František Kuřina
Hradec Kralove, Czechoslovakia

The Didactics of Mathematics and the Mathematics Teaching Practice

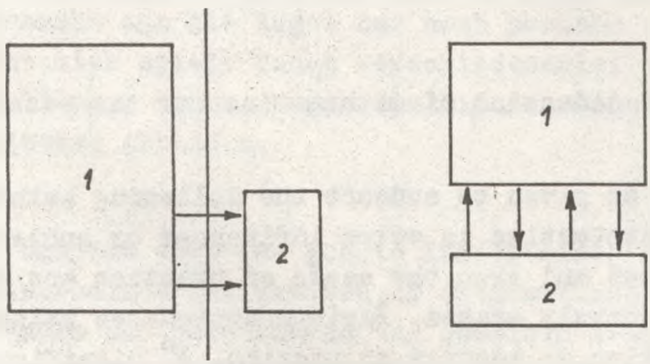
My contribution is based on my personal didactical experience and study of literature and does not represent any official standpoint of any institution. It consists of three parts. In the first part I deal with factors that influence the teaching of mathematics; in the second I formulate some didactical questions; and in the third I refer to the nineth Freudenthal's problem of mathematics education.

1. Significance of didactics of mathematics for the teaching practice

Examples can be given to support the following opinions: 1° didactics of mathematics is often influenced by subjective points of views and even the needs of practice are not sufficiently objectively stated, various approaches being conveyed by tradition or adopted by fashion, 2° significance of changes of the contents of teaching is overestimated and the real conditions under which the teaching takes place are not sufficiently respected.

What are the causes of this situation? One of the reasons is sure to be the fact that didactics of mathematics does not so far give sufficiently deep and objectively proved results which could be used for the practice of teaching.

Didactics of mathematics has been written and spoken of for a lot of years and there is quite a number of institutions connected with it: the world didactical production is quite rich. Yet we witness decisions supported rather by experience and empirical ideas than by a theory when theoretical or practical questions connected with the teaching of mathematics are dealt with. In this situation an extraordinary influence is brought about by personalities who connect the theoretical approaches with life experience, mathematical points of view with the opinions of a teacher, theory with practice. One of such personalities is undoubtedly Professor Anna Sophia Krygowska, who has contributed a lot to the constituting of the didactics of mathematics as an independent though adjacent scientific discipline.



1 SCHOOL PRACTICE

2 DIDACTICS OF MATHEMATICS

Fig. 1

Professor Krygowska in her work "Kierunki badań dydaktyki matematyki" stresses that didactics of mathematics deals with all questions connected with the teaching and learning of mathematics. This also includes questions of the practice of teaching. I should like to draw your attention to the fact that our J.A. Komenský in his "Analytical Didactics", published in the Polish town of Leszno in 1648, characterized didactics as a theory of correct teaching. According to him the theory of teaching is a system of methods of teaching and learning, which mastered will bring the pupils to thorough knowledge quickly, with a zest. So conceived, the theory of education is also connected with the teaching practice.

Although didactics gives to many questions no satisfying answer, it is the practice that must react to them as they are connected with the development of the society. Thus school practice becomes a large experimental field of work, unfortunately too large to allow for an evaluation of all its results in a proper way. Both experience of the past and that of the present practice of teaching mathematics in all types of schools do not seem to be made use of effectively. Therefore it is necessary to establish close cooperation of theoreticians and school teachers.

Hardly anywhere in the world solving questions of mathematics teaching seems to have been approached on the ground of a deep analysis of the causes of failures of the existing instruction. An analysis of the teaching process in the aspects of its effects and achieved aims should be the natural presumption for school reforms. The reforms built on naive premise that the new will be better appear not to be successful.

2. Some didactical problems

Characteristic feature of didactics of mathematics is, as Prof. Freudenthal has stressed at ICME 4, that didactical questions are not isolated but coherent. Solving one of them often conditions solving others, all of them being connected either directly or through their consequences for the practice.

The practice of teaching mathematics always demands an answer to the following questions:

What to teach at the basic school?

What to teach at the secondary school?

What to teach at Colleges and Universities?

How to teach?

What part does the experience of pupils play in the learning process?

The answers to questions of WHAT and HOW to teach are usually given by cultural tradition and needs of the society. Undoubtedly the contents and methods of teaching were set up rather on the ground of past experience and apparently not in an optimal way. Didactics of mathematics will become a scientific discipline only if it determines appropriate contents and methods anticipating the development of the society, anticipating its needs. Today we can hardly give qualified answers to the basic questions which would serve as a foundation for reshaping the contents of teaching and the methods of teacher's work in a class:

How is the school mathematics influenced by the needs of the society?

What part should be the role of the knowledge of mathematics, of its procedures and notions in the mental development of a child?

The following question is also significant for the practice:

How to organize instruction?

Here belong the problems of organizing working groups of pupils, their size, structure and stability. The possible answers are strongly influenced by tradition and by the economical potentials of the society.

Further let us mention the questions of organization of education: Education can be organized in either many separated subjects or in integrated units. The former is a consequence of historical development of sciences, the latter corresponds to the new trends influenced by interdisciplinary approaches to applications.

Each of possible approaches can be supported by a number of arguments. None of them seems to be proved reliably to be more effective. The trouble is that a reliable evaluation of results is long-termed and must be verified in practice and not only in laboratory conditions.

Last but not least, the questions of organization of pupils' work in single lessons belong to this thematic field. The best possible conception of teaching, elaborate conceptual structure of the discipline, system of knowledge,... all this is ineffective if the teacher fails to make the pupils work actively. The class-room practice is after all the criterion of successful handling of certain topic units. According to our experience, it is important to set up the succession of questions we are going to answer together with pupils. These questions must be understandable and should be possibly attractive for them, arousing curiosity, and motivating. So should be the answers to the questions, too.

The questions can be motivated by learning some phenomena of pupils' environment - these are analogies of problems formulated by science; but they can also be questions connected directly with the learning process, i.e. with pupils' inquisitiveness, with their effort to achieve certain aims.

The most frequent type of questions in instruction are the HOW-questions. They correspond most naturally to the active approach to mathematics, they are connected with algorithmic procedures, which usually play the leading role in learning, even though the closed scientific discipline puts in foreground the questions of existence (WHETHER-questions) and those of evidence (WHY-questions).

It may be the consequence of tradition that the most frequent type of school mathematics exercises corresponds to the WHICH-questions. These are in fact classical exer-

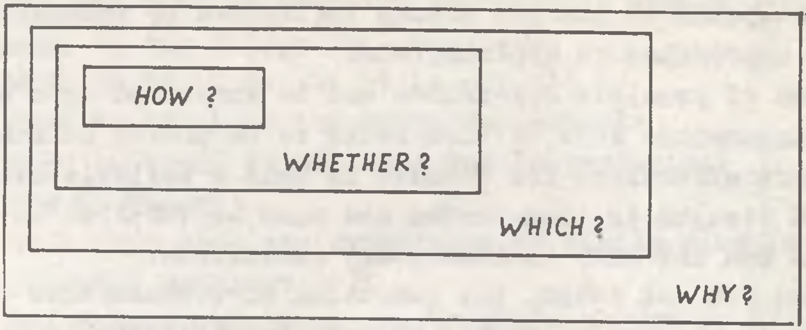


Fig. 2

cises called usually "problems of determining some sets" (solving equations, solving inequalities, constructional problems). The answers to the WHICH-questions require knowing or discovering the answers to one or a whole complex of HOW-questions. The HOW-questions emphasizing the constructional principles were neglected both in the past and in the most contemporary textbooks. These problems can be found at various levels of teaching mathematics. E.g. the authors of [3] point out the fact that the substance of troubles with mathematical induction lies in the incapability to construct the n-th (or (n+1)-th) member of the sequence, i.e., in the typical incapability to answer the HOW-question (How will the sequence continue?). At this example we can realize that the HOW-questions are often the foundation for finding the answer to the WHETHER-questions (Say whether it is valid for all natural numbers), to the WHICH-questions (For which numbers is it valid?), and to the WHY-questions (proof by means of mathematical induction).

Suitable HOW-type problems can contribute to the development of creative thinking (How else can it look like? Are all the possibilities exhausted?...). They can also prepare understanding of mathematical notions.

Let us exemplify the last two ideas by two examples from an experimental teaching of geometry to 8 and 9-years old pupils. The experiment was conducted at schools directed by the Laboratory for Didactics of Mathematics, Mathematics Institute of the Czechoslovak Academy of Sciences in Prague (details follow in the next section).

Example 1: A 9-years old pupil Marek K. was given a question:

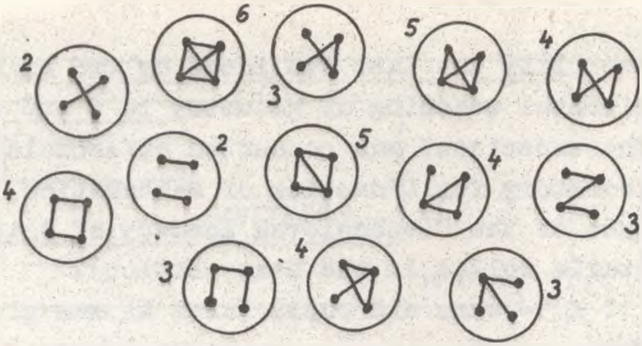
How can a button with four holes be sewn on?

An interesting discussion resulted in a natural agreement: 1. there must be a stitch through every hole, 2. it does not matter which way I look at the sewn on button. Marek gave then his answer: It is impossible to sew the button on by means of one stitch, it is possible to do so by means of two stitches. The button can be sewn on even by means of 6 stitches. The results of his investigation can be seen in fig.3.

We have enlisted this problem as a working sheet for the 2nd form. The results show that when we are working in a class with a great number of pupils it is difficult for the pupils to understand which two ways of sewing on the button can be considered to be the same; it is also difficult to discuss all the possibilities of sewing on the button.

Results obtained by speculation or by an individual pedagogical experiment always are likely to be seriously amended when implemented in current teaching.

Example 2: Pupils are made acquainted with the notion of segment by means of a succession of problems aiming at schematization of various activities, e.g., movement along the shortest path, the way of a light ray, the stretched string ... . To present the notions of ray and straight line it is suitable to use the constructional approach: we do not know what a ray is but we can imitate how it is originated by unlimited prolonging the segment. For this purpose we have set up a system of exercises such as:



How to sew on the button ?

Fig. 3

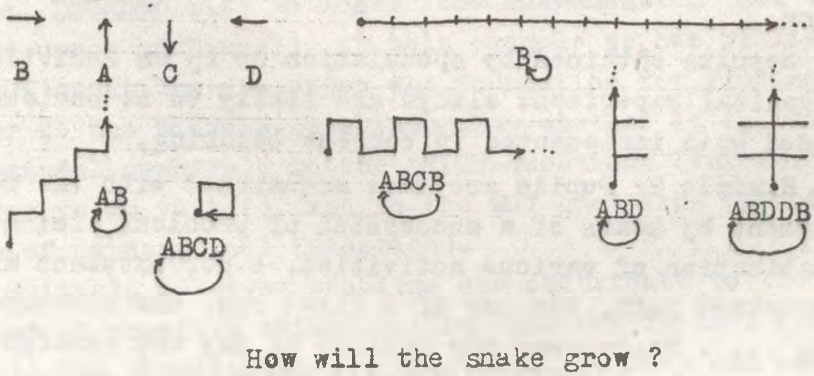
How does a snake grow?

which we give some examples of here.

Pupils work with a squared paper and draw snakes growing according to some settled laws. Figure 4 shows the growth of snakes according to the following rules:

B, AB, ABCB, ABCD, ABD, ABDDB.

The problems are interesting for pupils because interesting results are achieved. From a mathematical point of view it is a preparation for understanding a number of notions: ray, vector, inverse vector, periodicity,... Some pupils start



How will the snake grow ?

Fig. 4

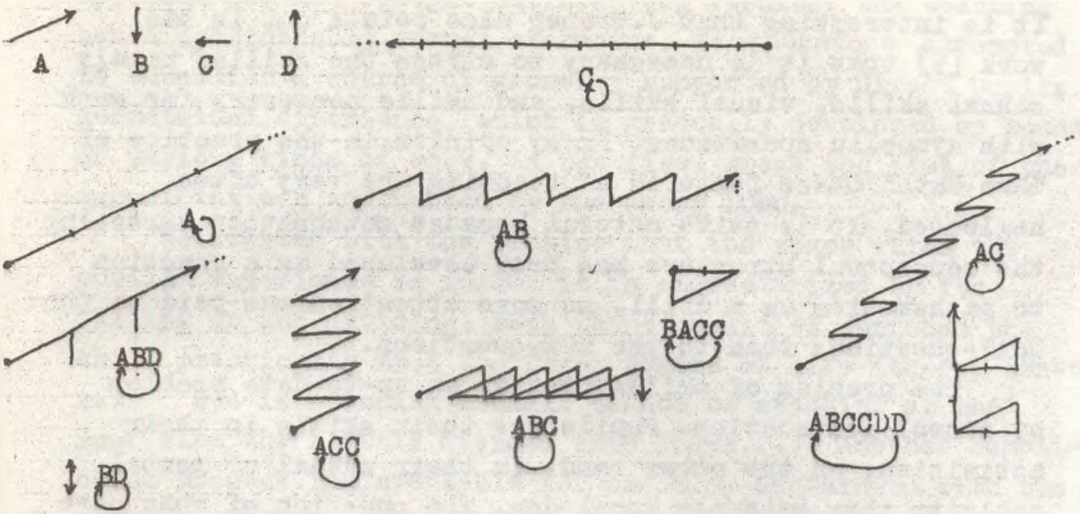


Fig. 5

to create freely symbolic "words" describing the growth of certain snakes according to given instructions. Figure 5 presents some results. The comments like "the snake is charmed away into a triangle" (BACC) or "the snake with knots" (ABCCDD) indicated that pupils were astonished by the fact that the "development had ceased" or that it went on in an unexpected way.

In my opinion, such HOW-questions are suitable for pupils 6-15 because the results of their work can usually be "seen"; moreover the problems are directly connected with a concrete action of the pupils and a stimulating competition. They provide for a variety of activities: from manual (drawing and technical drawing), through algorithmic (calculation according to rules by means of calculation technology), to mental reasoning leading to statements).

Mathematics is regarded here first of all as a method or technique, as it is stressed e.g. by Czechoslovak mathematician P. Vopěnka or Soviet mathematician A.D.Alexandrov [4].

It is interesting that J.Bruner also points out in his work [5] that it is necessary to stress the skills, namely manual skills, visual skills, and skills necessary for work with symbolic operations. In my opinion, in the practice of "New Math" these features of teaching are very often neglected. It is quite natural because mathematics stressing the conceptual structure has been developed as a reaction to mathematics as a drill, so more attention was paid to the WHAT-questions than to the HOW-questions.

The problem of skills remains an up-to-date problem of school instruction. Pupils use their skills in their activities; on the other hand, in their manual or mental activity they gain new knowledge. The question of what part can and should the experience of pupils play in the process of learning and teaching is so far an unanswered major question of the teaching practice and of didactics of mathematics.

3. An approach to geometry for pupils of age 8 - 10 years

At ICME 4 H.Freudenthal has, as you know, presented a list of main problems of mathematics teaching, the nineth of which concerned the teaching of geometry:

Is it possible to teach geometry by having the learner reflect on his spacial intuition?

I shall try to give a partial answer to this problem within the framework of this paper.

The incompleteness of my answer concerns its two aspects: 1. I shall restrict it to geometry teaching for pupils at the age of 8 - 12 only. 2. The positive answer to the given problem is supported by observation and evaluation of an experiment not yet finished.

In Czechoslovakia the course of new geometry was built within the framework of an axiomatic system with primitive notions of point and segment. Geometric figures are introduced on the ground of their set-theory definitions. This approach has not brought satisfactory results: the teaching has a considerably verbal character. We therefore attempted to establish a course of geometry supported by direct pupil's geometrical experience, which is gradually developed by means of various kinds of work. In the first stage the idea of the experiment was influenced by the group IOWO.

We started with the premise that the space which the pupils experience is formed in is characterized by its filling up and dividing. Both these qualities surround the child practically from the first months of life (there, where the bed is standing a chair cannot be standing at the same time, the flat is divided into rooms...). Another complicated process, indispensable for becoming acquainted with the space, is motion. On the basis of space and motion we stated the ranges of pupils' activities which our conception of geometry rests on. All activities to be set forth are closely connected, one conditions the other and they cannot be separated.

1. Filling up the space. The most natural children's activity leading to filling up the space is meccano. We need a meccano cube and its flat analogy - the mosaic. By means of these materials not only the geometrical fantasy of the pupils is increased but the preparation for presenting many geometrical notions is realized. This way the pupils are made acquainted with, e.g., the size of geometrical figures (the quantum of cubes for building something...), Cartesian structuralization of the space (description of the situation of a point...), and some geometric relations.

2. Dividing the space. This activity is in a certain sense inverse to the preceeding one. Just as the walls of the room divide the living space into parts, or a box divides it as well (interior - exterior), the plane is divided, e.g., by two circles, two lines etc.

3. Drawing and technical drawing. These activities are recordings of the motion of the pencil's point on the pro-

jection plane. The motion, subject to certain conditions, leads to modelling geometric figures like a segment or a circle. By motion we create or cancel a border dividing the space, not only in geometrical, but in the practical sense as well (drawing a square, opening and shutting a door). Technical drawing is a means of learning about reality.

4. Modelling. I mean, first of all, modelling with spatial or plane meccano and modelling by means of networks. The model of a geometrical shape is its picture. Pupils are gradually lead to representing points by ordered pairs of numbers.

5. Measuring. The traditional theme of school geometry is connected here with the maximum of pupils' activity. We develop simultaneously the technique of measuring as well as the operation of forming a scale.

In our experiment geometry was understood as a method of getting answers to questions arousing curiosity of pupils; answers given by means of technical drawing or modelling. It had partly a technical character. Some pupils reached astonishingly good results in it. While solving some problems it was possible to show geometry as an ample instrument for communicating a simple technical problem. Due to its vividness, geometry was a good field for pupils' experiments.

On the basis of more than 5 years of our experiment I am convinced that the teaching of geometry to pupils aged 6 - 10 years can and need be supported by their spatial experience. This experience is to be systematically developed in both the practical and the theoretical sense.

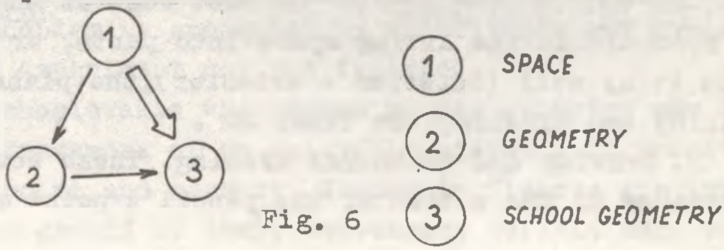


Fig. 6

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