\mathbb{R}_+$, exists and forms a continuous slowly varying solution of (S_{f,g_0}).

COROLLARY 1

If besides (H_1), $\frac{f(x)}{x}$, $x\in\mathbb{R}_+$, *is almost constant at* 0, *then the canonical Schröder equation* $\Phi(f(x)) = f'_+(0)\cdot\Phi(x)$, $x\in\mathbb{R}_+$, *possesses exactly one (up* *to a multiplicative constant) regularly varying solution; it equals the principal function, i.e.*

$$\Phi(x) = \Phi(y) \cdot \varphi(x|y) \qquad \textit{where } \varphi(x|y) := \lim_{n\to\infty} f^n(x)/f^n(y), \ x, y \in \mathbb{R}_+ .$$

COROLLARY 2

Under the assumptions of the above Lemma and Corollary 1, the linear equation $(\mathrm{S}_{f,g})$ *possesses exactly one regularly varying solution; it is given by the formula*

$$\Psi(x) = \Psi(y) \cdot (\varphi(x|y))^\rho \cdot \gamma(x|y), \quad x, y \in \mathbb{R}_+ ,$$

where $\rho = \frac{\log g(0^+)}{\log(f'_+(0))}$.

Tibor Farkas *On the associativity of algorithms*

Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{i=1}^\infty \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called *interval filling* if, for any $x \in [0, L(\lambda)]$, there exists a sequence (δ_n) such that $\delta_n \in \{0,1\}$ for all $n \in \mathbb{N}$ and $x = \sum_{i=1}^\infty \delta_n \lambda_n$. (This concept has been introduced and discussed in Daróczy-Járai-Kátai [1].)

An *algorithm* (with respect to an interval filling sequence λ) is defined as a sequence of functions $\alpha_n : [0, L(\lambda)] \to \{0,1\}$ $(n \in \mathbb{N})$ for which

$$x = \sum_{n=1}^\infty \alpha_n(x)\lambda_n \quad (x \in [0, L(\lambda)]).$$

The most important and frequently observed algorithms are the *regular* (or *greedy*), the *quasi-regular* and the *anti-regular* (or *lazy*) ones. In [2] Gy. Maksa introduced the following concept: an algorithm (α_n) is called *associative* if the binary operation $\circ : [0, L(\lambda)]^2 \to [0, L(\lambda)]$ defined by

$$x \circ y = \sum_{n=1}^\infty \alpha_n(x)\alpha_n(y)\lambda_n \quad (x, y \in [0, L(\lambda)])$$

is associative. In the same paper the author characterized the associative algorithms and proved the associativity of the regular algorithm with respect to any interval filling sequence and the non-associativity of the anti-regular algorithm in the case of a special class of interval filling sequences.

The purpose of this presentation is to prove the non-associativity of the anti-regular and the quasi-regular algorithms and the existence of associative algorithms different from the regular one.

[1] Z. Daróczy, A. Járai, I. Kátai, *Intervallfüllende Folgen und volladditive Funktionen*, Acta Sci. Math. **50** (1986), 337-350.

[2] Gy. Maksa, *On associative algorithm*, Acta Acad. Paed. Agriensis, Sectio Matematicae (to appear).

Carlos Finol *Inequalities arising from Schlumprecht's construction of an arbitrarily distortable Banach space*

T. Schlumprecht (Israel J. of Math., **76** (1991), 81-95) furnished an explicit construction of an arbitrarily distortable Banach space. The construction is accomplished by using a concrete submultiplicative function,

$$\frac{\log(1+x)}{\log 2}, \quad x \geqslant 1,$$

whose Matuszewska-Orlicz index, at infinity, is zero; along with other properties. That similar spaces can be constructed with functions which share some properties of that function is stated therein.

We single out those properties which characterize the 'Schlumprecht Class' of functions which produce such a spaces and derive a general inequality which all the functions in this class satisfy.

Margherita Fochi *Conditional functional equations in normed spaces*

Let X be a real normed vector space with $\dim X \geqslant 3$ and $f : X \to \mathbb{R}$. We study the exponential Cauchy equation

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \tag{1}$$

and its following conditional forms

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \text{ with } \|x\| = \|y\| \qquad (1)_1$$

and

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in X \text{ with } x \perp_I y \qquad (1)_2$$

where the James isosceles orthogonality $x \perp_I y$ is defined as follows

$$x \perp_I y \iff \|x+y\| = \|x-y\|.$$

Referring to recent results of Gy. Szabó on the additive Cauchy equation conditioned in the above domains, we prove the equivalence of equations (1), $(1)_1$ and $(1)_2$.

Roman Ger *Ring homomorphisms equation revisited*

We deal with a functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{$*$}$$

considered by J. Dhombres (*Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186-212) for functions f mapping a given ring into another one. In this paper both rings were supposed to have unit elements; additionally the division by 2 had to be performable.

Without these assumptions the study of equation (∗) becomes considerably more sophisticated (see author's paper *On an equation of ring homomorphisms*, Publicationes Math. **52** (1998), 397-417). At present, we deal with equation (∗) assuming that the domain is a unitary ring with no assumptions whatsoever upon the target ring.

Attila Gilányi *Hyers-Ulam stability of monomial functional equations on a general domain*

In this talk the Hyers-Ulam stability of mononomial functional equations for functions defined on a power-associative, power-symmetric groupoid is investigated.

Throughout the talk $(S, \circ)$ denotes a groupoid, that is, a nonempty set S with binary operation $\circ : S \times S \to S$. The powers of an element $x \in S$ are defined by $x^1 = x$ and, for a positive integer m, by $x^{m+1} = x^m \circ x$. An operation $\circ$ (or the groupoid $(S, \circ)$) is called *power-associative* if $x^{k+m} = x^k \circ x^m$ for all positive integers k, m and each $x \in S$, it is said to be l^{th}*-power-symmetric* (or simply *power-symmetric*) if $l \geqslant 2$ is a given integer such that $(x \circ y)^l = x^l \circ y^l$ for all $x, y \in S$. Using this notation we call a function f mapping from $(S, \circ)$ into a linear normed space X a *monomial function of degree* n if $\Delta_y^n f(x) - n! f(y) = 0$ for all $x, y \in S$, where Δ denotes the well-known difference operator.

Our main result reads as follows. If n is a positive integer, $(S, \circ)$ is a power-associative, power-symmetric groupoid, B is a Banach space, $f : S \to B$ is a function, and, for a nonnegative real number ε, we have

$$\|\Delta_y^n f(x) - n! f(y)\| \leqslant \varepsilon \quad (x, y \in S),$$

then there exists a unique monomial function $g : S \to B$ of degree n such that

$$\|f(x) - g(x)\| \leqslant \frac{1}{n!}\varepsilon \quad (x \in S).$$

In the special case when S is an Abelian group, this result yields the Hyers-Ulam stability of monomial functional equations in a well-known form, furthermore, if $n = 1$, we get the stability of the Cauchy equation.

Roland Girgensohn *Non-affine fractal interpolation functions*

Let $b \in \mathbb{N}$ and choose $b + 1$ data points (t_ν, y_ν), where $0 = t_0 < t_1 < \ldots < t_b = 1$ and $y_\nu \in \mathbb{R}$. Then the fractal interpolation functions of M.F. Barnsley, which are defined via certain iterated function systems, satisfy $f(t_\nu) = y_\nu$ and exhibit a fractal behaviour. The same functions can be defined as the solutions of systems of functional equations of the form

$$f((t_{\nu+1} - t_\nu)x + t_\nu) = a_\nu f(x) + g_\nu(x) \quad \text{for } \nu = 0, \ldots, b-1,$$

where $|a_\nu| < 1$ and $g_\nu : [0,1] \to \mathbb{R}$ are given, and $f : [0,1] \to \mathbb{R}$ is unknown. In the talk, we will point out a connection with certain Schauder bases on $C[0,1]$, we will give an explicit formula for the box dimension of these functions in the case of equidistant t_ν, and we will discuss certain singular solutions.

Andrzej Grząślewicz *On the functional equation $F(x,y) \bullet F(y,z) = F(x,z)$*

Let $(M, \bullet)$ be a "group with zero", $(B, \leqslant)$ a linearly ordered set. M. Fréchet in [1] and Z. Moszner in [2] characterized the solutions of equation

$$F(x,y) \bullet F(y,z) = F(x,z) \tag{1}$$

and their extensions in the case, where $M = \mathbb{R} = B$, $\bullet$ is the usual multiplication, $\leqslant$ is the usual order in $\mathbb{R}$ and F is a function defined on the set $\mathbb{R}\times\mathbb{R}\cap \leqslant$. A. Grząślewicz in [3] generalized these results assuming only, that F is defined on the set $B \times B \cap \leqslant$.

In our report we present the general solution of (1) assuming, that F is a function defined on the set $R_{A,B} := \{(x,y) \in A \times B : \ x \leqslant y\}$, where A is a subset of B. Moreover, as the common result with Angelo Grząślewicz, the extensions of considered solutions are characterized.

[1] M. Fréchet, *Solution continue la plus générale d'une équation fonctionelle de la théorie des probabilités en chaine,* Bull. Soc. Math. France **60** (1932), 232-280.
[2] Z. Moszner, *Ogólne rozwiązanie równania $F(x,y)\cdot F(y,z) = F(x,z)$ przy warunku $x \leqslant y \leqslant z$*, Rocznik Nauk.-Dydakt. WSP w Krakowie, **25**, Matematyka (1966), 123-138.
[3] A. Grząślewicz, *O pewnych homomorfizmach i homomorfizmach ciągłych grupoidu Brandta*, Rocznik Nauk.-Dydakt. WSP w Krakowie, **41**, Prace Matematyczne **6** (1970), 15-30.

Grzegorz Guzik *On embedding of a linear functional equation*

Let the iterative equation

$$\varphi(f(x)) = g(x)\varphi(x) + h(x), \tag{L}$$

where f, g, h are given continuous functions defined on an real interval X, have on X a continuous solution φ, and let f be embeddable in a continuous iteration group F defined on $\mathbb{R}\times X$. We say that (L) has embedding with respect to F if there exist functions G and H defined on $\mathbb{R}\times X$ and satisfying some functional equations such that $G(1,\cdot) = g$ and $H(1,\cdot) = h$ and each continuous solutions φ of (L) defined on X satisfies

$$\varphi(F(t,x)) = G(t,x)\varphi(x) + H(t,x). \tag{Lt}$$

We can prove that (L) has embedding with respect to F whenever it has a one parameter family of continuous solutions. We can prove moreover when embeddability is possible if the zero function is the only continuous solution of (L). Our results yield an answer (under assumptions of continuity) to the problem of L. Reich posed in 1997 on the 35-th ISFE (24. Remark).

Wojciech Jabłoński *On graph of non-affine continuous functions*

In 1970 Marek Kuczma and Roman Ger introduced a class

$$\mathcal{A}_n = \left\{ T \subset \mathbb{R}^n : \begin{array}{l} \text{every convex function } g : D \to \mathbb{R} \\ \text{where } T \subset D \subset \mathbb{R}^n,\ D \text{ is open and convex,} \\ \text{bounded from above on } T \text{ is continuous on } D \end{array} \right\}.$$

In 1973 Marek Kuczma [2] proved that for every continuous non-affine function $f : [a, b] \to \mathbb{R}$ we have $\mathrm{Gr}\, f \in \mathcal{A}_2$. This result has next been generalized to higher dimensions by Roman Ger. In a special case his result reads as follows

THEOREM G.
Let $D \neq \emptyset$ be an open and connected subset of $\mathbb{R}^n$ and let f be a non-affine real-valued function of class C^1, defined on D. Then $\mathrm{Gr}\, f \in \mathcal{A}_n$.

(Even in this special case the assumption that f is of class C^1 is necessary.)

That theorem does not contain the result proved by Marek Kuczma as a particular case, because of the regularity assumptions on f. Therefore there arises a question whether these assumptions are necessary. It appears that the assumptions on f can be weakened, and we have the following

THEOREM
Let $D \neq \emptyset$ be an open and connected subset of $\mathbb{R}^n$ and let f be non-affine continuous real-valued function defined on D. Then $\mathrm{Gr}\, f \in \mathcal{A}_n$.

[1] R. Ger, *Note on convex functions bounded on regular hypersurfaces,* Demonstratio Math. **6** (1973), 97-103.
[2] M. Kuczma, *On some set classes occurring in the theory of convex functions,* Annales Soc. Math. Pol., Comment. Math. **17** (1973), 127-135.

Witold Jarczyk *On mutual relations between Mulholland's and Tardiff's theorems*

Joint work with J. Matkowski.

We give a complete solution to the following problem posed by B. Schweizer (presented also by A. Sklar during the 37th ISFE in Huntington this year):

"Compare the assumptions imposed on $\varphi : [0, \infty) \to [0, \infty)$ in Mulholland's theorem [Proc. London Math. Soc. (2) **51** (1950), 294-307] and in Tardiff's theorem [Aequationes Math. **27** (1984), 308-316] which guarantee that φ satisfies the inequality

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all $x_1, x_2, y_1, y_2 \in [0, \infty)$."

Peter Kahlig *On the Dido functional equation*

Joint work with J. Matkowski.

By some geometrical considerations we formulate an equation which is related to the ancient isoperimetric problem of Dido. The continuous solution of this Dido functional equation depends on an arbitrary function. However, we show that in a class of functions of suitable asymptotic behavior at infinity, the Dido functional equation has a one-parameter family of "principal" solutions. Some applications are given.

Hans-Heinrich Kairies *On a Banach space automorphism and its connections to functional equation and cnd functions*

Denote by $\mathcal{H}$ the Banach space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are continuous, 1-periodic and even. It turns out that $F : \mathcal{H} \to \mathcal{H}$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

is a Banach space automorphism. Important properties of F are closely related to a de Rham type functional equation for $F[\varphi]$.

The class $F[\mathcal{H}]$ contains many continuous nowhere differentiable functions $F[\varphi]$. A large part of them can be identified by simple properties of the generating function φ.

Palaniappan Kannappan *On the stability of the generalized cosine functional equations*

This is a joint work with G.H. Kim.

We study among others the stability problem of the functional equations

$$f(x+y) + f(x-y) = 2f(x)g(y) \tag{1}$$

and

$$f(x+y) + f(x-y) = 2g(x)f(y) \tag{2}$$

for complex and vector valued functions.

Mikio Kato *Clarkson-type inequalities and their relations to type and cotype*

Joint work with Lars-Erik Persson and Yasuji Takahashi.

The celebrated Clarkson inequalities (**CI**) which might be regarded as the origin of the Banach space geometry, have been proved, originally for L_p and for various concrete Banach spaces. On the other hand, as a multi-dimensional global version of **CI**'s, "generalized Clarkson's inequality" (**GCI**) was given for L_p by M. Kato in connection with behavior of the operator norms of the Littlewood matrices. This includes Boas' and Koskela's inequalities; the former, considered in the context with uniform convexity, is the first one of this type (with two elements). **GCI** was further extended in parameters by L. Maligranda and L.-E. Persson. Also related to **GCI**, A. Tonge proved random Clarkson inequality (**RCI**).

In this talk we characterize all these inequalities by means of type and cotype in the general Banach space setting. As far as we know in literature, M. Milman first observed Clarkson's and type inequalities in the same frame work. Thus our results provide a conclusion to his observation (in an extended setting). Let X be a Banach space. Let $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

CI for L_p: *Denote* $\|\cdot\| = \|\cdot\|_p$. *Then for all* $x, y \in L_p$

$$\left(\|x+y\|^{p'} + \|x-y\|^{p'}\right)^{\frac{1}{p'}} \leqslant 2^{\frac{1}{p'}} \left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}} \quad \text{if } 1 \leqslant p \leqslant 2, \tag{1}$$

$$\left(\|x+y\|^{p} + \|x-y\|^{p}\right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}} \left(\|x\|^{p'} + \|y\|^{p'}\right)^{\frac{1}{p'}} \quad \text{if } 2 \leqslant p \leqslant \infty. \tag{2}$$

These (1) and (2) are characterized in a Banach space X as follows.

Theorem 1

(i) *X satisfies* (1) *if and only if X is of type p and 'type p constant' is 1.*

(ii) *X satisfies* (2) *if and only if X is of cotype p and 'cotype p constant' is 1.*

Let $1 \leqslant p \leqslant 2$. Then (1) for L_p and (2) for $L_{p'}$ are equivalent. Indeed, it is easily seen by duality argument that (1) holds in a Banach space X if and only if it holds in the dual space X'. Thus it is enough to treat **CI** (1) for the case $1 \leqslant p \leqslant 2$ in our discussion, which we refer to as (p, p')-*Clarkson inequality.*

Theorem 2

Let $1 \leqslant p \leqslant 2$. Then

(i) **GCI** *of Kato form, resp. of Maligranda-Persson form, holds in X, if and only if X is of type p and 'type p constant' is 1.*
In addition to this, the same implications for the dual space X' are equivalent.

(ii) **RCI** *holds in X with an 'absolute constant' K if and only if X is of type p.*

Theorem 3

Let $1 \leqslant p \leqslant 2$. **GCI** *holds in X if and only if* **RCI** *holds in $L_{p'}(X)$ with the absolute constant $K = 1$.*

Denis Khusainov *Stability of difference systems with rational right hand side*

The report describes the investigation of the stability of zero solution of difference equation systems with a rational right hand side with and without delay. The investigation uses the quadratic Lyapunov's function. The following system of difference equations is considered.

$$x(k+1) = [E + X(k)D]^{-1}[A + X(k)B]x(k), \quad k = 0, 1, \ldots \tag{1}$$

with $x(k) \in \mathbb{R}^n$, E – identity matrices, $X(k), B, D$ – block structure matrix with corresponding size. The following results were obtained.

THEOREM 1

Let A be asymptotically stable matrix (i.e. $\max_{i=\overline{1,n}} |\lambda_i(A)| < 1$. Then zero solution of system (1) is asymptotically stable. Stability region contains the sphere U_R with the radius

$$R = \frac{\sqrt{\gamma(H) + \Psi^2(H)} - \Psi(H)}{|A||D| + |B| + |D|\left[\sqrt{\gamma(H) + \Psi^2(H)} - \Psi(H)\right]} \cdot \frac{1}{\sqrt{\varphi(H)}},$$

where

$$\gamma(H) = \frac{\lambda_{\min}(H - A^T HA)}{\lambda_{\max}(H)}, \quad \Psi(H) = \frac{|HA|}{\lambda_{\max}(H)}, \quad \varphi(H) = \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)}.$$

The obtained results are extended to the system with several delays

$$x(k+1) = \left[E + \sum_{j=0}^{m} X(k-j)D_j\right]^{-1} \sum_{j=0}^{m} A_j x(k-j), \quad n = 0, 1, 2, \ldots. \tag{2}$$

Let us denote

$$\overline{A} = \sum_{j=0}^{m} \alpha_j A_j, \quad a(H) = \sum_{j=0}^{m} |A_j| \left(\alpha_j + \sqrt{\varphi(H)}\right),$$

$$d(H) = |D_0| + \sqrt{\varphi(H)} \sum_{j=0}^{n} |D_j|.$$

THEOREM 2

Let the constants $\alpha_j, j = \overline{0,m}$ and positive defined matrix H exist with the condition

$$\gamma(H) > a_2(H) + 2d(H)|\overline{A}|.$$

Then zero solution of system (2) is asymptotically stable. Stability region contains the sphere U_R with radius

$$R = \left[1 - \frac{a(H) + |\overline{A}|}{\sqrt{\gamma(H) + |\overline{A}|^2}}\right] \frac{1}{d(H)\sqrt{\varphi(H)}}.$$

Barbara Koclęga *On a generalized Cauchy equation*

This is a joint work with Professor Roman Ger.

A description of all continuous (resp. differentiable) solutions f mapping the real line $\mathbb{R}$ into a real normed linear space $(X, \|\cdot\|)$ (not necessarily strictly convex) of the functional equation

$$\|f(x+y)\| = \|f(x) + f(y)\|$$

has been presented by Peter Schöpf in [2]. Looking for more readable representations we have shown that any function f of that kind fulfilling merely very mild regularity assumptions has to be proportional to an odd isometry mapping $\mathbb{R}$ into X.

To gain a proper proof tool we have also established an improvement of Edgar Berz's [1] result on the form of Lebesgue measurable sublinear functionals on $\mathbb{R}$.

[1] E. Berz, *Sublinear functions on* $\mathbb{R}$, Aequationes Math. **12** (1975), 200-206.
[2] P. Schöpf, *Solution of* $\|f(\xi+\eta)\| = \|f(\xi) + f(\eta)\|$, Mathematica Pannonica **8/1** (1997), 117-127.

Zygfryd Kominek *On ε-convex functions*

Joint result with Bogdan Batko and Jacek Tabor.

We give a different proof of the known result of Hyers and Ulam on approximately convex functions getting somewhat better estimation. Moreover, we prove that in an arbitrary infinite-dimensional linear space this result is no longer true.

Aleksandar Krapež *Functional equations on almost quasigroups*

Quasigroups may be defined as groupoids in which all left and right translations are permutations. *Almost quasigroups* are groupoids in which some (but not all) of the translations may be constant mappings. If we additionally require that almost quasigroup has a unit, we get an *almost loop.* Similarly, associative (and commutative) almost quasigroup is an *almost group (almost Abelian group).*

A *quasizero* of a groupoid is a triple (p, q, r) of elements such that for all x, y $px = xq = r$. If $p = q = r$, the notion of the quasizero reduces to the familiar notion of zero.

A *quasigroup with quasizero* is a groupoid with quasizero (p, q, r), such that equation $xy = z$ is uniquely solvable in x for all $y \neq q$ and uniquely solvable in y for all $x \neq p$. Note that a quasigroup with quasizero is not a quasigroup.

A quasizero of a quasigroup with quasizero which has a unit, reduces to zero. Therefore we get notions of *loop with zero, group with zero* and *Abelian group with zero.* The last two are familiar from the semigroup theory, in particular the last one which is a multiplicative reduct of a field.

We have the following representation theorem:

THEOREM 1

Any almost quasigroup is either a quasigroup or a quasigroup with quasi-zero.

This result enables us to solve the two classical functional equations in the case of almost quasigroups:

THEOREM 2

If the four (six) almost quasigroup operations $A, B, C, D(, E, F)$ *satisfy the generalized associativity* (GA) *(the generalized bisymmetry* (GB)*) equation*

$$A(x, B(y, z)) = C(D(x, y), z) \tag{GA}$$

$$A(B(x, y), C(u, v)) = D(E(x, u), F(y, v)) \tag{GB}$$

then they are all isotopic to the same almost (Abelian) group.

The formulas of general solutions of these equations are also given.

In a similar way we can solve any generalized balanced functional equation on almost quasigroups.

Dorota Krassowska *A system of functional inequalities related to Cauchy's functional equation*

Joint work with J. Matkowski.

We consider the system of functional inequalities

$$f(a+x) \leqslant \alpha + f(x), \quad f(b+x) \leqslant \beta + f(x), \quad x \in \mathbb{R}.$$

Assuming the continuity of f at least at one point and some algebraic conditions on a, b, α, β, we show that every solution f of that system must be an affine function.

We also show that if the algebraic conditions are not satisfied, then the continuous solution of the system of functional equations

$$f(a+x) = \alpha + f(x), \quad f(b+x) = \beta + f(x), \quad x \in \mathbb{R}.$$

depends on an arbitrary function.

The relevant results for remaining three types of Cauchy's system of functional inequalities or equations are also considered.

Károly Lajkó *Further functional equations in the theory of conditionally specified distributions*

Let (X, Y) be an absolutely continuous bivariate random variable with support in the positive quadrant. Let us denote the joint, marginal, and conditional densities by $f_{X,Y}$, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$, respectively. We can write $f_{X,Y}$ in two ways and obtain the relationship

$$f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x) \qquad (x, y \in \mathbb{R}_+). \tag{1}$$

It is natural to inquire about all joint densities whose conditional densities satisfy

$$f_{X|Y}(x|y) = g_1(x(c_1 + c_2 y)); \quad f_{Y|X}(y|x) = g_2(y(d_1 + d_2 x)) \tag{2}$$

or

$$f_{X|Y}(x|y) = g_3\left(\frac{x - a_1 - a_2 y}{1 + cy}\right); \quad f_{Y|X}(y|x) = g_4\left(\frac{y - b_1 - b_2 x}{1 + dx}\right), \tag{3}$$

where $c_1, c_2, d_1, d_2, c, d \in \mathbb{R}_+$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$. In case (2) or (3) we have from (1) the functional equation

$$g_1(x(c_1 + c_2 y))f_Y(y) = g_2((d_1 + d_2 x)y)f_X(x) \quad (x, y \in \mathbb{R}_+) \tag{4}$$

or

$$g_3\left(\frac{x - a_1 - a_2 y}{1 + cy}\right) f_Y(y) = g_4\left(\frac{y - b_1 - b_2 x}{1 + dx}\right) f_X(x) \quad (x, y \in \mathbb{R}_+), \tag{5}$$

respectively, for functions $f_X, f_Y, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $g_3, g_4 : \mathbb{R} \to \mathbb{R}_+$. Solving these functional equations, it is possible to determine the nature of the joint distributions associated with (2) or (3).

Zbigniew Leśniak *Iterative roots of homeomorphisms of the plane*

We give a construction of iterative roots of a free mapping f of the plane. In particular, we deal with the case where f cannot be embedded in a flow.

Andrzej Mach *La solution générale de l'équation:*

$$\varphi(\varphi(\varphi(\ldots\varphi(\varphi(\alpha, x_1), x_2), \ldots), x_{n-1}), x_n) = \varphi(\alpha, x_1 \cdot \mu(x_2) \cdot \mu^2(x_3) \cdot \ldots \cdot \mu^{n-2}(x_{n-1}) \cdot x_n)$$

Dans l'équation considérée nous avons: n est un nombre naturel supérieur ou égal à deux, $\varphi : \Gamma \times G \to \Gamma$, Γ est un ensemble arbitraire non-vide, $\langle G; \cdot \rangle$ est un groupe binaire, $\mu \in \mathrm{Aut}\,(\langle G; \cdot \rangle)$ et $\mu^{n-1}(x) = x$, $x \in G$ (μ^ν dénote ν-ième itération).

Le travail [1] donne une construction générale des solutions de l'équation considérée et en conséquence donne une généralisation de la construction de l'équation de translation classique.

[1] A. Mach, *The construction of the solutions of the generalized translation equation,* submitted.

Elena N. Makhrova *Limit sets of continuous mappings of dendrites with closed periodic points set*

This is a joint work with L.S. Efremova.

In this report we consider piecewise monotone mappings of dendrites with countable ramification points set.

Let D be the class of dendrites such that for every $X \in D$ the next properties hold: (1) the ramification points set $R(X)$ is closed; (2) for any point $x \in R(X)$ the number of components of $X \setminus \{x\}$ is finite.

A continuous mapping f is called piecewise monotone if there exists a finite nonempty set $A = \{a_1, a_2, \dots, a_n\}$ such that for any component $C \subset X \setminus A$ the restriction $f|_C$ is monotone.

Let us formulate the main results.

THEOREM A.

Let f be a piecewise monotone mapping of a dendrite $X \in D$ into itself. Then the next statements are equivalent:

(A1) *the periodic points set $Per(f)$ is closed;*

(A2) $C(f) = Per(f)$, *where $C(f)$ is the center of f;*

(A3) *ω-limit set of any trajectory is a periodic orbit;*

(A4) $\Omega(f) = \cup_{z \in X} \omega(z, f) = Per(f)$, *where $\Omega(f)$ is f-nonwandering set, $\omega(z, f)$ is ω-limit set of the point z trajectory.*

COROLLARY

If f is a piecewise monotone mapping with closed set of periodic points of a dendrite $X \in D$ into itself, then the topological entropy of f equals zero.

Note that there are a dendrite $X \notin D$ and a continuous mapping $f : X \to X$ such that f has the closed set $Per(f)$ and a recurrent nonperiodic point.

THEOREM B.

For every unbounded set M of natural numbers there exist a dendrite $X \in D$ with countable ramification points set and continuous mapping $f : X \to X$ such that

(B1) *topological entropy of f equals* 0,

(B2) *the set of the least periods of f-periodic points coincides with M.*

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[1] L.S. Efremova, E.N. Makhrova, *On dynamics of monotone mappings of dendrites* (in Russian), Algebra & Analisis, (1999) (to appear).

Gyula Maksa *On a problem of Matkowski*

This work is joint with Z. Daróczy.

Let $I \subset \mathbb{R}$ be an open interval of positive length and let $CM(I)$ denote the class of all continuous and strictly monotonic real-valued functions defined on I. A function $M : I^2 \to I$ is called quasi-arithmetic mean if there exists $\phi \in CM(I)$ such that

$$M(x,y) = \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) =: A_\phi(x,y)$$

for all $x, y \in I$. J. Matkowski proposed the following problem: for which pair of functions $\phi, \psi \in CM(I)$ does the functional equation

$$A_\phi(x,y) + A_\psi(x,y) = x + y$$

hold for all $x, y \in I$.

In this talk we give a partial solution of this problem supposing comparability properties for A_ϕ and A_ψ in addition.

Lech Maligranda *The failure of the Hardy inequality and interpolation of intersections*

The main idea here is to clarify why it is sometimes incorrect to interpolate inequalities in a "formal" way. For this we consider two Hardy type inequalities, which are true for each parameter α different from 0 but they fail for the "critical" point $\alpha = 0$. This means that we cannot interpolate these inequalities between the noncritical points $\alpha = 1$ and $\alpha = -1$ and conclude that it is also true at the critical point $\alpha = 0$. Why? An accurate analysis shows that this problem is connected with the investigation of the interpolation of intersections $(N \cap L_p(w_0), N \cap L_p(w_1))$, where N is a linear space which consists of all functions with the integral equal to 0.

We calculate the K-functional for the couple $(N \cap L_p(w_0),\ N \cap L_p(w_1))$, which occurs to be essentially different from the K-functional for $(L_p(w_0), L_p(w_1))$, even for the case when $N \cap L_p(w_i)$ is dense in $L_p(w_i)$ $(i = 0, 1)$. This essential difference is the reason why the "naive" interpolation gives a wrong result.

[1] N. Krugljak, L. Maligranda, L.E. Persson, *The failure of the Hardy inequality and interpolation of intersections*, Arkiv Mat., to appear.

Janusz Matkowski *On a functional equation satisfied by pairs of exponential functions*

We prove that the functions $f, g : \mathbb{R} \to (0, \infty)$, satisfy the functional equation

$$f^{-1}[t(f(x)+f(y))] + g^{-1}(t[g(x)+g(y)]) = x + y, \quad x, y \in \mathbb{R},\ t > 0,$$

if, and only if, the function $\frac{f}{f(0)}$ is an exponential bijection, and the product fg is a constant function.

For $t = \frac{1}{2}$ this functional equation was considered recently by Z. Daróczy, Gy. Maksa, Zs. Páles, and the present author.

The solutions f, g of the above functional equation satisfy the functional equation

$$f^{-1}[tf(x) + (z-t)f(y)] + g^{-1}[(z-t)g(x) + tg(y)] = x + y$$

for all $x, y \in \mathbb{R}$, $z > 0$, $t \in (0, z)$. Specializing z and t we obtain some new functional equations. Open problems will be presented.

Janusz Morawiec *On compactly supported solutions of the two-coefficient dilation equation*

We consider the equation

$$\varphi(x) = a\varphi(2x) + b\varphi(2x-1) \tag{1}$$

and its compactly supported solutions $\varphi : \mathbb{R} \to \mathbb{R}$, where a and b are real parameters. In the present contribution we determine the sets $B_{a,b}$ and $C_{a,b}$ defined in the following way: Let $x \in [0,1]$. We say that $x \in B_{a,b}$ [resp. $x \in C_{a,b}$] if and only if the zero function is the only compactly supported solution of (1) which is bounded in a neighbourhood of x [resp. continuous at x].

Zenon Moszner *Sur les généralisations du wronskien*

Théorème

Les fonctions $f_1, \ldots, f_n$ réelles (complexes) différentiables jusqu'à l'ordre $n-1$ sur un intervalle réel I sont linéairement dépendantes sur cet intervalle si et seulement si le wronskien généralisé

$$\begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ f_1'(x_2) & \ldots & f_n'(x_2) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x_n) & \ldots & f_n^{(n-1)}(x_n) \end{vmatrix}$$

reste nul pour tous $x_1, \ldots, x_n$ dans I.

Théorème

Pour une fonction h réelle (complexe) de deux variables réelles (complexes), ayant les dérivées jusqu'à $h_{y^{n-1}x^{n-1}}$ sur $I \times J$, où I et J sont des ensembles connexes dans $\mathbb{R}$, le wronskien généralisé

$$\begin{vmatrix} h(x_1,y_1), & h_y(x_1,y_2), & \dots, & h_{y^{n-1}}(x_1,y_n) \\ h_x(x_2,y_1), & h_{yx}(x_2,y_2), & \dots, & h_{y^{n-1}x}(x_n,y_n) \\ \dots & \dots & \dots & \dots \\ h_{x^{n-1}}(x_n,y_1), & h_{yx^{n-1}}(x_n,y_2), & \dots, & h_{y^{n-1}x^{n-1}}(x_n,y_n) \end{vmatrix}$$

reste nul pour tous $x_1, \dots x_n$ *dans* I *et* $y_1, \dots y_n$ *dans* J *si et seulement si* $h(x,y) = a_1(x)b_1(y) + \dots + a_{n-1}(x)b_{n-1}(y)$, *où* $a_k : I \to \mathbb{R}$ *et* $b_k : J \to \mathbb{R}$ *pour* $k = 1, \dots, n-1$.

PROBLÈME

Est il vrai le théorème suivant: Si pour une fonction h comme dans le théorème 2 la matrice de Wronski de h

$$\begin{pmatrix} h(x,y), & h_y(x,y), & \dots, & h_{y^{n-1}}(x,y) \\ h_x(x,y), & h_{yx}(x,y), & \dots, & h_{y^{n-1}x}(x,y) \\ \dots & \dots & \dots & \dots \\ h_{x^{n-1}}(x,y), & h_{yx^{n-1}}(x,y), & \dots, & h_{y^{n-1}x^{n-1}}(x,y) \end{pmatrix}$$

a le rang égal à $p < n$ pour chaque point $(x,y) \in I \times J$, alors $h(x,y) = a_1(x)b_1(y) + \ldots + a_p(x)b_p(y)$ pour certaines $a_\nu : I \to \mathbb{R}$ et $b_\nu : J \to \mathbb{R}$ et $\nu = 1, \dots, p$?

THÉORÈME

La réponse est "oui" dans le cas reel si $p = 1$ *pour chaque* $n > 1$ *et si* $p = 2$ *et* $n = 3$ *et si* h *a les dérivées considèrées continues.*

František Neuman *Iteration groups and functional differential equations*

Iteration groups of continuous functions were studied by many authors in connection with flows, dynamical systems, fractional iterates, etc. At the beginning of the eighties the study of solutions of a system of Abel equations, or equivalently, embedding of a finite number of functions into an iteration group as its elements, was initiated by investigating functional differential equations. These questions became important when we considered transformations of functional differential equations with several deviating arguments into special, canonical equations with deviations in the form of shifts: $t + c_i$, c_i being constants (systems of Abel equations), or in the form of rays: $c_i \cdot t$ (systems of Schröder equations).

First we discovered several sufficient conditions for the existence of a solution of a system of Abel equations, then a systematic research was done by M.C. Zdun.

We explain why systems of Abel equations (or systems of Schröder equations) are important for transforming functional differential equations into their canonical forms.

Jolanta Olko *On an application of Banach–Steinhaus theorem*

Applying a set-valued version of Banach–Steinhaus theorem on the uniform boundedness, we generalize theorems concerning iteration semigroups of linear continuous set-valued functions.

Zsolt Páles *Solution and regularity theory of composite functional equations*

We deal with the functional equation

$$\begin{aligned} f(x+y) - f(x) + \phi(g(y+z) - g(y)) \\ = \psi(g(x+y+z) - g(y+z) - g(x+y) + g(y)) \\ (x, y, z > 0,\ x+y+z < \alpha), \end{aligned}$$

where $0 < \alpha \leq \infty$ and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $\phi : J \to \mathbb{R}$, $\psi : H \to \mathbb{R}$ are strictly monotonic functions defined on the sets

$$I :=]0, \alpha[, \qquad J := \{g(y+z) - g(y) \,|\, y, z > 0,\ y+z < \alpha\},$$

$$H := \{g(x+y+z) - g(y+z) - g(x+y) + g(y) \,|\, x, y, z > 0,\ x+y+z < \alpha\}.$$

The solution of the above equation is done in two steps. First, using the Bernstein–Doetsch theorem and the Lebesgue theorem on the almost everywhere differentiability of monotonic functions, we show that J, H are intervals and all the functions f, g, ϕ and ψ are everywhere differentiable. Then, after differentiation with respect to the variables x, y, z, we eliminate the parts where composite functions appear. Thus, an equation containing only f' and g' is obtained, which can be solved by using standard techniques.

Tomasz Powierża *Set-valued iterative square roots of bijections*

There are different ideas how to generalize the notion of an iterative root, especially when a function does not have such a root. We consider a multifunction as a substitute for this notion.

Following an idea of S. Łojasiewicz [Ann. Soc. Polon. Math. **24** (1951), 88-91] we show how to construct a set-valued iterative square root of a bijection which is single-valued if the function has a "real" square iterative root. We show also that every square iterative root of a bijection can be obtained using our construction.

Zbigniew Powązka *Functional equation connected with Schröder's equation*

Let a, b be positive real numbers. Let $f : [0, \infty) \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ fulfill equation

$$af(x) + bf(y) = f(ax + by)g(y - x). \tag{1}$$

There are mostly studied solutions of (1) in the class of locally integrable functions in $\mathbb{R}$.

Thomas Riedel *On some functional equations on the space of distance distribution functions*

Joint work with Kelly Wallace.

We present some lattice theoretic properties of the space of distance distribution functions which are then employed to solve various functional equations on this space. Some of the known results are reviewed and we will present new, joint work with Kelly Wallace on a Pexider type equation on the space of distance distribution functions.

Maciej Sablik *On a Chini's functional equation*

This is a report on a joint work with Thomas Riedel and Prasanna Sahoo.

In connection with some problems related to actuarial mathematics, M. Chini had considered in [1] the following equation

$$f(x+y) + f(x+z) = c\,f(x+h(y,z)), \tag{E}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions, and c is a nonzero fixed constant. Chini gave all differentiable solutions of the equation. We present the continuous solutions of (E) and of some more general equations.

[1] M. Chini, *Sopra un'equazione funzionale da cui discendono due notevoli formule di Matematica attuariale.* Periodico di Matematica **4** (1907), 264-270.

Alexander N. Sharkovsky *Asymptotical behavior of solutions of the simplest nonlinear q-difference equations*

Joint work with G.A. Derfel and E.Yu. Romanenko.

We consider nonlinear q-difference equations of the form

$$x(qt+1) = f(x(t)), \quad q > 1, \ t \in R^+.$$

The behavior of solutions is studied as $t \to +\infty$. The investigation of asymptotical properties of solutions is based, in particular, on the comparison of these with the properties of solutions of the difference equation $x(t+1) = f(x(t))$. We show that asymptotical properties of solutions of the q-difference equations are "similar" to those of the corresponding difference equations when $q > 1$ is not "very large".

Justyna Sikorska *On a functional equation related to the power means*

M.E. Kuczma in [1] has considered analytic solutions of the functional equation

$$x + g(y + f(x)) = y + g(x + f(y))$$

on the real line. In [2] solutions in the class of twice differentiable functions are given.

We present solutions in other classes of functions.

[1] M.E. Kuczma, *On the mutual noncompatibility of homogeneous analytic non-power means.* Aequationes Math. **45** (1993), 300-321.
[2] J. Sikorska, *Differentiable solutions of a functional equation related to the non-power means.* Aequationes Math. **55** (1998), 146-152.

Stanisław Siudut *Cauchy difference operator in some F^*-spaces*

Some abstract stability theorems with applications are presented. In particular, a necessary and sufficient condition of stability of the Cauchy equation in certain class of F^*-spaces is proved.

Fulvia Skof *On some mutually equivalent alternative quadratic equations*

The search of connections between the classes of solutions to different alternative equations stemming from the quadratic equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0$$

for operators f with values in a real normed space, points out a variety of situations in dependence on some peculiarities of the norm of the space (pre-Hilbert, strictly convex etc.). More specially, we consider here the property of pairwise equivalence of such equations, with special regard to the case that equivalence occurs if and only if the target space is endowed with a suitable norm.

Some remarks in this context, giving rise to new characterizations for the kind of norm involved, are presented in this talk.

Andrzej Smajdor *Concave iteration semigroups of linear set-valued functions and differential problems*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the set of all nonempty compact convex subsets of K. Suppose that $\{A^t : t \geqslant 0\}$ is a concave iteration semigroup of continuous linear functions $A^t : K \to \mathrm{cc}(K)$ such that $A^0(x) = \{x\}$. Then there exists a continuous linear set-valued function G such that

$$D_t A^t(x) = A^t(G(x)),$$

where D_t denotes the Hukuhara derivative of $A^t(x)$ with respect to t.

An existence and uniqueness theorem for the differential problem

$$D_t\Psi(t,x) = \Psi(t, G(x)),$$
$$\Psi(0,x) = \Psi_0(x)$$

is given.

Wilhelmina Smajdor *Entire solutions of a functional equation*

Joint work with Andrzej Smajdor.

All entire solutions of order less than 4 of the equation

$$|f(s+it)f(s-it)| = |f(s)^2 - f(it)^2|, \quad s, t \in \mathbb{R}$$

are

$$f(z) = az \ \text{ and } \ f(z) = a \sin bz,$$

where a, b are arbitrary complex constants.

Tomasz Szostok *Equation of Jensen type and orthogonal additivity in normed spaces*

A conditional functional inequality that is often considered in papers by authors dealing with Orlicz spaces is studied. Namely, under some assumptions on the arguments, the right-hand side of the Jensen inequality is multiplied by a constant. Related equation is considered. For functions defined on $(0, \infty)$ solutions of this equation are expressed in terms of multiplicative functions. After suitable modifications the same equation can be considered in normed spaces. Close connection between the resulting equation and that of orthogonal additivity is obtained.

László Székelyhidi *On a functional equation for a two-variable function*

This is a joint work with Prasanna Sahoo.

In this work we prove that the function $f : G \times G \to \mathbb{C}$, where G is a 2-divisible abelian group, satisfies the functional equation

$$f(x-t,y)+f(x+t,y+t)+f(x,y-t) = f(x-t,y-t)+f(x,y+t)+f(x+t,y)$$

for all x, y, t in G if and only if

$$f(x,y) = B(x,y) + \varphi(x) + \psi(y) + \chi(x-y),$$

where $B : G \times G \to \mathbb{C}$ is a biadditive function and $\varphi, \psi, \chi : G \to \mathbb{C}$ are arbitrary functions.

Jaromír Šimša *Some finite decompositions of three-place functions*

In the early 80's, F. Neuman found and stated a general criterion for a given two-place function $h : X \times X \to \mathbb{R}$ (or $\mathbb{C}$) to be decomposed in the form

$$h(x,y) = \sum_{i=1}^{m} a_i(x) b_i(y) \quad (x \in X, \ y \in Y) \tag{1}$$

(see e.g. the book Th.M. Rassias and J.Š. *Finite sums decompositions in mathematical analysis,* John Wiley and Sons, 1995.) Later on, M. Čadek and J.Š. showed that a tree-place function h can be represented as

$$h(x,y,z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \alpha_{ijk} a_i(x) b_j(y) c_k(z) \quad (x \in X, \ y \in Y, \ z \in Z)$$

if and only if h possesses the following three decompositions

$$h(x,y,z)=\sum_{i=1}^{m} a_i(x)u_i(y,z)=\sum_{j=1}^{n} b_j(y)v_j(x,z)=\sum_{k=1}^{p} c_k(z)w_k(x,y),$$

which are of type (1) and hence the criterion of F. Neuman applies to them.

In the present talk, we discuss decompositions of the form

$$h(x,y,z)=\sum_{i=1}^{m} a_i(x)u_i(y,z)+\sum_{j=1}^{n} b_j(y)v_j(x,z)+\sum_{k=1}^{p} c_k(z)w_k(x,y).$$

The main interest is devoted to the crucial case $m=n=p=1$.

Józef Tabor *Stability and the Chebyshev center*

We study the existence and uniqueness of the best approximate of a given function in classes of solutions of the Cauchy type functional equations. The notion of the Chebyshev center is applied to get the results.

Maryna B. Vereykina *Dynamics of solutions of a class of nonlinear boundary value problems*

We consider the simple boundary value problem, namely, the system of two equations with one spatial variable

$$\begin{aligned} \frac{\partial u}{\partial t} &= a\frac{\partial u}{\partial x}+b_1 u, \\ \frac{\partial v}{\partial t} &= -a\frac{\partial v}{\partial x}+b_2 v \end{aligned} \tag{1}$$

where $x\in[0,1]$, $t\in\mathbb{R}^+$, for $a,b_1,b_2\in\mathbb{R}$, with nonlinear boundary conditions

$$\begin{aligned} u|_{x=0} &= v|_{x=0}, \\ u|_{x=1} &= f(v(t))|_{x=1}, \quad t\in\mathbb{R}^+ \end{aligned} \tag{2}$$

and with the initial conditions

$$\begin{aligned} u|_{t=0} &= u_0(x), \\ v|_{t=0} &= v_0(x), \quad x\in[0,1] \end{aligned} \tag{3}$$

and assume that $a>0$ and f is a nonlinear function.

The solutions of the BVP (1) – (3) are represented as solutions of difference equations with continuous arguments

$$w(\tau+2)=e^{\frac{b_1+b_2}{a}} f(w(\tau)) \tag{4}$$

with initial conditions

$$w|_{\tau\in[-1,1)} = \begin{cases} v_0(-\tau)\cdot e^{\frac{b_2}{a}(\tau+1)} & \text{for } \tau\in[-1,0), \\ u_0(\tau)\cdot e^{\frac{b_1\tau+b_2}{a}} & \text{for } \tau\in[0,1). \end{cases}$$

Peter Volkmann *On a Cauchy equation in norm*

Jointly with Roman Ger we investigate the equation

$$\|f(x+y)\| = \|f(x)f(y)\| \tag{C}$$

for functions $f:\mathbb{R}\to C(K)$, K being compact, $K\neq\emptyset$. We have the theorem: Let $f:\mathbb{R}\to C(K)$ solve the inequalities

$$\|f(x)\|\cdot\|f(-x)\|\leqslant 1 \tag{I_1}$$

and (for $n\geqslant 2$)

$$\|f(x_1+\dots+x_n)\|\leqslant\|f(x_1)\cdots f(x_n)\|. \tag{I_n}$$

Then there are $\tau\in K$ and $\varphi:\mathbb{R}\to\mathbb{R}$, such that $\varphi(x+y)=\varphi(x)\varphi(y)$ and $\|f(x)\| = |f(x)(\tau)| = \varphi(x)$ (for $x,y\in\mathbb{R}$). The inequalities (I_1), (I_2) imply (C), and it is an interesting question, whether the theorem holds, when only requiring these two inequalities for f.

Anna Wach-Michalik *On special convex compositions with Euler's Gamma function*

Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a function satisfying the following properties:

$$\forall\, x\in\mathbb{R}_+ : f(x+1) = xf(x) \quad\text{and}\quad f(1)=1. \tag{$*$}$$

Thus $f(x)=p(x)\Gamma(x)$, where $p:\mathbb{R}_+\to\mathbb{R}_+$ is a periodic function of period 1 and $p(1)=1$ and Γ is the Euler Γ-function defined by the formula

$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}\,. \tag{Γ}$$

Prof. H. H. Kairies proposed to investigate the following set:

$$M = \left\{ g:\mathbb{R}_+\to\mathbb{R}_+ : \begin{array}{l} \text{if } f:\mathbb{R}_+\to\mathbb{R}_+ \text{ satisfies } (*) \\ \text{and } g\circ f \text{ is convex, then } f=\Gamma \end{array} \right\}.$$

By Bohr-Mollerup's theorem we know that $\log\in M$. We find some further elements of the set M and study its properties.

Janusz Walorski *On a problem connected with convexity of derivatives*

The aim of the talk is to present an answer to the question posed by Milan Merkle in [*Conditions for convexity of the derivative and some applications to the Gamma function,* Aequationes Math. **55** (1998), 273-280.]

Problems and Remarks

1. Remark. Let I be a real interval and f be a homeomorphisms mapping I onto I.

During the 6th International Conference on Functional Equations and Inequalities (Muszyna-Złockie, 1997) I proved that

If f has no fixed points, then it can be represented as a composition of at most 2 continuous involutions.

Now I can prove essentialy more:

If f is increasing [decreasing], then it can be represented as a composition of at most 4 [at most 3] continuous involutions.

The functions $(0,1) \ni x \mapsto x^2$, $(0,\infty) \ni x \mapsto x^2$, and $(0,1) \ni x \mapsto 1-x^2$ serve as examples showing that the numbers 2, 4, and 3 are the best possible here.

Witold Jarczyk

2. Remark and Problem. Inscribe a convex n-gon ($n \geqslant 3$) in the unit circle. Now, by drawing tangents, you get a circumscribed n-gon to the circle. László Fuchs and I proved fifty years ago (Compos. Math. **8** (1950), 61-67) that the sum of the areas of these two n-gons have the minimum 6, independent of n, realized by a pair of squares. The proof was analytical, using a function which is first strictly concave then strictly convex.

No elementary (not using calculus, say geometrical) proof has been found since. It seems very difficult to find one.

Clearly no minimal pair of n-gons exists for $n > 4$ because one can always slightly distort squares by joining (several) small additional sides.

Pál Erdös asked several years ago whether among pairs of triangles ("3-gons") constructed as above the pair of regular (equilateral) triangles has the minimal area-sum. I proved, by analytic tools similar to those used for the 1950 theorem, that this is true. I believe for this an elementary proof (geometric or at least without derivation) would be relatively easy to find.

János Aczél

3. Problem. Let $f :]0,\infty[\to \mathbb{R}$. If f is Jensen-convex, i.e.

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \qquad (x,y>0), \tag{1}$$

then the inequality

$$f\left(\frac{x+y+z}{3}\right) \leqslant \frac{f(x)+f(y)+f(z)}{3} \qquad (x,y,z>0)$$

also holds. Therefore, with $z = \sqrt{xy}$,

$$f\left(\frac{x+y+\sqrt{xy}}{3}\right) \leqslant \frac{f(x)+f(y)+f(\sqrt{xy})}{3} \qquad (x,y>0). \tag{2}$$

If f is continuous, then we can prove that (2) implies (1) as well. Is it true that (1) follows from (2) without any regularity assumptions?

Zoltán Daróczy and Zsolt Páles

4. Remark. In a recent issue of the Bulletin of the London Mathematical Society Braden and Byatt-Smith [1] have considered the equation

$$\begin{vmatrix} 1, & 1, & 1 \\ f(x), & f(y), & f(z) \\ f'(x), & f'(y), & f'(z) \end{vmatrix} = 0; \qquad x+y+z=0.$$

Solutions include $f(x) = x$, $f(x) = \exp x$, and $f(x) = p(x)$ where $p(x)$ is the Weierstrass pe function.

This equation arises from an equation of Sutherland (1974)

$$F(x)F(y)+F(y)F(z)+F(z)F(x) = G(x)+G(y)+G(z)$$

subject to $x+y+z=0$.

This was solved by Calogero (1970) at least in the physical situation that gave rise to Sutherland's equation.

[1] H.W. Braden, J.G.B. Byatt-Smith, *On a functional differential equation of determinental type*, Bull. London Math. Soc. **31** (1999), 463-470.

Thomas M.K. Davison

5. Problem. (Presented by János Aczél.)
What is known about the system of equations

$$F(x,x) = x, \quad F[F(x,y),z] = F(x,z) \qquad (x,y,z \in \mathbb{R};\ F:\mathbb{R}^2 \to \mathbb{R})?$$

I can prove that if $y \mapsto F(x,y)$ is differentiable, then $F(x,y) \equiv x$ and $F(x,y) \equiv y$ are the only solutions. The original question is also of interest in (general) vector spaces.

Günter Pickert (Giessen)

6. Remark. *Remarque au problème de G. Pickert.*
La théorème suivant est démontré dans la note [L. Piechowicz, S. Serafin, *Solution of the translation equation on some structures*, Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace Mat. **21** (1979), 109-114]:

Theorem

A mapping $F : M \times S \to M$ *is a solution of* $F(F(x,a),b) = F(x,b)$ *if and only if it is constructed as follows*

a) *We take a partition* $(M_i)_{i\in I}$ *of* M.

b) *We denote by* $\mathcal{F}$ *the set of all functions* $f : M \to M$ *such that*

$$\bigwedge_{i\in I} (f(M_i) \subset M_i \ \wedge \ \operatorname{card} f(M_i) = 1). \qquad (*)$$

c) *We take an arbitrary function* $\varphi : S \to \mathcal{F}$.

d) *We define* $F(x,a) := (\varphi(a))(x)$ *for* $(x,a) \in M \times S$.

Nous avons des relations suivantes:
$(*)$ $\iff$ f est stable sur chaque M_i et sa valeur sur M_i est dans M_i $\implies$ f est l'idéntité sur l'ensemble de ses valeurs $\iff$ $\bigwedge_{x\in M} f(f(x)) = f(x)$.
Passons au cas $M = S = \mathbb{R}$. Nous avons des équivalences suivantes:

$$F(a,a) = a \iff (\varphi(a))(a) = a \iff a \in \varphi(a)(\mathbb{R}).$$

La solution générale du problème de G. Pickert est donnée par la construction dans le théorème avec $\varphi(a)$ remplissante la condition: $\bigwedge_{a\in\mathbb{R}} a \in \varphi(a)(\mathbb{R})$.

EXEMPLES

$$\left.\begin{array}{ll} 1) & \bigwedge_{a,x\in\mathbb{R}} \varphi(a)(x) = x, \text{ donc } F(x,a) = x \\ 2) & \bigwedge_{a,x\in\mathbb{R}} \varphi(a)(x) = a, \text{ donc } F(x,a) = a \end{array}\right\} \text{les exemples de Pickert.}$$

3)

$$\varphi(a)(x) = \left\{\begin{array}{l} a \text{ pour } x \in [[a],[a]+1) \\ k \text{ pour } x \in [k,k+1) \text{ et } k \neq [a] \text{ et } k = 0, \pm 1, \dots \end{array}\right\} = F(x,a).$$

Zenon Moszner

7. Problem. (Presented by János Aczél).
What is the general solution of the integral equation (somewhat similar to the "integrated Cauchy equation")

$$2f(u) = 2\int_0^\infty f(x+u)f(x)\,dx \quad \text{(if the integral exists).}$$

$f(\lambda) = \lambda e^{-\lambda x}$ is a solution. There are applications in Statistics.

József Bukszár (Miskolc)

8. Problem. (Presented by János Aczél).
1. After the "fermatian" statement "I do not think it right to occupy space by a very full development of the demonstration: the following will be

enough for anyone who has an ordinary acquaintance with functional algebra and the differential calculus", A. De Morgan states in a paper that (with slightly changed notation)

$$\varphi(x+u)+\varphi(y+u)=\varphi[z(x,y)+u] \qquad (x,y,u \text{ and } \varphi \text{ nonnegative}) \tag{1}$$

implies that there exist nonnegative functions c and F such that

$$\varphi(x+u)=c(x)F(u). \tag{2}$$

Ingram Olkin and I were unable to fill in what was missing here. (There are applications to actuarial mathematics, among others.)

2. Could the injectivity assumption be weakened?

Albert W. Marschall (Lumnisland, WA, UBC, Canada)

9. Remark. *To A.W. Marschall's Problem 1.*
If φ is an injection, then we get from (1) with $u=0$: $z(x,y)=\varphi^{-1}[\varphi(x)+\varphi(y)]$. Putting this into (1) and writing

$$\varphi_u(x):=\varphi(x+u), \quad s=\varphi(x), \quad t=\varphi(y) \tag{3}$$

we get

$$\varphi_u\circ\varphi^{-1}(s+t)=\varphi_u\circ\varphi^{-1}(s)+\varphi_u\circ\varphi^{-1}(t) \qquad (s,t\in[\varphi(0),\lim_{x\to\infty}\varphi(x)[\,),$$

the Cauchy equation on a domain from which it can be extended to $[0,\infty[^2$ (or to $\mathbb{R}^2$). Since $\varphi_u\geqslant 0$ (cf. (3)): $\varphi_u\circ\varphi^{-1}(s)=c_u s$; thus $\varphi(x+u)=\varphi_u(x)=c_u\varphi(x)=c(u)\varphi(x)$. This proves (2), moreover $\varphi(x)=be^{A(x)}$ $(b\geqslant 0)$. Here A is an arbitrary injective additive function. This is not enought to guarantee $A(x)=ax$, thus $\varphi(x)=be^{ax}$ but local boundedness (on a small proper interval or on a set of positive measure) of φ is enough.

János Aczél

10. Remark. The following generalization of the regularity result presented in Zs. Páles' talk holds.
Consider the functional equation

$$\begin{aligned} f(x+y)-f(x)+\sum_{i=1}^{n}\phi_i[g_i(y+z_i)-g_i(y)] \\ =\sum_{i=1}^{n}\psi_i[g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y)] \\ (x,y,z_i>0,\ x+y+z_i<\alpha;\ i=1,\dots,n), \end{aligned} \tag{1}$$

where $0<\alpha\leqslant\infty$, $f,g_i:I\to\mathbb{R}$, $\phi_i:J_i\to\mathbb{R}$, $\psi_i:H_i\to\mathbb{R}$ and

$$I=(0,\alpha),$$

$$J_i = \{g_i(y+z_i) - g_i(y) \mid y, z_i > 0,\ y+z_i < \alpha\},$$
$$H_i = \{g_i(y+x+z_i) - g_i(y+x) - g_i(y+z_i) + g_i(y) \mid x, y, z_i > 0,\ y+x+z_i < \alpha\},$$

for $i = 1, \dots, n$. Suppose, that the functions $\phi_1, \dots, \phi_n$, $\psi_1, \dots, \psi_n$, $g_1, \dots, g_n$ and f satisfy (1), furthermore, $\phi_1, \dots, \phi_n$; $\psi_1, \dots, \psi_n$ and $g_1, \dots, g_n$ are strictly monotonic in the same sense, respectively.
Then

- f is strictly convex or strictly concave and continuously differentiable on I;
- $g_1, \dots, g_n$ are strictly convex or strictly concave on I;
- $J_1, \dots, J_n$ and $H_1, \dots, H_n$ are open intervals;
- $\phi_1, \dots, \phi_n$ and $\psi_1, \dots, \psi_n$ are differentiable on $J_1, \dots, J_n$ and $H_1, \dots, H_n$, respectively.

Attila Gilányi and Zsolt Páles

11. Problem. Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing convex function such that $\varphi(0) = 0$. The complementary function in the sense of Young to φ is defined by
$$\varphi^*(u) = \sup\{uv - \varphi(v) :\ v > 0\}, \quad u \geqslant 0.$$
We can describe φ^* also by the formula $\varphi^*(u) = \int_0^u p^{-1}(t)\,dt$, where p^{-1} denotes the inverse of p from the integral representation of φ, i.e. $\varphi(u) = \int_0^u p(t)\,dt$.
Consider a new function $h_\varphi : (0, \infty) \to [1, 2]$ given by
$$h_\varphi(u) = \frac{\varphi^{-1}(u)(\varphi^*)^{-1}(u)}{u}, \quad u > 0.$$
The function h_φ stems from the theory of Orlicz spaces. In fact
$$h_\varphi(u) = \frac{\|\chi_{(0,\frac{1}{u})}\|^o_{L^\varphi}}{\|\chi_{(0,\frac{1}{u})}\|_{L^\varphi}}$$
where $\|\cdot\|^o_{L^\varphi}$ denotes the Orlicz norm and $\|\cdot\|_{L^\varphi}$ the Luxemburg norm (cf. [1], [2]).
Since $\|x\|_{L^\varphi} \leqslant \|x\|^o_{L^\varphi} \leqslant 2\|x\|_{L^\varphi}$ for any x in the Orlicz space L^φ it follows that $1 \leqslant h_\varphi(u) \leqslant 2$ for all $u > 0$.
Notice also that if $\varphi(u) = u^p$, $1 \leqslant p < \infty$, then $h_\varphi(u) = p^{\frac{1}{p}}(p')^{\frac{1}{p'}}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. It is easy to show that if $h_\varphi(u) = 2$ for all $u > 0$, then $\varphi(u) = cu^2$ for some $c > 0$ (cf. [2]). I have a rather complicated proof that if
$$h_\varphi(u) = a \quad (1 < a < 2) \quad \text{for all } u > 0$$
then

$$\varphi(u) = cu^p$$

for some $c > 0$ and $p > 1$.

My question is: to find a simple proof of the last statement.

[1] M.A. Krasnoselskii, Ya.B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff Groningen 1961.
[2] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. **5**, Univ. of Campinas, Campinas SP, Brazil. 1989.

Lech Maligranda

12. Problem. Let $\varphi_n, \psi_n, \chi_n : \mathbb{Z} \to \mathbb{C}$ be functions, $B_n : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ biadditive functions, and $F_n : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ $2\mathbb{Z}$-periodic functions in both variable, that is

$$F_n(x + 2z, y) = F_n(x, y + 2z)$$

is satisfied for all x, y, z in $\mathbb{Z}$. let

$$f_n(x, y) = \varphi_n(x) + \psi_n(y) + \chi_n(x - y) + B_n(x, y) + F_n(x, y)$$

for all x, y in $\mathbb{Z}$ and for $n = 1, 2, \ldots$. Suppose that the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{Z}$ to the limit f. Is it true, that f has the form

$$f(x, y) = \varphi(x) + \psi(y) + \chi(x - y) + B(x, y) + F(x, y)$$

where $\varphi, \psi, \chi : \mathbb{Z} \to \mathbb{C}$ are arbitrary, $B : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is biadditive and F is $2\mathbb{Z}$-periodic in the above sense?

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13. Remark. *On A. W. Marschall's problem.*
The equation $\varphi(x+u)+\varphi(y+u) = \varphi[z(x, y)+u]$ is Chini's equation (cf. [Talk by M. Sablik, p. 187], [2]). Using [3], it suffices to assume that z is continuous in each variable and φ is locally bounded (either from above or below) and non constant to obtain the continuity of φ. This allows us to use [4] to reduce the equation to

$$\varphi(x + z) = M(x)\varphi(z) + P(x) \qquad (x, z \geqslant 0)$$

and using $z = 0$ and $g(x) = \varphi(x) - \varphi(0)$ we obtain

$$g(x + z) = M(x)g(z) + g(x) \qquad (x, z \geqslant 0).$$

By Corollary 2 in Chapter 15 of [1], we obtain, after the elimination of extraneous solutions, that

$$\varphi(x) = Ke^{ax} \quad \text{and} \quad z(x, y) = \frac{1}{a}\ln(e^{ax} + e^{ay}) \quad \text{for } K > 0,\ a \neq 0.$$

We further note that with the direct methods presented in M. Sablik's talk, it suffices to assume that z is continuous in one variable, but we then need to assume the continuity of φ.

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14. Problem. Characterize

$$\mathcal{F} := \{p \in \mathbb{Z}[X] :\ \forall\, x \in \mathbb{R}\ \ |p(x)| \leqslant 1 \ \Leftrightarrow\ |x| \leqslant 1\}.$$

Comments. $X^n, T_n(X) \in \mathcal{F}$. If $p \in \mathcal{F}$, then $-p \in \mathcal{F}$. If $p, g \in \mathcal{F}$ then $p \circ q \in \mathcal{F}$. But there are more than these: $kX^4 - kX^2 + 1 \in \mathcal{F}$ for $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Thomas M.K. Davison

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