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13

# Annales Academiae Paedagogicae Cracoviensis

## Studia Mathematica II

13

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## Studia Mathematica II

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# Contents

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Bogdan Choczewski and Zygfryd Kominek

# A proof of S. Rolewicz's conjecture

**Abstract.** S. Rolewicz's conjecture concerning solutions of the functional-differential inequality (P) is proved.

## 1. Introduction

We aim at showing the following conjecture of S. Rolewicz:

*The only even, nonnegative and differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the inequality*

$$f(t) - f(s) - f'(s)(t-s) \geq f(t-s), \quad t, s \in \mathbb{R}, \tag{P}$$

*are given by the formula* $f(t) = Ct^2$, $t \in \mathbb{R}$, $C \geq 0$.

The problem was posed by S. Rolewicz (oral comunication) as arising in his study of $\mathbf{\Phi}$-subdifferentials of a real-valued function defined on a metric space and strongly monotone multifunctions, cf. [7] and also [3] for more details.

If, moreover, $f$ is twice differentiable in a neighborhood of the origin and it satisfies an initial condition, S. Rolewicz's conjecture was confirmed by the first author, cf. [1].

## 2. Consequences of (P)

Most of the assertions that follow were essentially proved in [1]. For the sake of completeness of the argument we repeat them here with suitable comments, changes and additions.

If an even and differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a solution to (P), then it also satisfies the inequality

$$f(s) - f(t) + f'(t)(t-s) \geq f(t-s), \quad s, t \in \mathbb{R}. \tag{1}$$

(exchange $t$ with $s$ in (P)). If, moreover, $f$ is nonnegative then

$$f(0) = f'(0) = 0. \tag{2}$$

(put $t = s = 0$ in (P) to get $f(0) = 0$ and then $s = 0$ in (P) to obtain $f'(0)t \leq 0$, $t \in \mathbb{R}$, yielding $f'(0) = 0$) and the following inequalities hold true:

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$$[f'(t) - f'(s)](t - s) \geq 2f(t - s), \quad t, s \in \mathbb{R}, \tag{3}$$

(add the corresponding sides of (P) and (1)),

$$f(t) - f(s) + f'(s)(t + s) \geq f(t + s), \quad t, s \in \mathbb{R}, \tag{4}$$

(put $-s$ in place of $s$ in (P) and use the oddness of $f'$),

$$2t\, f'(t) \geq f(2t), \quad t \in \mathbb{R}, \tag{5}$$

(put $s = t$ in (4)), and

$$f'(s) \geq \frac{2}{s} f(s), \quad s > 0, \tag{6}$$

(thanks to (2), with $t = 0$ in (3)).

Now we are in position to prove

Lemma 1

*Let $f : \mathbb{R} \to \mathbb{R}$ be an even, nonnegative and differentiable function satisfying inequality* (P). *Then*

a) *$f$ is of class $C^1$ in $\mathbb{R}$,*

b) *$f$ is either the zero function or it does not vanish in $\mathbb{R} \setminus \{0\}$,*

c) *if, moreover,*

$$f(1) = 1. \tag{T}$$

*then the following inequalities are satisfied:*

$$f(x) \leq x^2, \quad x \in [-1, 1]; \qquad f(x) \geq x^2, \quad x \in \mathbb{R} \setminus [-1, 1]. \tag{7}$$

*Proof.* a) Inequality (P) and nonnegativity of $f$ imply (see (3)) that $f'$ is increasing in $\mathbb{R}$. Thus $f$ is convex, whence $f'$ is continuous (cf. M. Kuczma [5], Th. 3, p. 157).

b) Assume that $f(s_0) = 0$ for some $s_0 > 0$. Since $f'$ is increasing and $f'(0) = 0$ (cf. (2)) we infer that $f(s) = 0$ for every $s \in [0, s_0]$. Consequently $f(s) = 0$ for all $s \in [-s_0, s_0]$, because $f$ is even. Inequality (5) and oddness of $f'$ then yield $f(2t) \leq 0$ for $|t| \leq s_0$, i.e., $f(s) = 0$ for $|s| \leq 2s_0$. On repeating this argument we get $f(s) = 0$ for $|s| \leq 2^n s_0$, $n \in \mathbb{N}$, and $f$ vanishes in $\mathbb{R}$. If $s_0 < 0$, we have $f(-s_0) = 0$ as well, and the above argument works.

c) Because of (T) and b) we have $f(s) > 0$ for $s > 0$ and (6) yields

$$\frac{f'(s)}{f(s)} \geq \frac{2}{s} \quad s > 0.$$

Integrating this inequality first in the interval $[1, x]$, $x > 1$, then in $[x, 1]$, $x \in (0, 1)$, by virtue of (1) we obtain

$$f(x) \geq x^2, \quad x > 1; \qquad f(x) \leq x^2, \quad x \in (0, 1).$$

Since $f$ is even, we get inequalities (7), and the proof of the lemma is finished.

## 3. Key lemma

The following lemma is crucial for the proof of our theorem. It says that if $f$ is a solution of (P) which is not a quadratic function then $f'$ exceeds by a constant the derivative of the latter, at least over an interval contained in $\left(\frac{1}{2}, +\infty\right)$.

LEMMA 2

*Let $f : \mathbb{R} \to \mathbb{R}$ be an even, nonnegative and differentiable function satisfying inequality* (P) *and condition* (T). *Then we have the following assertions.*

a) *Either*

$$f(x) = x^2, \quad x \in \mathbb{R}, \tag{8}$$

*or there are: an $\varepsilon > 0$ and $a, b \in \mathbb{R}$, $\frac{1}{2} < a < b$, such that*

$$f'(x) > 2x + \varepsilon, \quad x \in [a, b]. \tag{9}$$

b) *If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to zero; $x_n > 0$, $n \in \mathbb{N}$; such that*

$$f'(x_n) \geq 2x_n\,, \quad n \in \mathbb{N}, \tag{10}$$

*then* (8) *holds true.*

*Proof.* a) Assume that the "or" statement is not true. Then in every interval $[a, b] \subset \left(\frac{1}{2}, \infty\right)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$f'(x_n) \leq 2x_n + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Since the sequence is bounded, we may assume that $(x_n)_{n\in\mathbb{N}}$ converges to an $x_0 \in [a, b]$.

By Lemma 1a) $f'$ is continuous. Therefore $f'(x_0) \leq 2x_0$. On the other hand, it follows from (5) and (7) that $f'(x_0) \geq \frac{1}{2x_0} f(2x_0) \geq 2x_0$. This proves that the set $A := \{x : \ f'(x) = 2x\}$ is dense in $\left(\frac{1}{2}, \infty\right)$. From the continuity of $f'$ we infer that $f'(x) = 2x$ for all $x \in \left(\frac{1}{2}, \infty\right)$, whence

$$f'(1) = 2. \tag{T'}$$

According to (4) and (7) we have

$$f(s) - f(t) + f'(t)(t + s) \geq (t + s)^2, \quad t + s \geq 1.$$

After exchanging $s$ with $t$ and summing up the corresponding sides of the two inequalities we get

$$f'(s) - 2s \geq 2t - f'(t), \quad t + s \geq 1,$$

On putting $t = 1$ here, on account of (T'), we obtain

$$f'(s) \geq 2s, \quad s \geq 0.$$

Integrating this inequality in the interval $[0,x]$, $x>0$, we have (cf. (2))

$$f(x) \geq x^2, \quad x \geq 0.$$

Define the function $g:\mathbb{R}\to\mathbb{R}$ by the formula

$$g(x) = f(x) - x^2, \quad x \in \mathbb{R}. \tag{11}$$

It is seen that $g$ is an even, nonnegative and differentiable solution of inequality (P) fulfilling the condition $g(1)=0$. By Lemma 1b) we have $g(x)=0$ for every $x\in\mathbb{R}$, which means that (8) is satisfied.

b) Define the function $g:\mathbb{R}\to\mathbb{R}$ by formula (11). This is an even, differentiable solution of inequality (P). The thing to prove is that $g$ is nonnegative. For, from Lemma 1c) we know that $f$ satisfies inequalities (7), whence

$$g(x) \leq 0 \quad \text{for } x \in [-1,1] \qquad \text{and} \qquad g(x) \geq 0 \quad \text{for } x \in \mathbb{R}\setminus[-1,1].$$

Putting $t=u+v$ and $f=g$ in (P) we have

$$g(u+v) - g(u) - g(v) \geq g'(u)v, \qquad u,v \in \mathbb{R}, \tag{12}$$

whence

$$\frac{g(v+x_n)-g(v)}{x_n} - \frac{g(x_n)}{x_n} \geq \frac{g'(x_n)}{x_n}v, \quad v \in \mathbb{R}.$$

Consequently, for any $v>0$ we get $g'(v)\geq 0$, as $g(0)=g'(0)=f'(0)=0$, cf. (2), and $g'(x_n)=f'(x_n)-2x_n\geq 0$, cf. (10). This means that $f'(x)\geq 2x$, $x\geq 0$ , whence $f(x)\geq x^2$, $x>0$, which implies that

$$g(x) \geq 0, \quad x \in \mathbb{R},$$

as $g$ is even. Moreover, by (T) and (11) we have $g(1)=0$. According to Lemma 1b) the function $g$ vanishes everywhere, and (8) follows.

## 4. Solution of (P)

Now we are in position to prove the theorem announced.

Theorem 1

*The only solutions $f:\mathbb{R}\to\mathbb{R}$ of problem (P) are given by the formula*

$$f(x) = Cx^2, \quad x \in \mathbb{R}, \tag{S}$$

*where $C$ is a nonnegative constant.*

*Proof.* Assume (T) and that (8) does not hold. By Lemma 2b) it is enough to show that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to zero, $x_n>0$, $n\in\mathbb{N}$, such that condition (10) is fulfilled. Suppose it is not the case. Then there is a $\delta>0$ such that

$$f'(x) < 2x, \qquad x \in (0,\delta). \tag{13}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be function defined by (11).

According to (12) and (13) we have

$$g(u+v) - g(u) - g(v) \geq 0, \quad u \in (0,\delta),\ v < 0.$$

Therefore

$$\frac{g(v) - g(u+v)}{u} \leq -\frac{g(u)}{u}, \quad u \in (0,\delta),\ v < 0.$$

Since (8) does not hold, by Lemma 2a) there are an $\varepsilon > 0$ and a proper interval $[a,b] \subset \left(\frac{1}{2}, \infty\right)$ such that inequality (9) is fulfilled. From (2) and (11) we infer that $g'(0) = 0 = g(0)$, so there exists a $\gamma \in (0,\delta)$ such that

$$-\frac{g(u)}{u} < \varepsilon, \quad u \in (0,\gamma).$$

Then

$$\frac{g(v) - g(u+v)}{u} < \varepsilon, \quad u \in (0,\gamma),\ v < 0. \tag{14}$$

Fix a $u \in \left(0, \min\left\{\gamma, \frac{1}{2}(b-a)\right\}\right)$ and put here $v := -(n+1)u$, with $n \in \mathbb{N}$ so chosen that $nu, (n+1)u \in [a,b]$, to get

$$\frac{g((n+1)u) - g(nu)}{u} < \varepsilon.$$

because $g$ is even. By virtue of the Mean-Value Theorem the left hand side of the above inequality equals $g'(\xi)$ for some $\xi \in [nu, (n+1)u] \subset [a,b]$. Hence on account of (9) and (14)

$$\varepsilon < f'(\xi) - 2\xi = g'(\xi) < \varepsilon,$$

a contradiction. Thus (8) holds true which is (S) with $C = 1$, as claimed.

If (T) does not hold, then we put $C := f(1)$. When $C = 0$, formula (S) follows by Lemma 2b). Assume that $C > 0$. Then the function $f^* : \mathbb{R} \to \mathbb{R}$ given by the formula

$$f^*(x) = \frac{1}{C} f(x), \quad x \in \mathbb{R},$$

is an even, nonnegative and differentiable solution of the inequality (1) satisfying the condition $f^*(1) = 1$ and our assertion follows from the first part of the proof.

## 5. Remarks

a) A shorter proof of Rolewicz's conjecture was found by R. Girgensohn when he got acquainted with our result. A joint work which contains also solutions of a Pexider type inequality and a functional equation related to (P) is to appear online in SIAM Problems & Solutions.

b) (added in proof). The paper mentioned above (and presented on the 39th International Symposium on Functional Equations, Denmark, August 2001; cf. [2]) is that by R. Girgensohn and the present authors [4]. It was to be published online together with [3] containing the formulation of the problem only. However, it was not, supposedly because of file transmission problems, so that [4] appeared for some time with the label "unsolved". This caused that prior to [4] there is published in SIAM P&S another solution of Rolewicz's problem, due to M. Renardy [6]. His proof is a shortened version of that of Theorem 1 in [4].

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# Spectra of certain operators and iterative functional equations

**Abstract.** We discuss spectral properties of the operator $F : \mathcal{D} \to F[\mathcal{D}]$, defined by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x).$$

$\mathcal{D}$ is the vector space of real functions $\varphi$ such that the sum above converges for all $x \in \mathbb{R}$. The point spectrum and the eigenspaces of $F$ and of its restriction to the vector space $\mathcal{U}$ of ultimately bounded functions are given. Moreover we compute the point spectrum and eigenspaces, the continuous spectrum and the residual spectrum of $F$, restricted to the Banach spaces $\mathcal{B}$ of bounded functions and $\mathcal{C}$ of bounded and continuous functions.

## 1. Background

We first consider the operator $F : \mathcal{D} \to F[\mathcal{D}]$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x), \tag{1}$$

where

$$\mathcal{D} = \{\varphi : \mathbb{R} \to \mathbb{R};\ F[\varphi] : \mathbb{R} \to \mathbb{R}\}.$$

So $\varphi \in \mathcal{D}$ iff the right hand side of (1) converges for every $x \in \mathbb{R}$.

There are several reasons to study this operator:

1. $F$ generates continuous nowhere differentiable functions from very simple ones, like the Takagi function $F[d]$ from $d(x) = \operatorname{dist}(x, \mathbb{Z})$ or the Weierstrass function $F[c]$ from $c(x) = \cos 2\pi x$. See [4].
2. $F$ plays a role in the stability theory of functional equations. See [1] and [2].
3. The study of operator theoretical properties of $F$ exhibits interesting connections with the theory of iterative functional equations. See [3],

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[4] and [5]. For general facts concerning iterative functional equations see the monograph [7]. For the convenience of the reader, the solutions of some iterative functional equations connected to the operator $F$ are constructed explicitely, although they could as well be deduced from [7].

The structure of $\mathcal{D}$ is investigated in [6]. In this paper we consider $F$ and its restrictions to certain subspaces of $\mathcal{D}$, namely

$\mathcal{U} := \{\varphi : \mathbb{R} \to \mathbb{R};$ there are positive numbers $M(\varphi),\ \omega(\varphi)$ such that $|x| \geq \omega(\varphi)$ implies$|\varphi(x)| \leq M(\varphi)\}$,

$\mathcal{B} := \{\varphi : \mathbb{R} \to \mathbb{R};$ there is a positive number $M(\varphi)$ such that $|\varphi(x)| \leq M(\varphi)$ for every $x \in \mathbb{R}\}$,

$\mathcal{C} := \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi$ is bounded and continuous$\}$.

The following statements are in part proved in [3]-[6].

$F_1 := F : \mathcal{D} \to F[\mathcal{D}]$ is a vector space isomorphism,
$F_2 := F|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ is a vector space automorphism,
$F_3 := F|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a Banach space automorphism,
$F_4 := F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a Banach space automorphism.

An important tool for the proofs is the following fact which connects $F$ with a first iterative functional equation.

PROPOSITION 1

a) *Assume that $\varphi \in \mathcal{D}$. Then $F[\varphi]$ satisfies the de Rham – type functional equation*

$$f(x) - \frac{1}{2}f(2x) = \varphi(x) \quad \text{for every } x \in \mathbb{R}. \tag{2}$$

b) *Assume that $\varphi \in \mathcal{D}$ is given. Then equation (2) has at most one solution $f \in \mathcal{U}$, namely $f = F[\varphi]$.*

Proposition 1 is contained in [6]. Part b) reveals the importance of the subspace $\mathcal{U}$ of ultimately bounded functions. The statement is no longer true, if we allow unbounded solutions. So far, $F_2 : \mathcal{U} \to \mathcal{U}$ has not been examined in the literature.

For $F_3$ and $F_4$ the concepts of classical spectral theory for linear continuous operators $T \in L(X, X)$ on a Banach space $X$ over $\mathbb{R}$ apply. We have the resolvent

$$\rho(T) := \{\lambda \in \mathbb{R};\ (\lambda I - T)^{-1} \in L(X, X)\}$$

$(I := \mathrm{id}_X)$ and the spectrum

$$\sigma(T) := \mathbb{R} \setminus \rho(T).$$

The spectrum can be partitioned into the point spectrum

$$\sigma_p(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ not injective}\},$$

the continuous spectrum

$$\sigma_c(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ injective, not surjective, } (\lambda I - T)(X) \ \text{ dense}\}$$

and the residual spectrum

$$\sigma_r(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ injective, not surjective, } (\lambda I - T)(X) \text{ not dense}\}.$$

Thus $\sigma_p(T)$ is the set of eigenvalues of $T$. We denote by $E\ (T, \lambda) := \{x \in X;\ Tx = \lambda x\}$ the eigenspace corresponding to $\lambda \in \sigma_p(T)$.

For $F_1$ and $F_2$ (where no topology is involved) the defining properties given above for the set $\sigma_p(F_\nu)$ of eigenvalues and the corresponding eigenspaces $E(F_\nu, \lambda)$ still make sense, whereas the continuous spectrum and the residual spectrum are not defined.

The aim of this paper is a complete description of $\sigma_p(F_\nu)$ and $E(F_\nu, \lambda)$ for $1 \leqslant \nu \leqslant 4$, which is given in Section 2 and a complete description of $\sigma_c(F_\nu)$ and $\sigma_r(F_\nu)$ for $3 \leqslant \nu \leqslant 4$, which is given in Section 3. Our systematic treatment extends some auxiliary results from [4] and [6]. In addition to (2) some other iterative functional equations will enter the scene.

## 2. Point spectra and eigenspaces of $F_\nu$

Let $F_\nu$, $1 \leqslant \nu \leqslant 4$, be defined as in Section 1 and, to unify notation, write $\mathcal{D}_\nu$ for the domain of $F_\nu$, i.e. $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_2 = \mathcal{U}$, $\mathcal{D}_3 = \mathcal{B}$, $\mathcal{D}_4 = \mathcal{C}$.

Before dealing with the individual spectra and eigenspaces we collect some facts which are true for all $F_\nu$.

Proposition 2

a) *We have* $0 \notin \sigma_p(F_\nu)$ *and* $1 \notin \sigma_p(F_\nu)$ *for* $1 \leqslant \nu \leqslant 4$.

b) *Fix* $\nu \in \{1, 2, 3, 4\}$, *and assume that* $\varphi$ *belongs to the eigenspace* $E(F_\nu, \lambda)$. *Then* $\varphi$ *satisfies the Schröder functional equation*

$$f(x) = \gamma(\lambda) f(2x) \quad \textit{for every } x \in \mathbb{R}, \tag{3}$$

*where*

$$\gamma(\lambda) := \frac{1}{2}\frac{\lambda}{\lambda - 1}. \tag{3a}$$

c) $\lambda \leqslant \frac{1}{2}$ *implies* $\lambda \notin \sigma_p(F_\nu)$ *for* $1 \leqslant \nu \leqslant 4$.

*Proof.* a) and b). Assume that $\varphi \in E(F_\nu, \lambda)$ and $\varphi \neq \mathbf{o}$ (the zero function defined on $\mathbb{R}$) for some fixed $\nu \in \{1, 2, 3, 4\}$, i.e., $F_\nu[\varphi] = \lambda\varphi$. Then $\varphi \in \mathcal{D}_\nu$ and by Proposition 1 a)

$$F_\nu[\varphi](x) - \frac{1}{2}F_\nu[\varphi](2x) = \varphi(x)$$

for every $x \in \mathbb{R}$. Hence

$$(\lambda - 1)\varphi(x) = \frac{1}{2}\lambda\varphi(2x)$$

for every $x \in \mathbb{R}$. Since $\lambda = 1$ or $\lambda = 0$ would imply $\varphi = \mathbf{o}$, assertions a) and b) are proved.

c) Assume that $\varphi \in E(F_\nu, \lambda) \setminus \{\mathbf{o}\}$ for some fixed $\nu \in \{1, 2, 3, 4\}$. Then $\varphi \in \mathcal{D}_\nu$ and therefore $F_\nu[\varphi](x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$ converges for every $x \in \mathbb{R}$. By b) we have

$$\varphi(x) = \gamma\varphi(2x) = \gamma^m\varphi(2^m x) \tag{4}$$

for every $x \in \mathbb{R}$ and $m \in \mathbb{N}$, where $\gamma = \gamma(\lambda)$. We obtain $F_\nu[\varphi](x) = \sum_{k=0}^{\infty}(2\gamma)^{-k}\varphi(x)$ and because of $\varphi \neq \mathbf{o}$, $\sum_{k=0}^{\infty}(2\gamma)^{-k}$ must converge. Therefore necessarily $\left|\frac{1}{2\gamma}\right| < 1$, so that $\left|\frac{\lambda-1}{\lambda}\right| < 1$, i.e., $\lambda > \frac{1}{2}$.

Remark

The general solution $g : \mathbb{R} \to \mathbb{R}$ of the Schröder equation (3), which will be from now on referred to by $(S_\lambda)$, is constructed as follows.

Choose any $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$ and extend it uniquely by $(S_\lambda)$ to a function $g_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

Then extend $g_1$ to a function $g : \mathbb{R} \to \mathbb{R}$ by $g(0) := 0$ in case of $\lambda \neq 2$ and by $g(0) := \alpha \in \mathbb{R}$ arbitrarily in case of $\lambda = 2$.

Proposition 2 says that the eigenvalues of $F_\nu (1 \leqslant \nu \leqslant 4)$ are contained in the set

$$J = (\frac{1}{2}, \infty) \setminus \{1\}.$$

It turns out that in fact $\sigma_p(F_1)$ is the full set $J$. The point spectra $\sigma_p(F_\nu)$ then shrink in a remarkable way according to the shrinking of the domain $\mathcal{D}_\nu$ of $F_\nu$, namely:

$$\sigma_p(F_2) = [\tfrac{2}{3}, 2] \setminus \{1\}, \quad \sigma_p(F_3) = \{\tfrac{2}{3}, 2\}, \quad \sigma_p(F_4) = \{2\}.$$

Moreover it turns out that all the eigenspaces $E(F_\nu, \lambda)$ for $\lambda \in \sigma_p(F_\nu)$ can be characterized as solution sets of equation $(S_\lambda)$ under certain constraints. They are all of infinite dimension with exactly one exception: $E(F_4, 2)$. These facts are proved in the following

Proposition 3

*We have*

a) $\sigma_p(F_1) = (1/2, \infty) \setminus \{1\}$,

b) $\sigma_p(F_2) = [2/3, 2] \setminus \{1\}$,

c) $\sigma_p(F_3) = \{2/3, 2\}$,

d) $\sigma_p(F_4) = \{2\}$.

*The eigenspaces*

$$E(F_\nu, \lambda) = \{\varphi \in F_\nu;\ \varphi \ satisfies\ (S_\lambda)\}$$

*are explicitely described (characterized) in the proof.*

*Proof.* a) By Proposition 2, for any $\lambda \in \sigma_p(F_1)$ we have $\lambda \in J$.

Now let $\lambda \in J$. Take any nonzero solution $g : \mathbb{R} \to \mathbb{R}$ of the Schröder equation $(S_\lambda)$ as described in Remark. Then $g$ satisfies (4) with $\gamma$ given by (3a), whence

$$\begin{aligned} F_1[g](x) &= \sum_{k=0}^{\infty} \frac{1}{2^k}\ g(2^k x) = \sum_{k=0}^{\infty} \frac{1}{(2\gamma)^k}\ g(x) = \frac{2\gamma}{2\gamma - 1}\ g(x) \\ &= \lambda g(x), \end{aligned}$$

which means that $\lambda$ is an eigenvalue of $F_1$. Moreover, for $\lambda \in J$

$$E(F_1, \lambda) = \{\varphi \in \mathcal{D};\ \varphi \text{ satisfies } (S_\lambda)\} = \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ satisfies } (S_\lambda)\}.$$

The construction in Remark characterizes the elements of $E(F_1, \lambda)$ and clearly $\dim E(F_1, \lambda) = \infty$ for every $\lambda \in \sigma_p(F_1)$.

b) By a), for any $\lambda \in \sigma_p(F_2)$, we have $\lambda \in J$.

Now let $\lambda \in J$ and take, as in Remark, any bounded nonzero initial function $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$. Then its extension $g_1$ by $g(2x) = \frac{1}{\gamma} g(x)$ is ultimately bounded iff $\left|\frac{1}{\gamma}\right| = \left|2\frac{\lambda - 1}{\lambda}\right| \leqslant 1$, i.e., iff $\frac{2}{3} \leqslant \lambda \leqslant 2$.

As in a), this shows that any $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ is an eigenvalue of $F_2$. All elements of $E(F_2, \lambda)$ for $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ are generated from bounded $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$, because the extension $g_1$ of an unbounded $g_0$ would remain unbounded in the vicinity of $+\infty$ or $-\infty$, hence it does not belong to $\mathcal{U}$. Note that a function $g \in E(F_2, \lambda)$ is not necessarily bounded around zero. Clearly $\dim E(F_2, \lambda) = \infty$ for every $\lambda \in \sigma_p(F_2)$.

c) By b), for any $\lambda \in \sigma_p(F_3)$, we have $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$.

Now let $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ and take, as in Remark, any bounded nonzero initial function $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$. Then its extension $g_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = \gamma g(2x)$ stays bounded iff $|\gamma| = 1$, i.e. iff $\lambda = \frac{2}{3}$ ($\gamma = -1$) or $\lambda = 2$ ($\gamma = 1$).

This shows as in a), that any $\lambda \in \{\frac{2}{3}, 2\}$ is an eigenvalue of $F_3$. As all elements of $E(F_3, \lambda)$ for $\lambda \in \{\frac{2}{3}, 2\}$ are bounded, the generating initial function has to be bounded as well.

Clearly $\dim E(F_3, \lambda) = \infty$ for $\lambda = \frac{2}{3}$ and for $\lambda = 2$.

d) By c), for any $\lambda \in \sigma_p(F_4)$, we have $\lambda \in \{\frac{2}{3}, 2\}$.

The elements of $E(F_4, \lambda)$ are the continuous and bounded solutions of $(S_\lambda)$. The only continuous solution of $(S_{\frac{2}{3}}) : f(x) = -f(2x)$ is the function $\mathbf{o}$. Therefore $\lambda = \frac{2}{3}$ is not an eigenvalue of $F_4$. The only continuous solutions of $(S_2) : f(x) = f(2x)$ are the constant functions. So $\sigma_p(F_4) = \{2\}$ and $E(F_2, 2) = \text{span}\ \{\mathbf{1}\}$, where $\mathbf{1}(x) = 1$ for every $x \in \mathbb{R}$.

## 3. The spectra of $F_3$ and $F_4$ in the Banach space setting

We start with $F_3 \in L(\mathcal{B}, \mathcal{B})$. We have seen in Proposition 3 that $\sigma_p(F_3) = \{\frac{2}{3}, 2\}$. Therefore $\frac{2}{3}I - F_3$ and $2I - F_3$ are not injective. The remaining details on the spectrum of $F_3$ are given in

THEOREM 1
*The point spectrum $\sigma_p(F_3)$ is $\{\frac{2}{3}, 2\}$. For every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, the operator $\lambda I - F_3$ is bijective. Consequently, the continuous spectrum $\sigma_c(F_3)$ and the residual spectrum $\sigma_r(F_3)$ are both empty.*

*Proof.* For $\lambda \neq \frac{2}{3}$ and $\lambda \neq 2$ the operator $\lambda I - F_3$ is injective by Proposition 3 c). It remains to show that for any given $f \in \mathcal{B}$ and $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, the operator equation

$$(\lambda I - F_3)[\varphi] = f \tag{5}$$

has a solution $\varphi \in \mathcal{B}$. To do so, first assume that (5) has a solution $\varphi \in \mathcal{B}$. Then

$$f(x) = \lambda\varphi(x) - \left\{\varphi(x) + \frac{1}{2}\ \varphi(2x) + \frac{1}{2^2}\ \varphi(2^2x) + \cdots\right\},$$
$$\frac{1}{2}\ f\ (2x) = \frac{1}{2}\lambda\varphi(2x) - \left\{\frac{1}{2}\varphi(2x) + \frac{1}{2^2}\ \varphi(2^2x) + \cdots\right\},$$

hence

$$(\lambda - 1)\varphi(x) - \frac{1}{2}\ \lambda\varphi(2x) = f(x) - \frac{1}{2}\ f(2x). \tag{6}$$

For $\lambda = 1$, we define, according to (6),

$$\Phi(x) := f(x) - 2f\left(\frac{x}{2}\right). \tag{6a}$$

A simple calculation shows that this function $\Phi$ is in fact a bounded solution of (5). Excluding from now on the case $\lambda = 1$, we write equation (6) in the equivalent form

$$\varphi(x) - \frac{\lambda}{2(\lambda - 1)}\varphi(2x) = \frac{1}{\lambda - 1}\{f(x) - \frac{1}{2}f(2x)\}$$

or

$$\varphi(x) - \gamma\varphi(2x) = g(x) \tag{7}$$

with $\gamma$ given by (3a) and

$$g(x) = \frac{1}{\lambda - 1}\left\{f(x) - \frac{1}{2}\, f(2x)\right\}.$$

Clearly $g \in \mathcal{B}$ and $|\gamma| \neq 1$ (as $\lambda \neq \frac{2}{3}$, $\lambda \neq 2$). Iteration of (7) gives

$$\varphi(x) = \gamma\varphi(2x) + g(x) = \cdots = \gamma^m\varphi(2^m x) + \sum_{k=0}^{m-1}\gamma^k g(2^k x)$$

for every $x \in \mathbb{R}$, $m \in \mathbb{N}$. Consequently, for $|\gamma| < 1$ there is at most one bounded solution $\Phi$ of (7), given by

$$\Phi(x) = \sum_{k=0}^{\infty}\gamma^k g(2^k x). \tag{8}$$

A direct substitution shows that (7) is satisfied with $\varphi = \Phi$. Going back further to (6), we see that

$$f(x) - \frac{1}{2}\, f(2x) = (\lambda - 1)\Phi(x) - \frac{1}{2}\,\lambda\Phi(2x) =: h(x).$$

As $f$ and $h$ are bounded, we have $f = F[h]$ by Proposition 1 b), hence

$$\begin{aligned} f(x) &= (\lambda - 1)F[\Phi](x) - \frac{1}{2}\,\lambda F[\Phi](2x) \\ &= (\lambda - 1)\left\{\Phi(x) + \frac{1}{2}\Phi(2x) + \frac{1}{2^2}\Phi(2^2 x) + \cdots\right\} \\ &\qquad - \frac{1}{2}\lambda\left\{\Phi(2x) + \frac{1}{2}\Phi(2^2 x) + \frac{1}{2^2}\Phi(2^3 x) + \cdots\right\} \\ &= \lambda\,\Phi(x) - F[\Phi](x). \end{aligned}$$

So in case $|\gamma| < 1$, i.e., $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, equation (5) has a solution $\Phi \in \mathcal{B}$, given by (8).

Now let $|\gamma| > 1$. We write equation (7) in the equivalent form

$$\varphi(x) = \frac{1}{\gamma}\,\varphi\left(\frac{x}{2}\right) - \frac{1}{\gamma}\,g\left(\frac{x}{2}\right). \tag{9}$$

Iteration of (9) gives

$$\varphi(x) = \left(\frac{1}{\gamma}\right)^n \varphi\left(\frac{x}{2^n}\right) - \sum_{k=1}^{n}\left(\frac{1}{\gamma}\right)^k g\left(\frac{x}{2^k}\right)$$

for every $x \in \mathbb{R}$, $n \in \mathbb{N}$. Because of $|\gamma| > 1$, there is at most one bounded solution $\Phi$ of (9), given by

$$\Phi(x) = -\sum_{k=1}^{\infty} \left(\frac{1}{\gamma}\right)^k g\left(\frac{x}{2^k}\right). \tag{10}$$

On the other hand, this function $\Phi$ satisfies equation (9):

$$\begin{aligned}\Phi(x) - \frac{1}{\gamma}\Phi\left(\frac{x}{2}\right) &= -\left\{\frac{1}{\gamma}g\left(\frac{x}{2}\right) + \frac{1}{\gamma^2}g\left(\frac{x}{2^2}\right) + \cdots\right\} \\ &\quad + \frac{1}{\gamma}\left\{\frac{1}{\gamma}g\left(\frac{x}{2^2}\right) + \frac{1}{\gamma^2}g\left(\frac{x}{2^3}\right) + \cdots\right\} \\ &= -\frac{1}{\gamma}\, g\left(\frac{x}{2}\right).\end{aligned}$$

Hence $\Phi$ satisfies equation (6) and, as in the case $|\gamma| < 1$, also equation (5).

The case $|\gamma| > 1$ corresponds to the remaining values $\lambda \in (\frac{2}{3}, 1) \cup (1, 2)$. Recall that $\lambda = 1$ has already been treated.

Finally, we discuss the spectrum of $F_4 \in L(\mathcal{C}, \mathcal{C})$.

By Proposition 3 d), the point spectrum of $F_4$ is just the singleton $\{2\}$. The eigenvalue $\frac{2}{3}$ of $F_3$ is no longer an eigenvalue of $F_4$. It turns out that $\frac{2}{3}$ belongs to the residual spectrum of $F_4$ and that the resolvent of $F_4$ coincides with the resolvent of $F_3$.

Theorem 2

*The point spectrum and the residual spectrum of $F_4$ are singletons: $\sigma_p(F_4) = \{2\}$, $\sigma_r(F_4) = \{\frac{2}{3}\}$. For every $\lambda \in \mathbb{R} \setminus \{2/3, 2\}$ the operator $\lambda I - F_4$ is bijective. Consequently $\sigma_c(F_4)$ is empty.*

*Proof.* For $\lambda \neq 2$, the operator $\lambda I - F_4$ is injective by Proposition 3 d). Next we show, that for any $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$ and for any given $f \in \mathcal{C}$, the operator equation

$$(\lambda I - F_4)[\varphi] = f \tag{11}$$

has a solution $\varphi \in \mathcal{C}$. To see this we argue exactly as in the proof of Theorem 1.

Let the value $\Phi(x)$ of the function $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by (6a) when $\lambda = 1$, by (8) when $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$ and by (10) when $\lambda \in (\frac{2}{3}, 1) \cup (1, 2)$. Then $\Phi$ is a bounded solution of equation (11), as we have seen in the proof of Theorem 1. Moreover, $\Phi$ is continuous, as $x \mapsto f(x) - 2f\left(\frac{x}{2}\right)$ is continuous and because the series (8) and (10) are uniformly convergent on $\mathbb{R}$ with continuous terms. So $\mathbb{R} \setminus \{\frac{2}{3}, 2\}$ belongs to the resolvent $\rho(F_4)$.

The only remaining case to be checked is $\lambda = \frac{2}{3}$ (not covered by the argument in the proof of Theorem 1). If equation (11) has a solution $\varphi \in \mathcal{C}$ for $\lambda = \frac{2}{3}$, then necessarily

$$\varphi(x) + \varphi(2x) = 3\left\{\frac{1}{2}f(2x) - f(x)\right\} = g(x). \tag{12}$$

This corresponds to equation (7), recall that $\lambda = \frac{2}{3}$ iff $\gamma = -1$. Note further, that $g = -3F_4^{-1}[f]$ or equivalently $f = -\frac{1}{3}F_4[g]$. Iteration of (12) gives

$$\varphi(x) = (-1)^n \varphi(2^n x) + \sum_{k=0}^{n-1} (-1)^k g(2^k x) \tag{13}$$

for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
Now take any $\tilde{g} \in \mathcal{C}$ such that $\tilde{g}(2^k) = (-1)^k$ and $\| \tilde{g} \| = \sup \{|\tilde{g}(t)|;\ t \in \mathbb{R}\} = 1$. Then for the corresponding $\tilde{\varphi}$ we would obtain by (13)

$$\begin{aligned} \tilde{\varphi}(1) &= (-1)^n \tilde{\varphi}(2^n) + \sum_{k=0}^{n-1} (-1)^k (-1)^k \\ &= (-1)^n \tilde{\varphi}(2^n) + n \end{aligned}$$

for every $n \in \mathbb{N}$, which is impossible for a bounded $\tilde{\varphi}$. Hence (12) has no solution $\varphi = \tilde{\varphi} \in \mathcal{C}$ for $g = \tilde{g}$.

Consequently, equation (11) with $\lambda = \frac{2}{3}$, i.e.

$$(\frac{2}{3} I - F_4)[\varphi] = f \tag{14}$$

has no solution $\varphi = \tilde{\varphi} \in \mathcal{C}$ for $f = \tilde{f} = -\frac{1}{3}F_4[\tilde{g}]$.

Now let $f^* \in \mathcal{C}$ such that $\| \tilde{f} - f^* \| \leqslant \frac{1}{9}$.

If (14) has a solution $\varphi = \varphi^* \in \mathcal{C}$ for $f = f^*$, then (12) and (13) are satisfied with $g = g^*$ and we have, using $\| F_4^{-1} \| = \frac{3}{2}$ (which is proved in [4])

$$\begin{aligned} \| \tilde{g} - g^* \| &= \| -3F_4^{-1}[\tilde{f}] + 3F_4^{-1}[f^*] \| \\ &= 3 \| F_4^{-1}[\tilde{f} - f^*] \| \\ &\leqslant 3 \| F_4^{-1} \| \| \tilde{f} - f^* \| \\ &= \frac{1}{2}. \end{aligned}$$

This implies $|\tilde{g}(2^k) - g^*(2^k)| = |(-1)^k - g^*(2^k)| \leqslant \frac{1}{2}$, hence $g^*(2^k) = (-1)^k \cdot \varepsilon_k$ with $\varepsilon_k \in [\frac{1}{2}, \frac{3}{2}]$. Now by (13)

$$\varphi^*(1) = (-1)^n \varphi^*(2^n) + \sum_{k=0}^{n-1} \varepsilon_k$$

for every $n \in \mathbb{N}$ which is again impossible for $\varphi^* \in \mathcal{C}$. This contradiction shows that no element of the closed ball with center $\tilde{f}$ and radius $\frac{1}{9}$ belongs to the image $(\frac{2}{3}I - F_4)[\mathcal{C}]$, so that this set is not dense in $\mathcal{C}$, hence $\frac{2}{3} \in \sigma_r(F_4)$.

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# La fonction d'indice et la fonction exponentielle

**Résumé.** On montre quelles solutions de l'équation fonctionnelle conditionnelle

$$f(c) \cdot f(d) \neq 0 \Longrightarrow f(c+d) = f(c) \cdot f(d), \tag{$*$}$$

où

$$f : \mathbb{R}(p) = \{(x_1, \dots, x_p) \in \mathbb{R}^p : \, x_i \geqslant 0 \text{ pour } i = 1, \dots, p\} \setminus \{\underline{0}\} \longrightarrow \mathbb{R}(p),$$

$\underline{0} := (0, \dots, 0) \in \mathbb{R}^p$ et si $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_p)$, $x + y = (x_1 + y_1, \dots, x_p + y_p)$ et $x \cdot y = (x_1 y_1, \dots, x_p y_p)$, sont en même temps des solutions de l'équation

$$f(c+d) = f(c) \cdot f(d) \tag{$**$}$$

et on donne la solution générale de $(**)$ et toutes les solutions de $(**)$ et de la condition

$$\exists\, r \in \mathbb{R},\, 0 < r \neq 1 \;\; \forall\, x \in \mathbb{R}(p) : \; f(rx) = f(x).$$

On indique aussi la construction de toutes les solutions de $(*)$ presque-mesurables et on considère le problème du prolongement des solutions de $(**)$.

## 1. Introduction

F.S. Roberts dans [8], en applicant la mathématique au processus du choix, a introduit la notion de la fonction d'indice, dont la généralisation

$$f : \mathbb{R}(p) = \{(x_1, \dots, x_p) \in \mathbb{R}^p : \, x_i \geqslant 0 \text{ pour } i = 1, \dots, p\} \setminus \{\underline{0}\} \longrightarrow \mathbb{R}(p)$$

remplit la condition suivante (nommée l'équation conditionnelle de Cauchy)

$$\forall\, c, d \in \mathbb{R}(p) \;\; [f(c) \cdot f(d) \neq \underline{0} \Longrightarrow f(c+d) = f(c) \cdot f(d)], \tag{1}$$

où $\underline{0} := (0, \dots, 0) \in \mathbb{R}^p$ et si $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_p)$, $x + y = (x_1 + y_1, \dots, x_p + y_p)$ et $x \cdot y = (x_1 y_1, \dots, x_p y_p)$.

Nous comprenons ici sous la fonction exponentielle la fonction $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ pour laquelle

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$$\forall\, c, d \in \mathbb{R}(p) :\ f(c+d) = f(c) \cdot f(d). \tag{2}$$

Ils existent pour $p \geqslant 2$ des functions qui satisfont à (1) et qui ne remplissent pas (2). Par exemple pour $p = 2$ la fonction $f = (f_1, f_2)$, où

$$f_1(x_1, x_2) = \begin{cases} 1 & \text{pour } (x_1, x_2) \in \mathbb{R}(2) \text{ et } x_1 \leqslant x_2, \\ 0 & \text{pour } (x_1, x_2) \in \mathbb{R}(2) \text{ et } x_1 > x_2 \end{cases}$$

et

$$f_2(x_1, x_2) = 1 - f_1(x_1, x_2),$$

satisfait à (1) (voir [6]) et ne remplit pas (2). Remarquons que cette situation ne peut pas avoir lieu pour $p = 1$ puisque dans ce cas $f(x) \neq 0$, donc $f(c) \cdot f(d) \neq 0$ toujours.

Ils se posent donc les deux questions

(a) sous quelle condition nécéssaire et suffisante la fonction satisfaisante à (1) remplit aussi (2)

et

(b) quelle est la solution générale de (2)?

## 2. La fonction d'indice et la fonction exponentielle

La réponse à la question (a) donne le

Théoreme 1

*La fonction* $f = (f_1, \ldots, f_p) : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ *remplissante* (1) *satisfait à* (2) *si et seulement s'il existe* $k \in \{1, \ldots, p\}$ *tel que* $f_k \neq 0$ *sur* $\mathbb{R}(p)$.

Nous allons démontrer au commencement les deux lemmes.

Lemme 1

*Désignons par* $Z_\nu := \{x \in \mathbb{R}(p) : f_\nu(x) \neq 0\}$ *pour* $\nu = 1, \ldots, p$. *La fonction* $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ *remplissante* (1) *satisfait à* (2) *si et seulement si*

$$\mathbb{R}(p) \times \mathbb{R}(p) \subset (Z_1 \times Z_1) \cup (Z_2 \times Z_2) \cup \ldots \cup (Z_p \times Z_p). \tag{3}$$

*Démonstration.* La fonction $f$ remplissante (1) satisfait à (2) si et seulement s'il n'existe pas la paire $(c, d) \in \mathbb{R}(p) \times \mathbb{R}(p)$ pour laquelle $f(c) \cdot f(d) = 0$, c. à d. si et seulement si pour chaque paire $(c, d) \in \mathbb{R}(p) \times \mathbb{R}(p)$ nous avons $f(c) \cdot f(d) \neq \underline{0}$. Cela signifie qu'il existe un $k$ tel que $(c, d) \in Z_k \times Z_k$, ou, équivalentement, que $(c, d) \in (Z_1 \times Z_1) \cup (Z_2 \times Z_2) \cup \ldots \cup (Z_p \times Z_p)$. L'inclusion (3) est donc démontrée.

LEMME 2

*Si pour une fonction* $g : \mathbb{R}(p) \longrightarrow \mathbb{R}$ *remplissante* (2) *il existe un* $c \in \mathbb{R}(p)$ *tel que* $g(c) = 0$*, alors*

$$g = 0 \text{ sur } \operatorname{Int}\mathbb{R}(p) := \{(x_1, \ldots, x_p) \in \mathbb{R}(p) : x_\nu > 0 \text{ pour } \nu = 1, \ldots, p\}.$$

*Démonstration.* Nous avons $g(x + c) = g(x)g(c) = 0$ pour chaque $x \in \mathbb{R}$. Supposons qu'il existe un $d \in \operatorname{Int}\mathbb{R}(p)$ tel que $g(d) \neq 0$. Il en résulte que $g(2d) = g(d)g(d) \neq 0$ et par l'induction $g(nd) \neq 0$ pour chaque $n$ entier et positif. Puisque $d \in \operatorname{Int}\mathbb{R}(p)$ il existe un $n$ tel que $nd - c \in \mathbb{R}(p)$, alors

$$0 \neq g(nd) = g((nd - c) + c) = g(nd - c)g(c) = 0,$$

donc une contradiction.

*Démonstration du théorème 1.* S'il existe un $k$ tel que $f_k \neq 0$ sur $\mathbb{R}(p)$, dans ce cas $Z_k = \mathbb{R}(p)$, alors (3) à lieu. Si pour chaque $k$ il existe $c_k$ tel que $f_k(c_k) = 0$, donc, d'après le lemme 2, $f_k = 0$ sur $\operatorname{Int}\mathbb{R}(p)$ pour chaque $k$ et de là $f(1, \ldots, 1) = \underline{0}$. Nous avons une contradiction avec $f(\mathbb{R}(p)) \subset \mathbb{R}(p)$.

COROLLAIRE 1

*Si la fonction* $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ *remplissante* (1) *est continue, elle satisfait à* (2).

Cela résulte du théorème 3 dans [6], p. 171, où est démontré que pour la fonction $f = (f_1, \ldots, f_p)$ remplissante (1) et continue on a $Z_\nu = \emptyset$ ou $Z_\nu = \mathbb{R}(p)$ pour $\nu = 1, \ldots, p$ et il existe au moins un $k$ tel que $Z_k = \mathbb{R}(p)$.

Remarquons que le corollaire 1 n'est pas vrai pour la fonction mesurable au sens de Lebesgue au lieu de la fonction continue (voir exemple plus haut).

## 3. Solution généerale de (2)

La réponse à la question (b) donne le

THÉOREME 2

*Nous recevons chaque solution* $f = (f_1, \ldots, f_p) : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ *de* (2) *et seulement une solution de* (2) *par la construction suivante:*

(i) *pour chaque* $k = 1, \ldots, p$ *prenons une fonction* $\phi_k : \{1, \ldots, p\} \longrightarrow \{0, 1\}$ *d'une maniere qu'il existe au moins un* $j$ *tel que* $\phi_j = 1$,

(ii) *posons*

$$f_k(x) = \phi_k(i_1)\phi_k(i_2) \ldots \phi_k(i_m) \exp a_k(x) \tag{4}$$

*pour* $x \in R(i_1, \ldots, i_m)$ *et* $m, k = 1, \ldots, p$*, ou* $\{i_1, \ldots, i_m\}$ *est une combinaison arbitraire de* $m$ *éléments de l'ensemble* $\{1, \ldots, p\}$, $R(i_1, \ldots, i_m)$ *est l'ensemble des éléments* $(x_1, \ldots, x_p) \in \mathbb{R}(p)$ *pour lesquels* $x_{i_j} > 0$

*pour* $j = 1, \ldots, m$ *et* $x_n = 0$ *pour* $n \neq i_1, \ldots, i_m$ *et* $a_k : \mathbb{R}^p \longrightarrow \mathbb{R}$ *pour* $k = 1, \ldots, p$ *sont des fonctions additives.*

Remarquons que $f_k(c)$ de la formule (4) est pour $c \in R(i_1, \ldots, i_m)$ égale à 0 ou à $\exp a_k(c)$ lorsque $\phi_k$ prend sur l'ensemble $\{i_1, \ldots, i_m\}$ la valeur 0 ou non. Il en résulte que $f_k = 0$ sur $R(i_1, \ldots, i_m)$ ou $f_k \neq 0$ sur $R(i_1, \ldots, i_m)$ et de la, puisque $R(i_1, \ldots, i_m)$ forment des cônes sur $\mathbb{R}$, les ensembles $Z_\nu$, comme les réunions quelques-uns de $R(i_1, \ldots, i_m)$, forment aussi des cônes sur $\mathbb{R}$.

*Démonstration.* La fonction $f$ est bien définie puisque les ensembles $R(i_1, \ldots, i_m)$ et $R(j_1, \ldots, j_n)$ sont disjoints pour $\{i_1, \ldots, i_m\} \neq \{j_1, \ldots, j_n\}$ et la réunion des ensembles $R(i_1, \ldots, i_m)$ pour toutes les combinaisons ($m = 1, \ldots, p$) est égale à $\mathbb{R}(p)$. De plus puisque $\phi_j = 1$ nous avons $f_j \neq 0$ sur $\mathbb{R}(p)$, donc $f(\mathbb{R}(p)) \subset \mathbb{R}(p)$.

Soit $x \in R(i_1, \ldots, i_m)$ et $y \in R(j_1, \ldots, j_n)$ et soit $(s_1, \ldots, s_r)$ la combinaison composée des éléments $i_1, \ldots, i_m, j_1, \ldots, j_n$. Dans ce cas $x + y \in R(s_1, \ldots, s_r)$ et

$$\phi(i_1) \ldots \phi(i_m)\phi(j_1) \ldots \phi(j_n) = \phi(s_1) \ldots \phi(s_r),$$

d'où $f_k(x)f_k(y) = f_k(x + y)$, c.q.f.d.

Supposons à present que $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ remplit (2) et définissons $\phi_k : \{1, \ldots, p\} \longrightarrow \{0, 1\}$ par l'équivalence

$$\phi_k(j) = 1 \Longleftrightarrow f_k(0, \ldots, \underset{j}{1}, 0, \ldots, 0) \neq 0. \tag{5}$$

Soit $(i_1, \ldots, i_m)$ une combinaison des elements $1, \ldots, p$. Nous pouvons supposer que $i_1 < i_2 < \ldots < i_m$. L'ensemble $R(i_1, \ldots, i_m)$ est fermé par rapport à l'addition et $f$ rétréssante à cet ensemble remplit (2). Fixons $k$ de l'ensemble $\{1, \ldots, p\}$ et remarquons que

$$\begin{aligned} &f_k(0, \ldots, 0, \underset{i_1}{1}, 0, \ldots, 0, \underset{i_2}{1}, 0, \ldots, 0, \underset{i_m}{1}, 0, \ldots, 0) \\ &\quad = \prod_{j=1}^{m} f_k(0, \ldots, 0, \underset{i_j}{1}, 0, \ldots, 0). \end{aligned} \tag{6}$$

Considérons les deux cas:

($\alpha$) il existe $x_0 \in R(i_1, \ldots, i_m)$ tel que $f_k(x_0) = 0$,

($\beta$) $f_k(x) \neq 0$ pour chaque $x$ de $R(i_1, \ldots, i_m)$.

Dans le cas ($\alpha$) $f_k(x) = 0$ sur $R(i_1, \ldots, i_m)$ d'après le lemme 2 (appliqué pour $p = m$). Supposons que $\phi_k(i_1) = \phi_k(i_2) = \phi_k(i_m) = 1$, alors d'après (5) $f_k(0, \ldots, 0, \underset{i_j}{1}, 0, \ldots, 0) \neq 0$ pour $j = 1, \ldots, m$, d'où d'après (6)

$$f_k(0, \ldots, 0, \underset{i_1}{1}, 0, \ldots, 0, \underset{i_2}{1}, 0, \ldots, 0, \underset{i_m}{1}, 0, \ldots, 0) \neq 0.$$

Nous avons une contradiction puisque

$$(0,\dots,0,\underset{i_1}{1},0,\dots,0,\underset{i_2}{1},0,\dots,0,\underset{i_m}{1},0,\dots,0)\in R(i_1,\dots,i_m).$$

Nous avons donc démontré que au moins un des nombres $\phi_k(i_1),\dots,\phi_k(i_m)$ est égal à zéro, alors (4) a lieu avec $a_k(x)$ additives quelconques.

Dans le cas $(\beta)$ on a $\phi_k(i_1)=\dots=\phi_k(i_m)=1$ d'après (5) et (6) et $f_k(x)>0$ sur $R(i_1,\dots,i_m)$. En posant $b_k(x)=\ln f_k(x)$ nous avons (4) avec la fonction additive $b_k : R(i_1,\dots,i_m)\longrightarrow\mathbb{R}$. Soit $Z_k=\{x\in\mathbb{R}(p) : f_k(x)\neq 0\}$. Puisque l'ensemble $Z_k\cup\{\underline{0}\}$ forme un cône sur l'ensemble $\mathbb{Q}$ des nombres rationnels et la fonction

$$a_k(x)\begin{cases}\ln f_k(x) & \text{pour } x\in Z_k,\\ 0 & \text{pour } x=0\end{cases}$$

est additive (et donc $\mathbb{Q}$-homogène), on peut prolonger cette fonction à la fonction additive sur $\mathbb{R}^p$ tout entier (voir [5], Th. 2, p. 86 ou [1], Th. 5, p. 88).

La fonction $f$ en considération, comme remplissante (2), est en même temps une fonction qui satisfait aussi à (1), donc d'après le théorème 1 il doit exister un $j$ tel que $f_j\neq 0$ sur $\mathbb{R}(p)$, d'où $\phi_j=1$ sur $\{1,\dots,p\}$.

La démonstration du théorème 2 est donc terminée.

Remarque 1

On a ici

$$Z_k=\{(x_1,\dots,x_p)\in\mathbb{R}(p) : \forall j\in\phi^{-1}(\{0\}) : x_j=0\}.$$

On peut donc donner la description suivante des solutions de (2).

Corollaire 2

*On peut recevoir chaque solution $f=(f_1,\dots,f_p):\mathbb{R}(p)\longrightarrow\mathbb{R}(p)$ de (2) et seulement une solution de (2) par la formule*

$$f_k(x)=\begin{cases}\exp a_k(x) & \textit{pour } x\in Z_k=\{(x_1,\dots,x_p)\in\mathbb{R}(p) : \forall j\in M_k : x_j=0\},\\ 0 & \textit{pour } x\in\mathbb{R}(p)\setminus Z_k,\end{cases}$$

*ou $M_k$ est, pour $k=1,\dots,p$, un sous-ensemble de l'ensemble $\{1,\dots,p\}$ tel que au moins un $M_k=\emptyset$ et $a_k:\mathbb{R}^p\longrightarrow\mathbb{R}$ est une fonction additive.*

On peut aussi remplacer la formule (4) par la suivante

$$f_k(x)=\begin{cases}0 & \text{pour } x\in\bigcup_{i\in\phi_k^{-1}(\{0\})}E[i],\\ \exp a_k(x) & \text{pour } x\in\mathbb{R}(p)\setminus\bigcup_{i\in\phi_k^{-1}(\{0\})}E[i],\end{cases}$$

ou $E[i]=\{x=(x_1,\dots,x_p)\in\mathbb{R}(p) : x_i>0\}$.

Corollaire 3

*Si la solution $f$ de (2) est continue par rapport à chaque variable, alors pour chaque $k \in \{1, \dots, p\}$ on a $\phi_k = 1$ ($f_k = \exp a_k$) ou $\phi_k = 0$ ($f_k = 0$) et $f$ est continue.*

*Démonstration.* Supposons qu'ils existent $k, l, m \in \{1, \dots, p\}$ tels que $\phi_k(1) = 1$ et $\phi_k(m) = 0$. Nous avons $f_k = 0$ sur $R(m, l)$ et $f_k = \exp a_k$ sur $R(l)$. Si $l < m$, pour $x_0 = (0, \dots, 0, \underset{l}{a}, 0, \dots, 0)$, où $a > 0$, et

$$x_n = (0, \dots, 0, \underset{l}{a}, 0, \dots, 0, \underset{m}{n^{-1}}, 0, \dots, 0), \quad \text{où } n = 1, 2, \dots,$$

nous avons $x_n \longrightarrow x_0$ et $f_k(x_n) \not\longrightarrow f_k(x_0) = \exp a_k(x_0) \neq 0$, donc une contradiction. Pour $m < l$ le raisonnement est analogue.

Puisque pour chaque $k \in \{1, \dots, p\}$ on a $f_k = \exp a_k$ où $f_k = 0$ et la fonction additive $a_k$, continue par rapport à chaque variable, est continue, donc $f_k$ sont continues sur $\mathbb{R}(p)$.

Remarquons que la solution de (1) peut être mesurable, n'étant pas continue (voir les exemples plus bas, notamment l'exemple 1° (ii)).

Remarque 2

Chaque solution $f = (f_1, \dots, f_p)$ de (1) doit être de la forme $f_k = F_k \exp a_k$, où $F = (F_1, \dots, F_p)$ est aussi une solution de (1), ayante les valeurs dans l'ensemble $\{0, 1\}^p \setminus \{\underline{0}\}$ et $a_k : \mathbb{R}^p \longrightarrow \mathbb{R}$ est une fonction additive [6]. Cette solution peut être non-mesurable au sens de Lebesgue, même si les functions $a_k$ ($k = 1, \dots, p$) sont mesurables (donc continues). En effet pour $p = 2$ prenons un élément $(a, b)$ de la base de Hamel de $\mathbb{R}^2$ et pour $Z_1$ prenons l'ensemble de ces éléments de $\mathbb{R}(2)$ qui ont des coefficients positives auprès "$a$" dans le développement de ces éléments par rapport à cette base de Hamel et posons $Z_2 = \mathbb{R}(2) \setminus Z_1$. La fonction $f = (f_1, f_2)$ telle que

$$f_1(c_1, c_2) = \begin{cases} 1 & \text{si } (c_1, c_2) \in Z_1, \\ 0 & \text{si } (c_1, c_2) \in Z_2 \end{cases}$$

et

$$f_2(c_1, c_2) = 1 - f_1(c_1, c_2)$$

est une solution de (1) (voir [6]) avec $a_1(c_1, c_2) = a_2(c_1, c_2) = 0$ qui n'est pas mesurable L, puisque les ensembles $Z_1$ et $Z_2$ ne sont pas mesurables.

Cette situation ne peut pas avoir lieu pour la solution de (2) puisque tous les ensembles $R(\dots)$ sont mesurables L (ils ont la mesure zéro à l'exception de $R(1, \dots, p)$ qui a cette mesure égale à $+\infty$.)

## 4. Exemples des solutions de (2)

1° Nous donnerons toutes les solutions de (2) pour $p = 2$. Nous avons 4 possibilités pour les fonctions $\phi_k$ $(k = 1, 2)$

| | $\phi_a$ | $\phi_b$ | $\phi_c$ | $\phi_d$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 2 | 1 | 1 | 0 | 0 |

et 3 ensembles

$$\begin{aligned} R(1) &= \{(x_1, x_2) \in \mathbb{R}(2) : x_1 > 0 \text{ et } x_2 = 0\}, \\ R(2) &= \{(x_1, x_2) \in \mathbb{R}(2) : x_1 = 0 \text{ et } x_2 > 0\}, \\ R(1,2) &= \{(x_1, x_2) \in \mathbb{R}(2) : x_1 > 0 \text{ et } x_2 > 0\}. \end{aligned}$$

Et voila toutes les solutions $f = (f_1, f_2)$ de (2).

(i) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_a : f_2(x) = \exp a_2(x)$, ici (voir le corollaire 3) $M_1 = M_2 = \emptyset$.

(ii) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_b$ et $x = (x_1, x_2) \in \mathbb{R}(2)$

$$f_2(x_1, x_2) = \begin{cases} \exp a_2(x) & \text{pour } x_1 = 0 \text{ et } x_2 > 0, \\ 0 & \text{pour } x_1 > 0 \text{ et } x_2 \geqslant 0, \end{cases}$$

ici $M_1 = \emptyset$ et $M_2 = \{1\}$.

Si $a_1$ et $a_2$ sont continues, $f$ est mesurable, n'étant pas continue ($f_2$ n'est pas continue). Nous avons la même situation dans beaucoup d'exemples plus bas.

(iii) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_c$:

$$f_2(x_1, x_2) = \begin{cases} \exp a_2(x) & \text{pour } x_1 > 0 \text{ et } x_2 = 0, \\ 0 & \text{pour } x_1 \geqslant 0 \text{ et } x_2 > 0, \end{cases}$$

ici $M_1 = \emptyset$ et $M_2 = \{2\}$.

(iv) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_d : f_2(x) = 0$, ici $M_1 = \emptyset$ et $M_2 = \{1, 2\}$.

(v) $f_2(x) = \exp a_2(x)$ et pour $\phi_1 = \phi_b$:

$$f_1(x_1, x_2) = \begin{cases} \exp a_1(x) & \text{pour } x_1 = 0 \text{ et } x_2 > 0, \\ 0 & \text{pour } x_1 > 0 \text{ et } x_2 \geqslant 0, \end{cases}$$

ici $M_2 = \emptyset$ et $M_1 = \{1\}$.

(vi) $f_2(x) = \exp a_2(x)$ et pour $\phi_1 = \phi_c$:

$$f_1(x_1, x_2) = \begin{cases} \exp a_1(x) & \text{pour } x_1 > 0 \text{ et } x_2 = 0, \\ 0 & \text{pour } x_1 \geqslant 0 \text{ et } x_2 > 0, \end{cases}$$

ici $M_2 = \emptyset$ et $M_1 = \{2\}$.

(vii) $f_2(x) = \exp a_2(x)$ et pour $\phi_1 = \phi_d$ : $f_1(x) = 0$, ici $M_2 = \emptyset$ et $M_1 = \{1, 2\}$.

2° Nous allons donner quelques exemples des solutions de (2) pour p=3 (pas toutes! ). Nous avons 8 possibilités pour $\phi_k$ ($k = 1, 2, 3$):

| | $\phi_z$ | $\phi_b$ | $\phi_c$ | $\phi_d$ | $\phi_e$ | $\phi_f$ | $\phi_g$ | $\phi_h$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 3 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |

et 7 ensembles $R(1)$, $R(2)$, $R(3)$, $R(1, 2)$, $R(1, 3)$, $R(2, 3)$, $R(1, 2, 3)$.

Et voilà des solutions de (2).

(i) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_3 = \phi_b$ et $x = (x_1, x_2, x_3) \in \mathbb{R}(2)$:

$$f_2(x_1, x_2, x_3) = \begin{cases} \exp a_2(x) & \text{pour } x_1 = 0, \\ 0 & \text{pour } x_1 > 0, \end{cases}$$

$$f_3(x_1, x_2, x_3) = \begin{cases} \exp a_3(x) & \text{pour } x_1 = 0, \\ 0 & \text{pour } x_1 > 0, \end{cases}$$

ici $M_1 = \emptyset$, $M_2 = \{1\}$ et $M_3 = \{1\}$.

(ii) $f_1(x) = \exp a_1(x)$ et pour $\phi_2 = \phi_c$ et $\phi_3 = \phi_g$:

$$f_2(x_1, x_2, x_3) = \begin{cases} \exp a_2(x) & \text{pour } x_2 = 0, \\ 0 & \text{pour } x_2 > 0, \end{cases}$$

$$f_3(x_1, x_2, x_3) = \begin{cases} \exp a_3(x) & \text{pour } x_1 = x_2 = 0, \\ 0 & \text{pour } x_1 > 0 \text{ ou } x_2 > 0, \end{cases}$$

ici $M_1 = \emptyset$, $M_2 = \{2\}$ et $M_3 = \{1, 2\}$.

(iii) $f_2(x) = \exp a_2(x)$ et pour $\phi_2 = \phi_d$ et $\phi_3 = \phi_f$:

$$f_1(x_1, x_2, x_3) = \begin{cases} \exp a_1(x) & \text{pour } x_3 = 0, \\ 0 & \text{pour } x_3 > 0 \end{cases}$$

$$f_3(x_1, x_2, x_3) = \begin{cases} \exp a_3(x) & \text{pour } x_1 = x_3 = 0, \\ 0 & \text{pour } x_1 > 0 \text{ ou } x_3 > 0, \end{cases}$$

ici $M_2 = \emptyset$, $M_1 = \{3\}$ et $M_3 = \{1, 3\}$.

(iv) $f_3(x) = \exp a_3(x)$ et pour $\phi_1 = \phi_c$ et $\phi_2 = \phi_h$:

$$f_1(x_1, x_2, x_3) = \begin{cases} \exp a_1(x) & \text{pour } x_2 = 0, \\ 0 & \text{pour } x_2 > 0 \end{cases}$$

et $f_2(x) = 0$, ici $M_3 = \emptyset$, $M_1 = \{2\}$ et $M_2 = \{1, 2, 3\}$.

3° Pour $p = 4$ nous avons $2^4 = 16$ possibilités pour les fonctions $\phi_k$ ($k = 1, 2, 3, 4$) et 15 ensembles du type $R(\ldots)$.

Et voilà un exemple pour la solution : $f_2(x) = \exp a_2(x)$ et pour

a) $\phi_1$ prise comme il suit: $\phi_1(1) = \phi_1(2) = 0$ et $\phi_1(3) = \phi_1(4) = 1$,
b) $\phi_3$ prise comme il suit: $\phi_3(1) = \phi_3(3) = 0$ et $\phi_3(2) = \phi_3(4) = 1$,
c) $\phi_4$ prise comme il suit: $\phi_4(4) = 0$ et $\phi_4(1) = \phi_4(2) = \phi_4(3) = 1$

et pour $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}(4)$:

$$f_1(x_1, x_2, x_3, x_4) = \begin{cases} \exp a_1(x) & \text{pour } x_1 = x_2 = 0, \\ 0 & \text{pour } x_1 > 0 \text{ ou } x_2 > 0, \end{cases}$$

$$f_3(x_1, x_2, x_3, x_4) = \begin{cases} \exp a_3(x) & \text{pour } x_1 = x_3 = 0, \\ 0 & \text{pour } x_1 > 0 \text{ ou } x_3 > 0, \end{cases}$$

$$f_4(x_1, x_2, x_3, x_4) = \begin{cases} \exp a_4(x) & \text{pour } x_4 = 0, \\ 0 & \text{pour } x_4 > 0, \end{cases}$$

ici $M_2 = \emptyset$, $M_1 = \{1, 2\}$, $M_3 = \{1, 3\}$ et $M_4 = \{4\}$.

## 5. Solution générale de (2) et de (7)

COROLLAIRE 4

*Si pour une fonction $f = (f_1, \ldots, f_p) : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ remplissante* (2) *il existe un $r \neq 1$ positif tel que pour chaque $x$ de $\mathbb{R}(p)$*

$$f(rx) = f(x), \tag{7}$$

*dans ce cas*

$$\forall\, x \in \mathbb{R}(p)\ \forall\, k \in 1, \ldots, p:\ f_k(x) \neq 0 \Longrightarrow f_k(x) = 1.$$

*donc on peut prendre $a_k = 0$ dans* (4) *pour chaque $k = 1, \ldots, p$ (c. à d. $f_k(\mathbb{R}(p)) \subset \{0, 1\}$). De plus $f$ satisfait à* (7) *pour chaque $r > 0$.*

*Démonstration.* La condition (7) implique que $f(x) = f(\frac{1}{r}x)$ pour chaque $x \in \mathbb{R}(p)$, nous pouvons donc supposer que $r > 1$. Soit $n$ tel que $rn > 2$ et $k$ fixé dans l'ensemble $\{1, \ldots, p\}$. Puisque pour $x \in \mathbb{R}(p)$

$$f_k\left(\frac{r^n-2}{r-1}x\right) f_k\left(\frac{1}{r-1}x\right) = f_k\left(\frac{r^n-1}{r-1}x\right) = f_k\left(\sum_{j=0}^{n-1} r^j x\right) = [f_k(x)]^n,$$

donc si $f_k(x) \neq 0$, on a $f_k(\frac{1}{r-1}x) \neq 0$ et de là

$$f_k(x)\, f_k\left(\frac{1}{r-1}x\right) = f_k\left(\frac{r}{r-1}x\right) = f_k\left(\frac{1}{r-1}x\right) \neq 0,$$

d'où $f_k(x) = 1$, c. à d. on peut prendre $a_k(x) = 0$. Si $f_k(x) = 0$ on peut aussi poser $a_k(x) = 0$, c.q.f.d.

Remarquons qu'il ne doit pas être $a_k = 0$ dans (4) pour la solution de (1) et de (7). En effet par exemple la fonction $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ donnée par la formule (4), où $\phi_1 = 1$, $a_1 = 0$, $\phi_k = 0$ et $a_k$ arbitraire pourvu que additives pour $k = 2, \ldots, p$, remplit (1) et (7) pour chaque $r > 0$.

Le corollaire 4 est une conclusion simple du théorème 1 dans [3], puisque d'après le théorème 2 les ensembles $Z_\nu$ forment des cônes sur $\mathbb{R}$, alors aussi les conditions $Z_\nu \subset (r-1)\,\mathrm{lin}\, Z_\nu$ sont remplies.

Remarquons aussi que le corollaire 4 n'est pas vrai pour les solutions de (1) pour $p \geqslant 3$ et $r$ transcendant (voir [6], Th. 2, p. 179, et [2]), étant vrai pour $p = 1, 2$ et pour chaque $p$ et chaque $r$ algébrique [7].

## 6. Solutions de (1) et de (7) presque-mesurables

On à démontré dans [2] que pour chaque $r$ transcendant et $p \geqslant 3$ il existe une solution de (1) et (7) qui a au moins une valeur en dehors de l'ensemble $\{0,1\}^p \backslash \{0\}$. On fait usage de l'axiome du choix de Zermelo dans la construction de cette solution. Le théorème suivant montre qu'on ne peut pas faire cette construction sans l'ensemble non mesurable au sens de Lebesgue.

Théoreme 3

(a) *Si pour une solution $f = (f_1, \ldots, f_p)$ de (1) les ensembles $A_i(c) = \{tc \in \mathbb{R}(p) : t \in \mathbb{R}$ et $f_i(tc) \neq 0\} = Z_i \cap D(c)$ pour $i = 1, \ldots, p$ et pour un $c \in \mathbb{R}(p)$ fixé, où $D(c) = \{tc : t \in \mathbb{R}(1)\}$ et $Z_i = \{x \in \mathbb{R}(p) : f_i(x) \neq 0\}$, sont mesurables linéairement au sens de Lebesgue, alors $A_i(c) = D(c)$ pour $c \in Z_i$.*

(b) *Si de plus $f$ satisfait à (7) avec un $r \neq 1$ pour chaque $x \in D(c)$, elle a toutes les valeurs sur $D(c)$ dans l'ensemble $\{0,1\}^p \backslash \{\underline{0}\}$, donc elle remplit (7) sur $D(c)$ pour chaque $r > 0$.*

(c) *Si $A_i(c)$ sont mesurable pour chaque $c \in \mathbb{R}(p)$ et chaque $i = 1, \ldots, p$ ($f$ est dite presque-mesurable dans ce cas), $Z_i = D(c)$, donc $Z_i \cup \{\underline{0}\}$ forme un cône sur $\mathbb{R}$. Inversement si $Z_i \cup \{\underline{0}\}$ forme un cône sur $\mathbb{R}$, les ensembles $A_i(c)$ sont évidemment mesurables pour chaque $c \in \mathbb{R}(p)$.*

(d) *Évidemment si $f$ presque-mesurable satisfait à (7) avec un $r \neq 1$ pour chaque $x \in \mathbb{R}(p)$, elle a toutes les valeurs dans l'ensemble $\{0,1\}^p \setminus \{\underline{0}\}$, donc elle remplit (7) pour chaque $r > 0$.*

*Démonstration.* En désignant $Z_i^1 = Z_i$, $Z_i^0 = \mathbb{R}(p) \setminus Z_i$, $A_i^1(c) = A_i(c)$ et $A_i^0(c) = \mathbb{R}(1) \setminus A_i(c)$, nous savons [6] que

$$(i_1 j_1, \ldots, i_p j_p) \neq \underline{0} \Longrightarrow Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_p^{i_p j_p} \quad (8)$$

pour tous $i_1, \ldots, i_p, j_1, \ldots, j_p \in \{0,1\}$ et

$$Z_1 \cup \ldots \cup Z_p = \mathbb{R}(p), \quad (9)$$

donc $Z_1^0 \cap \ldots \cap Z_p^0 = \emptyset$. De là

$$A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c) + A_1^{j_1}(c) \cap \ldots \cap A_p^{j_p}(c) \subset A_1^{i_1 j_1}(c) \cap \ldots \cap A_p^{i_p j_p}(c)$$

pour tels $i_1, \ldots, i_p, j_1, \ldots, j_p$ que $(i_1 j_1, \ldots, i_p j_p) \neq \underline{0}$ et $A_1^0(c) \cap \ldots \cap A_p^0(c) = \emptyset$. Les ensembles $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ sont mesurables et disjoints pour les suites $i_1, \ldots, i_p$ différentes et de plus $D(c) = A_1(c) \cup \ldots \cup A_p(c)$ est la somme de tous ces ensembles, alors il existe une suite $(i_1, \ldots, i_p) \neq \underline{0}$ telle que la mesure de l'ensemble $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ est positive. Puisque

$$A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c) + A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c) \subset A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c),$$

donc d'après le théorème de Steinhaus ([5], p. 69) il existe un segment de $D(c)$ de la longueur positive contenu dans $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ et puisque $Z_i \cup \underline{0}$, $i = 1, \ldots, p$, est un cône sur le corps des nombres rationnels ([6]), l'ensemble $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c) \cup \{\underline{0}\}$ est le même, donc il doit être

$$D(c) = A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c) = Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} \cap D(c).$$

Si $c \in Z_k$, alors $i_k \neq 0$, d'où $Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} \cap D(c) \subset Z_k \cap D(c)$, donc $Z_k \cap D(c) = D(c)$, alors $tc \in Z_k$ pour chaque $t$ de $\mathbb{R}(1)$. Il en résulte que $A_i(c) = D(c)$ pour $c \in Z_i$. Si $f$ est presque-mesurable, donc $Z_i = \bigcup_{c \in Z_i} A_i(c) = \bigcup_{c \in Z_i} D(c)$.

Si (7) a lieu avec un $r \neq 1$ pour $x \in D(c)$, nous pouvons supposer dans (7) que $r > 1$. Si $x \in A_k(c)$ ($f_k(x) \neq 0$), alors $f_k(\frac{1}{r-1}x) \neq 0$ et

$$f_k(x)\, f_k\left(\frac{1}{r-1}x\right) = f_k\left(\frac{r}{r-1}x\right) = f_k\left(\frac{1}{r-1}x\right),$$

d'où $f_k(x) = 1$. Le théorème 3 est donc démontré.

Remarques

1. Il résulte de la démonstration plus haut qu'il suffit supposer dans le théorème 3 que pour un $c \in \mathbb{R}(p)$ (pour chaque $c \in \mathbb{R}(p)$) il existe une suite

$i_1, \ldots, i_p$ telle que l'ensemble $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ (voir la démonstration) a la mesure intérieure de Lebesgue positive au lieu de la mesurabilité de $A_i(c)$ pour ce $c$ (pour chaque $c$) et $i = 1, \ldots, p$ (ça suffit dans le théorème de Steinhaus). Cette supposition nouvelle est pourtant moins simple. De même il suffit supposer dans la partie (c) du théorème 3 que les ensembles $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ soient mesurables pour chaque $c \in \mathbb{R}(p)$ et chaque suite $i_1, \ldots, i_p$ au lieu de la mesurabilité de $A_i(c)$ pour chaque $c \in \mathbb{R}(p)$ et chaque $i = 1, \ldots, p$, mais ces deux suppositions sont équivalentes puisque $A_j(c)$ est la réunion de ces $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ pour lesquels $i_j = 1$.

2. La mesurabilité de $A_i(c)$ est équivalente à la mesurabilité de l'ensemble $B_i(c) = \{t \in \mathbb{R}(1) : f_i(tc) \neq 0\}$ puisque la fonction linéaire $t \mapsto tc$, pour $t$ de $\mathbb{R}(1)$ et $c \in \mathbb{R}(p)$, transforme bijectivement $B_i(c)$ sur $A_i(c)$ et tient la mesurabilité.

   De même les mesurabilités des ensembles $A_i(c)$ et $C_i(c) = f_1^{-1}(\{0\}) \cap D(c)$ sont équivalentes puisque $A_i(c) = D(c) \cap C_i(c)$.

3. La phrase dernière du théorème 3 ((7) avec un $r \neq 1 \Rightarrow$ (7) avec chaque $r > 0$) n'est pas vraie pour chaque solution de (1) (c. à d. sans la supposition de la mesurabilité des ensembles $A_i(c)$). En effet d'après [2] pour chaque $r$ transcendant il existe une solution de (1) et de (7) qui a au moins une valeur hors de l'ensemble $\{0,1\}^p \setminus \{\underline{0}\}$) et cette solution ne peut remplir (7) avec aucun $r$ algébrique différent de 1 (voir [7]).

4. La mesurabilité des ensembles $A_i(c)$, étant la condition suffisante pour que la fonction $f : \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ remplissante (1) et (7) avec un $r \neq 1$ ait les valeurs dans l'ensemble $\{0,1\}^p \setminus \underline{0}\}$, n'est pas une condition nécéssaire à cet effet, puisqu'il existe une fonction $f = (f_1, f_2) : \mathbb{R}(2) \longrightarrow \mathbb{R}(2)$ qui remplit (1) et (7) avec un $r \neq 1$ et qui a les valeurs dans l'ensemble $\{0,1\}^2 \setminus \{(0,0)\}$, pour laquelle l'ensemble $Z_1$ ne forme pas un cône sur $\mathbb{R}$. En effet soit $B$ une base de Hamel de $\mathbb{R}$ sur le corps $\mathbb{Q}$ des nombres rationnels telle que $1, \sqrt{2} \in B$ et soit $Z_1$ l'ensemble de ces éléments $(x, 0)$ de $\mathbb{R}(2)$ qui ont des coefficients positifs aupres $\sqrt{2}$ dans le développement de $x$ par rapport à cette base $B$. Posons $Z_2 = \mathbb{R}(2) \setminus Z_1$. La fonction $f = (f_1, f_2)$ définie comme il suit

$$f_1(x,y) = \begin{cases} 1 & \text{pour } (x,y) \in Z_1, \\ 0 & \text{pour } (x,y) \in Z_2, \end{cases} \qquad f_2(x,y) = \begin{cases} 1 & \text{pour } (x,y) \in Z_2, \\ 0 & \text{pour } (x,y) \in Z_1 \end{cases}$$

   a les propriétés exigées si $r = 2$ (voir [6]) et $Z_1$ ne forme pas du cône sur $\mathbb{R}$ $((\sqrt{2}-1, 0) \in Z_1$ et $\sqrt{2}\,(\sqrt{2}-1, 0) = (2-\sqrt{2}, 0) \notin Z_1)$. On peut constater que seulement les ensembles $A_1((1,0))$ et $A_2((1,0))$ ne sont pas mesurables ici.

5. La presque-mesurabilité de $f$ remplissante (1) entraîne que les ensembles $Z_i$, pour $i = 1, \ldots, p$, comme les cônes sur $\mathbb{R}^p$,, sont $p$-mesurables dans l'espace $\mathbb{R}^p$, puisque $Z_i \cup \{\underline{0}\}$ forment des cônes sur $\mathbb{R}$. L'exemple plus haut montre que l'implication inverse n'est pas vraie et que la mesurabilité de $f$ n'implique pas sa presque-mesurabilité.

6. L'exemple de la solution de (1) dans l'introduction montre que la mesurabilité de $A_i(c)$ n'implique pas de la continuité de cette solution. Une simple modification de cette solution (le remplacement de la fonction $f_1$ par $f_1 \exp a$, où $a : \mathbb{R}^2 \longrightarrow \mathbb{R}$ est une fonction additive discontinue) donne une solution de (1) non mesurable sur la plaine avec les mêmes ensembles $A_i(c)$ mesurables linéairement. Si $a(0, x_2)$ est discontinue dans cet exemple, la fonction $f$ n'est pas mesurable sur la demi-droite $\{(0, x_2) : x_2 \in \mathbb{R}(1)\}$.

7. Le corollaire 3 est une conclusion du théorème 3 puisque dans ce cas $A_i(c) = D(c)$ pour $c \in Z_i$ et $A_i(c) = \emptyset$ si $c \notin Z_i$.

8. On donne dans [4] la construction (pas simple et par l'itération) de toutes les solutions de (1) et de (7) avec *chaque* $r > 0$. La même construction nous donne toutes les solutions de (1) presque-mesurables. En effet nous savons d'après [5] que chaque solution $f = (f_1, \ldots, f_p)$ de (1) est de la forme $f_i = F_i \exp a_i$, où $F = (F_1, \ldots, F_p)$ est une solution de (1) ayante les valeurs dans l'ensemble $\{0, 1\}^p \setminus \{\underline{0}\}$ et $a_i : \mathbb{R}^p \longrightarrow \mathbb{R}$ est additive. La mesurabilité de $A_i(c)$ pour $f$ entraîne que les ensembles $Z_i \cup \{\underline{0}\}$ forment les cônes sur $\mathbb{R}$ et puisque

$$\{x \in \mathbb{R}(p) : f_i(x) \neq 0\} = \{x \in \mathbb{R}(p) : F_i(x) \neq 0\},$$

on a $F_i(x) = 1$ pour $x \in Z_i$, d'où $F_i(rx) = F_i(x)$ pour chaque $r > 0$. Il suffit donc construire $F$ comme dans [4] pour avoir $f$.

Il résulte de nos considérations que la construction dans [4] nous permet donner toutes les solutions de (1) pour lesquelles les ensembles $Z_i \cup \{\underline{0}\}$ forment des cônes sur $\mathbb{R}$ et seulement ces solutions.

9. On peut se borner dans toutes nos considération à $c = (c_1, \ldots, c_p)$ tel que $|c| = 1$ ou tel que $c_1 + \ldots + c_p = 1$ puisque pour ces $c$ les ensembles $D(c)$ sont les mêmes que pour $c$ précédents.

10. Considérons sur les droites de $\mathbb{R}^p$ la topologie donnée par la distance simple des points. On peut remplacer dans le théorème 3 et dans les remarques plus haut la presque-mesurabilité de $f$ par la condition que les ensembles $A_i(c)$, pour $i = 1, \ldots, p$ et chaque $c \in \mathbb{R}(p)$, jouissent de la propriété de Baire, en remplaçant en même temps dans la démonstration du théorème 3 le théorème de Steinhaus par le théorème de Piccard ([5],

p. 48).Cette condition et la presque-mesurabilité sont équivalentes pour les fonctions remplissantes (1) (c. à d. pour les cônes $Z_i \cup \{\underline{0}\}$, $i = 1, \ldots, p$, sur le corps $\mathbb{Q}$ des nombres rationnels, satisfaisants aux relations (8) et (9)), puisque elle sont équivalentes à la condition que les ensembles $Z_i \cup \{\underline{0}\}$ forment les cônes sur $\mathbb{R}$. On peut aussi remplacer ici la condition que les ensembles $A_i(c)$ jouissent de la propriété de Baire pour $i = 1, \ldots, p$ et un $c \in \mathbb{R}(p)$ (chaque $c \in \mathbb{R}(p)$) par la supposition que pour un $c \in \mathbb{R}(p)$ (pour chaque $c \in \mathbb{R}(p)$) il existe une suite $i_1, \ldots, i_p$ telle que l'ensemble $A_1^{i_1}(c) \cap \ldots \cap A_p^{i_p}(c)$ jouit de la propriété de Baire et il est de la deuxième catégorie. On ajoute ici cette dernière supposition puisque elle se trouve dans le théorème de Piccard et dans ce cas ne résulte pas des autres suppositions, par contre elle est remplie sans la supposition par au moins un des ensembles $A_i(c)$, car $D(c) = \bigcup_{i=1}^{p} A_i(c)$.

11. Il résulte de la remarque après le théorème 2 que les ensembles $Z_\nu \cup \{\underline{0}\}$ pour la solution arbitraire de (2) forment des cônes sur $\mathbb{R}$, donc cette solution est toujours presque-mesurable.

## 7. Prolongements

On peut facilement démontrer qu'il existe toujours une solution $g$ de (2) sur $\mathbb{R}(p) \cup \{\underline{0}\}$ qui est un prolongement de la solution $f = (f_1, \ldots, f_p)$ de (2) sur $\mathbb{R}(p)$ et

1° ce prolongement est unique ($g(\underline{0}) = (1, \ldots, 1)$) si et seulement s'il n'existe pas $k \in \{1, \ldots, p\}$ tel que $f_k(x)$ est idéntiquement égale à zero

et

2° dans le cas contraire chaque fonction

$$g = (g_1, \ldots, g_p) : \mathbb{R}(p) \cup \{\underline{0}\} \longrightarrow \mathbb{R}(p) \cup \{\underline{0}\}$$

telle que $g = f$ sur $\mathbb{R}(p)$ et pour laquelle $g_k(\underline{0}) = 1$ si $f_k$ n'est pas idéntiquement zéro et $g_k(\{\underline{0}\}) \in \{0, 1\}$ pour $f_k = 0$, forme une solution de (2) sur $\mathbb{R}(p) \cup \{\underline{0}\}$ qui est un prolongement de $f$ et inversement chaque prolongement doit être de cette forme.

S'il s'agit du prolongement à $\mathbb{R}^p$ de la solution $f$ de (2) sur $\mathbb{R}(p)$ ce prolongement existe si et seulement si $f = (f_k, \ldots, f_p)$ remplit la condition

$$\forall\, k \in \{1, \ldots, k\}\, [\exists\, x_0 \in \mathbb{R}(p) :\ f_k(x_0) = 0] \Longrightarrow f_k = 0,$$

c. à d. la condition

$$\forall\, k \in 1, \ldots, p :\ Z_k = \emptyset \text{ ou } Z_k = \mathbb{R}(p)$$

ou la condition

$$\forall k \in \{1, \ldots, p\} \, [\forall x \in \mathbb{R}(p) : f_k(x) = 0 \text{ ou } \forall x \in \mathbb{R}(p) : f_k(x) = \exp a_k(x)]$$

et dans ce cas ce prolongement $g = (g_1, \ldots, g_p)$ est unique et

$$g_k = 0 \text{ sur } \mathbb{R}^p \text{ si } f_k = 0 \text{ sur } \mathbb{R}(p)$$

et

$$g_k = \exp a_k \text{ sur } \mathbb{R}^p \text{ si } f_k = \exp a_k \text{ sur } \mathbb{R}(p).$$

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# Some consequences of a theorem of Liouville

**Abstract.** Let $E_n$ denote the $n$-dimensional Euclidean space and $S$ the group of Euclidean similarities. It is shown that the group $\langle g, S\rangle$ generated by $S$ and a single diffeomorphism $g$ outside $S$ has an orbit which is dense in $(E_n)^{n+1}$.

## 1. Introduction

By the theorem referred to in the title we mean Liouville's theorem on conformal mappings in space, that is, mappings which preserve all angles between smooth curves. It may be stated as follows.

Liouville's Theorem

*Any sufficiently smooth conformal mapping between connected open regions of a Euclidean space of dimension at least three is induced by a Möbius transformation acting on the whole space together with a point at infinity.*

In textbooks on differential geometry and related subjects this theorem is usually proved for differentiable mappings of class $C^3$ (see e.g. [3], p. 140, [4], vol. I, p. 373, or [12], vol. III, p. 310 and vol. IV, p. 13). However in 1958, Philip Hartman has proved it for $C^1$ mappings (see [6]) and more recently Yu. Rešetnyak has proved an even stronger theorem in which he makes no differentiability assumptions at all (see [8], [9]).

Let $E_n$ denote the $n$-dimensional Euclidean space and $S$ its group of automorphisms comprising Euclidean motions as well as similarities. Thus with respect to Cartesian coordinates $S$ consists of all mappings of the form

$$X \mapsto \lambda X \cdot M + V, \quad 0 \neq \lambda \in \mathbb{R}, \quad M \cdot M^{\top} = I \quad (M \text{ an orthogonal matrix}).$$

A matrix of the form $\lambda M$ where $M$ is orthogonal will be called *quasi-orthogonal.* For an arbitrary number $m$ let us denote by $(E_n)^{m+1}$ the set $E_n \times E_n \times \ldots \times E_n$ where the factor appears $m+1$ times. As $S$ acts on $E_n$ it also acts on the sets $(E_n)^{m+1}$.

For an $(m+1)$-tuple $(P_0, P_1, \ldots, P_m) \in (E_n)^{m+1}$ we shall denote by $[P_0, P_1, \ldots, P_m]$ the affine subspace spanned by these points. An $(m+1)$-

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tuple of points not contained in any subspace of dimension less than $m$ will be called *independent* or will be referred to as a non-degenerate $m$-simplex or just as an *$m$-simplex* for brevity.

THEOREM 1

*Let $m \leqslant n$ and let $g$ denote a differentiable $C^1$ mapping of $E_n$ onto itself which has a differentiable inverse. If $g \notin S$ then the group $G = \langle g, S \rangle$ generated by $g$ and the group $S$ of Euclidean similarities has an orbit $\Omega$ which is dense in $(E_n)^{m+1}$. Precisely speaking, any orbit containing a non-degenerate $m$-simplex is dense in $(E_n)^{m+1}$.*

The main tool in the proof is Liouville's theorem in the case where the mapping is defined in the whole of $E_n$, combined with the fact that the Euclidean group $S$ is maximal within the affine group (see [5], [7], or [11]). Although Liouville's theorem is not valid for planar regions in general, its analogue for the case when the region considered is the whole plane is still true. Therefore Theorem 1 holds also for $n = 2$.

Let $G$ be a group acting on $E_n$ and $\mathcal{D}$ a subset of $(E_n)^{m+1}$ which is invariant under the action of $G$ induced on $(E_n)^{m+1}$. We consider functions $f : \mathcal{D} \to \mathbb{R}$ where $\mathbb{R}$ denotes the field of real numbers satisfying the following functional equation:

$$f(g(P_0), g(P_1), \ldots, g(P_m)) = f(P_0, P_1, \ldots, P_m), \quad \text{for all } g \in G \tag{i.1}$$

Any function $f$ satisfying (i.1) with respect to a given group $G$ is called an invariant with respect to the group $G$. An invariant with respect to the group $S$ of Euclidean similarities will be called a *Euclidean invariant*. As an immediate consequence of the theorem above we obtain:

COROLLARY 1

*Let $m \leqslant n$. Let the bijective mapping $g : E_n \to E_n$ and its inverse be of class $C^1$ and assume* (i.1) *holds for a continuous Euclidean invariant $f$ and for $G = \langle S, g \rangle$. Assume further that $f$ is defined on a set which contains a non-degenerate $m$-simplex. Then either $g \in S$ or else $f$ is constant.*

Corollary 1 is a slight improvement of theorem 2 of [10], p. 107, which was proved without any reference to transitivity properties of the group $G$ that is, independently of Theorem 1. Let us call an invariant $f$ trivial if it assumes distinct values only on tuples $(P_0, P_1, \ldots, P_m)$ and $(Q_0, Q_1, \ldots, Q_m)$ which are not mapped onto each other by any bijection since they can be distinguished by means of the identity relation. Evidently non-trivial $(m+1)$-ary invariants exist with respect to a group $G$ if and only if the group is not $(m+1)$-fold transitive. The example of the affine group shows that the theorem and the corollary cannot be improved in a certain sense. For, by the maximality of $S$ within the affine group, the affine group is generated by any affine mapping not in $S$ together with $S$. Also the affine group is transitive on $m$-simplexes but not on arbitrary $(m+1)$-tuples. Hence non-trivial $(n+1)$-ary invariants exist for the affine group so that we cannot omit the continuity in the corollary. But the situation may change if we require additionally that $g$ is not an affine mapping. It is not known to the author whether there exist differentiable mappings $g$ other than affine ones such that $G = \langle g, S\rangle$ is not $(n+1)$-fold transitive.

Also the assumption that $m \leqslant n$ cannot be omitted. Here again the affine group provides us with a counterexample. Consider an $r$-simplex $P_0, P_1, \ldots, P_r$. Recall that each point $X$ of the subspace $[P_0, P_1, \ldots, P_r]$ can be written in a unique way as

$$X = \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_r P_r$$

where $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$ and $\lambda_0, \lambda_1, \ldots, \lambda_r$ are called the *barycentric coordinates of* $X$ with respect to $P_0, P_1, \ldots, P_r$. As a function of $P_0, P_1, \ldots, P_r$, and $X$ the $i$-th barycentric coordinate $\lambda_i = \lambda_i(P_0, P_1, \ldots, P_r, X)$ is a continuous affine invariant which is not constant. Let us now take $r = n$. From the fact that the point $X$ is uniquely determined by its barycentric coordinates $\lambda_0, \ldots, \lambda_n$ it follows easily that the affine group has no dense orbits on $(E_n)^{n+2}$.

Note however, that the invariants $\lambda_i = \lambda_i(P_0, P_1, \ldots, P_r, X)$ are no counterexamples against Corollary 1 since they do not contain any $(r+1)$-simplexes in their domain of definition. Hence the condition of the existence of a simplex in the domain of definition is also essential and cannot be omitted from the hypotheses of the corollary.

## 2. Proof of Theorem 1

The following Lemmas 2.1-2.3 will serve as preliminary steps towards the proof.

We start with an elementary fact about the Euclidean group, namely that there are no groups in between the Euclidean group and the affine group. Thus if $S \subset H \subseteq A$ where $H$ is a subgroup, then $H = A$. Denote by $GL_n(\mathbb{R})$ the group of all real $n \times n$ matrices with non-vanishing determinant and by $\mathbb{R}^* O_n(\mathbb{R})$ the group of all quasi-orthogonal matrices. Then it is easily seen that the above assertion is equivalent with the following:

Lemma 2.1

*Let* $H_n(\mathbb{R})$ *be a subgroup of* $GL_n(\mathbb{R})$ *properly containing the group* $\mathbb{R}^* O_n(\mathbb{R})$. *Then* $H_n(\mathbb{R}) = GL_n(\mathbb{R})$.

We consider a bijective mapping $g : E \to E$ such that $g$ and its inverse are of class $C^1$. As in the theorem let $G = \langle g, S\rangle$ and assume $g \notin S$. We express $g$ in terms of coordinates in the form

$$g(X) = (u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2, \ldots, x_n), \ldots, u_n(x_1, x_2, \ldots, x_n)).$$

Then with respect to an arbitrary point $P_0 = (x_{01}, x_{02}, \dots, x_{0n})$ we have (with $P = (x_1, x_2, \dots, x_n)$)

$$u_i(P) = u_i(P_0) + \sum_1^n \frac{\partial u_i}{\partial x_j}(x_j - x_{0j}) + o_i(P)$$

where

$$\lim_{P \to P_0} \frac{o_i(P)}{|P - P_0|} = 0,$$

where $|P - P_0|$ stands for the Euclidean distance between $P$ and $P_0$. Let $D$ denote the Jacobian matrix $D = \left(\frac{\partial u_i}{\partial x_j}\right)$. Then the above equations can be rewritten as

$$g(P) = g(P_0) + (P - P_0) \cdot D^\top + o(P), \tag{1}$$

where $o(P)$ is a vector valued function such that $\frac{o(P)}{|P-P_0|} \to 0$ as $|P - P_0| \to 0$.

When $n \geqslant 3$ from Liouville's theorem it follows:

Lemma 2.2

*There exists a point $P$ where $D$ is not quasi-orthogonal.*

Indeed, assume the contrary: the Jacobian matrix $D$ of $g$ is quasi-orthogonal at each point $P$.

An arbitrary $C^1$-mapping preserves angles between smooth curves going through a point $P$ if, and only if, its Jacobian matrix $D$ at $P$ is quasi-orthogonal. Hence $g$ preserves all angles between smooth curves, i.e. $g$ is a conformal mapping. When $n \geqslant 3$ it follows by Liouville's theorem that $g$ is induced by a Möbius transformation. Let us recall that Möbius transformations are bijective mappings of the set $E_n \cup \{\infty\}$ onto itself which can be composed from inversions at spheres or hyperplanes (see [2]). Alternatively, they can be characterized as maps preserving the system of point sets which are either given by spheres of $E_n$ or by hyperplanes of $E_n$ together with the point $\infty$. Since $g$ is defined on the whole space $E_n$ and maps $E_n$ onto itself we may identify $g$ with the Möbius transformation in question, which fixes the point $\infty$. It is well-known that any Möbius transformation fixing $\infty$, as a mapping of $E_n$ must belong to $S$. Thus $g \in S$ contrary to our assumption that $g \notin S$.

The case $n = 2$ needs some special attention since there exist conformal mappings of planar regions which are not induced by Möbius transformations. As is well-known however, it follows from the theorem of Casorati-Weierstraß that a conformal and one-to-one mapping of the entire complex plane is necessarily of one of the forms $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$. Since such mappings belong to $S$ it follows that Lemma 2.2 holds also when $n = 2$.

If $g(P) \neq P$ consider the mapping $h : E_n \to E_n$ given by

$$h(X) = g(X) - g(P) + P.$$

This mapping belongs to $G = \langle g, S \rangle$, fixes the point $P$ and has the same Jacobian matrix as $g$. Let $G_P$ denote the subgroup of $G$ fixing the point $P$. The Jacobian matrices of elements of $G_P$ taken at $P$ form a group containing the matrix $D$ and also all quasi-orthogonal matrices. It follows that this group is the full linear group $GL_n(\mathbb{R})$. If $Q$ is another point then $G_Q$ is conjugate to $G_P$ by a translation $t$ taking $P$ to $Q$, i.e. $G_Q = tG_Pt^{-1}$. This implies that the Jacobian matrices of elements of $G_Q$ also exhaust the full linear group.

LEMMA 2.3

*Let $P$ be an arbitrary point. Then for each invertible matrix $M$ there exists an element $h$ contained in the stabilizer $G_P$ having $M$ as its Jacobian matrix.*

Let $\Delta_1$ and $\Delta_2$ be $n$-simplexes with the vertices $P_0, P_1, \ldots, P_n$ and $Q_0, Q_1, \ldots, Q_n$, respectively. In order to prove the theorem it suffices to show that for a given $\varepsilon > 0$ there exists an $n$-simplex $\Delta_3$ in the orbit of $\Delta_1$ with vertices $P'_0, P'_1, \ldots, P'_n$ such that $|Q_i - P'_i| < \varepsilon$, $i = 0, 1, \ldots, n$. Since $G$ contains arbitrary translations we may assume here that $P_0 = Q_0 = O$ where $O$ denotes the origin. By Lemma 2.3 there exists a mapping $h \in G_O$ with the Jacobian matrix $A$ satisfying $P_i \cdot A^\top = Q_i$, $i = 1, \ldots, n$. For $h$ formula (1) becomes

$$h(P) = P \cdot A^\top + o(P), \quad P \to O.$$

Let $d = \max |P_i|$, $i = 1, \ldots, n$. Choose $\delta$ such that $|o(P)| < \frac{\varepsilon}{d}|P|$ if $|P| < \delta$. Choose $\lambda$ such that $\lambda|P_i| \leqslant \delta$. Then $|o(\lambda P_i)| < \frac{\varepsilon}{d}\lambda|P_i| \leqslant \varepsilon\lambda$. Let $f : E_n \to E_n$ denote the similarity, $f(X) = \lambda X$. Then $f^{-1}hf \in G$ and $f^{-1}(h(f(P_i)) = Q_i + Y_i$ where $|Y_i| = \lambda^{-1}|o(\lambda P_i)| < \varepsilon$. This proves the theorem.

REMARK 1

Theorem 1 can easily be extended to arbitrary differentiable mappings. To do this one has to use the stronger result of Yu. Rešetnyak rather than Liouville's theorem for $C^1$ mappings. All that needs to be done here is to show that Rešetnyak's condition of conformality (see [8]) is satisfied by a differentiable mapping at a point $P$ provided the Jacobian matrix of the mapping is quasi-orthogonal at $P$. Then Rešetnyak's result will ensure that a differentiable mapping which has quasi-orthogonal Jacobian matrix everywhere is a Möbius transformation. Hence Lemma 2.2 can be proved in a similar way as we have done it for $C^1$ mappings. The rest of the proof goes unchanged.

## 3. Continuous extensions of the affine group

In this section we consider extensions of the group $A$ of all affine mappings $X \mapsto X \cdot M + B$ of $E_n$ by a continuous transformation $f$, i.e., a bijective mapping $f : E_n \to E_n$ which is continuous in both directions.

THEOREM 2

*For any continuous transformation $f$ of $E_n$ either $f \in A$ or the group $G = \langle f, A \rangle$ is $(n+1)$-fold transitive on $E_n$.*

*Proof.* The proof will be achieved through the following elementary Lemmas 3.1-3.10 below.

Assume that for $1 \leqslant m \leqslant n$ there exist mutually distinct points $Q_0, Q_1, \ldots, Q_m$, $R$ such that $[Q_0, Q_1, \ldots, Q_m]$ has dimension $m$ and contains $R$ and that none of the transforms of the tuple $(Q_0, Q_1, \ldots, Q_m, R)$ spans a subspace of higher dimension. For brevity let us write

$$\rho(Q_0, Q_1, \ldots, Q_m, R)$$

if this is true. By this definition we may permute the points $Q_0, Q_1, \ldots, Q_m$ arbitrarily without disturbing the relation $\rho$. It should be clear that there exist relations of this type which are not empty. To see this we need only take $m = n$.

LEMMA 3.1

*If $m$ is the smallest number for which a non-empty relation $\rho$ exists then $G$ is $(m+1)$-fold transitive.*

Let $Q_0, Q_1, \ldots, Q_{m-1}, Q_m$ be distinct points which span a subspace of dimension less or equal $m-1$. Since the subgroup $A$ of $G$ is transitive on $m$-simplexes it suffices to show that there is an $h \in G$ such that $h(Q_0), h(Q_1), \ldots, h(Q_{m-1}), h(Q_m)$ is an $m$-simplex. Choose a maximal independent subset among these points. Since the ordering plays no role here, we may assume that $Q_0, Q_1, \ldots, Q_r$, $r < m$, are the points of this subset. Then since $m$ was assumed minimal there exists an $h_1 \in G$ such that

$$h_1(Q_0), h_1(Q_1), \ldots, h_1(Q_r), h_1(Q_{r+1})$$

are independent. If $r+2 \leqslant m$ and $h_1(Q_0), h_1(Q_1), \ldots, h_1(Q_{r+1}), h_1(Q_{r+2})$ are dependent we may find $h_2 \in G$ such that

$$h_2[h_1(Q_0)], h_2[h_1(Q_1)], \ldots, h_2[h_1(Q_{r+1})], h_2[h_1(Q_{r+2})]$$

are independent. Continuing in this way we find the required mapping $h \in G$.

LEMMA 3.2

*Let $h \in G$. Then*

$$\rho(Q_0, Q_1, \ldots, Q_m, R)$$

*implies*

$$\rho(h(Q_0), h(Q_1), \ldots, h(Q_m), h(R))$$

*provided that the points $h(Q_0), h(Q_1), \ldots, h(Q_m)$ are independent.*

This follows immediately from the definition of the relation $\rho$. Since all affine mappings are contained in $G$ it follows that if the tuples $(Q_0, Q_1, \ldots, Q_m, R)$ and $(Q'_0, Q'_1, \ldots, Q'_m, R')$ are affinely equivalent then the assertions

$$\rho(Q_0, Q_1, \ldots, Q_m, R) \quad \text{and} \quad \rho(Q'_0, Q'_1, \ldots, Q'_m, R')$$

are equivalent.

Lemma 3.3

*Assume that $\rho(Q_0, Q_1, \ldots, Q_m, R_\nu)$ holds for all elements of a sequence $(R_\nu)$ converging to a point $R$ distinct from $Q_0, Q_1, \ldots, Q_m$. Then*

$$\rho(Q_0, Q_1, \ldots, Q_m, R)$$

*holds as well.*

Let $h$ be an arbitrary element of $G$. Denote by $L_\nu$ the subspace spanned by the points $h(Q_0), h(Q_1), \ldots, h(Q_m), h(R_\nu)$ and by $L$ the subspace spanned by the points $h(Q_0), h(Q_1), \ldots, h(Q_m), h(R)$. Let $m_1 = m_1(h)$ be the largest from among those numbers $\dim L_\nu$ that occur inifinitely often in the sequence and choose a subsequence $R_\lambda$ such that $\dim L_\lambda = m_1$ for all $\lambda$. Note that $m_1 \leqslant m$ because of the assumption that $\rho(Q_0, Q_1, \ldots, Q_m, R_\nu)$ is true.

If $h(R)$ depends on the points $h(Q_0), h(Q_1), \ldots, h(Q_m)$ then obviously $\dim L = \dim[h(Q_0), h(Q_1), \ldots, h(Q_m)] \leqslant \dim L_\lambda = m_1$.

If $h(R)$ is independent of the remaining points $h(Q_0), h(Q_1), \ldots, h(Q_m)$ then the same is true of $h(R_\nu)$ at least for all $\nu$ larger than a certain number $N$. Then $\dim(L) = \dim(L_\lambda) = m_1$ for all $\lambda > N$.

Since $m_1(h) \leqslant m$ for all $h \in G$ it follows that $\rho(Q_1, Q_2, \ldots, Q_m, R)$ is true.

From now on we shall work with barycentric coordinates.

Lemma 3.4

*If $S = (Q_0, Q_1, \ldots, Q_m)$ and $S' = (Q'_0, Q'_1, \ldots, Q'_m)$ span subspaces of dimension $m$ and if $R, R'$ are points such that the barycentric coordinates of $R$ with respect to $S$ coincide with the barycentric coordinates of $R'$ with respect to $S'$ then $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is equivalent with $\rho(Q'_0, Q'_1, \ldots, Q'_m, R')$.*

This follows from the fact that the tuples

$$(Q_0, Q_1, \ldots, Q_m, R) \quad \text{and} \quad (Q'_0, Q'_1, \ldots, Q'_m, R')$$

are affinely equivalent if and only if the barycentric coordinates of $R$ with respect to $Q_0, Q_1, \ldots, Q_m$ are the same as those of $R'$ with respect to $Q'_0, Q'_1, \ldots, Q'_m$.

Lemma 3.5

*If $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true and $R$ is not contained in the subspace spanned by $Q_0, Q_1, \ldots, Q_{m-1}$ then $\rho(Q_0, Q_1, \ldots, Q_{m-1}, R, Q_m)$ is also true.*

This is true since $(Q_0, Q_1, \ldots, Q_{m-1}, R)$ is still an $m$-simplex if $(Q_0,\ Q_1, \ldots, Q_m)$ is, and since $Q_m \in [Q_0, \ldots, Q_{m-1}, R]$ if $R \in [Q_0, \ldots, Q_m]$. Apart from that for $\rho(Q_0, Q_1, \ldots, Q_{m-1}, R, Q_m)$ to be true we only require that the dimension of the subspace spanned by $h(Q_0),\ h(Q_1), \ldots,\ h(Q_m),\ h(R)$ never exceeds $m$ as $h$ runs through $G$. This follows from the assumption that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true.

Lemma 3.6

*If $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true and $R$ is contained in the subspace $S_{m-1}$ spanned by $Q_0, Q_1, \ldots, Q_{m-1}$ then $\rho(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$ is true for any point $P$ outside the subspace $S_{m-1}$.*

This follows from the fact that the tuple $(Q_0, Q_1, \ldots, Q_{m-1}, Q_m, R)$ is affinely equivalent with $(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$.

Let us now assume that $m$ is minimal so that $G$ is $(m+1)$-fold transitive.

Lemma 3.7

*If $m = 1$ then $f \in A$.*

If $m = 1$ then there exist points $Q_0, Q_1, R$ such that $\rho(Q_0, Q_1, R)$ holds, i.e., the points $Q_0$, $Q_1$, $R$ are distinct and collinear, and $g(Q_0)$, $g(Q_1)$, $g(R)$ are collinear for all $g \in G$. Let $L$ denote the line going through $Q_0, Q_1$, and $R$. We thus have a set $M_0 = \{Q_0, Q_1, R\}$ of collinear points such that for each $g \in G$ the set $g(M_0)$ will again be collinear. By interchanging the points $Q_0, Q_1, R$ if necessary, we may assume that $R$ belongs to the segment $Q_0Q_1$.

Let $R = \sigma Q_0 + \tau Q_1$ where $\sigma + \tau = 1$ and $\sigma, \tau$ are the barycentric coordinates of $R$ with respect to $Q_0$ and $Q_1$. If $\tau S = -\sigma Q_0 + Q_1$ then $Q_1 = \sigma Q_0 + \tau S$ whence $\rho(Q_0, S, Q_1)$. Similarly if $\sigma T = Q_0 - \tau Q_1$ then $Q_0 = \sigma T + \tau Q_1$ whence we have $\rho(T, Q_0, Q_1)$. The points $S$ and $T$ lie outside the segment $Q_0Q_1$ and on different sides of it. We are now going to enlarge the set $M_0$ successively.

First we may add the points $S, T$ to get $M_1 = \{T, Q_0, R, Q_1, S\}$ which still has the property that $g(M_1)$ is a set of collinear points for all $g \in G$. Secondly, for all pairs of consecutive points $C_1, C_2$ in such a set we may add the point $C = \sigma C_1 + \tau C_2$. Combining these steps alternately we arrive at a sequence $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_\nu \subseteq \ldots$ of sets such that $g(M_\nu)$ consists of collinear points for all $g \in G$ and $M = \bigcup M_\nu$ is dense on the line $L$. Consequently, also the points of $g(M)$ are collinear for all $g \in G$. Since the set $M$ is dense on $L$ and the mappings $g$ are continuous, it follows that the line $L$ is mapped onto another line by every $g \in G$. The same must be true for any other line $L_1$ since on $L_1$ we can construct a set of points $M_1$ analogous to the set $M$ on $L$.

Thus each $g \in G$ preserves lines and hence it is a semi-affine mapping of $E_n$. Since the field of real numbers has no automorphisms other than the identity, it follows that $G$ consists entirely of affine mappings, i.e., $G = A$ and consequently $f \in A$.

We shall now assume that $m > 1$ which is only possible if $G \neq A$, i.e., $f \notin A$.

LEMMA 3.8

*If there exists a point $R$ contained in $S_{m-1}$ such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true then $\rho(Q_0, Q_1, \ldots, Q_m, X)$ is true for every point $X \in S_{m-1}$ which is distinct from $Q_0, Q_1, \ldots, Q_{m-1}$.*

We can find a mapping $h \in G$ which fixes $Q_0, Q_1, \ldots, Q_{m-1}$ and maps $R$ to another arbitrarily chosen point $R_1$ inside $S_{m-1}$ which is distinct from $Q_0, Q_1, \ldots, Q_{m-1}$. Since $h$ is a homeomorphism it cannot transform all points of $E_n \setminus S_{m-1}$ into $S_{m-1}$. Let $Q$ be a point which remains outside, i.e., $h(Q) \in E_n \setminus S_{m-1}$. Choose an affine map $a$ which fixes $S_{m-1}$ pointwise and maps $h(Q)$ to $Q$. Then

$$ah(R) = R_1,\ ah(Q_0) = Q_0,\ ah(Q_1) = Q_1, \ldots,\ ah(Q_{m-1}) = Q_{m-1},$$

and

$$ah(Q) = Q.$$

By Lemmas 3.2 and 3.6 it follows that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$ is true and again by Lemma 3.6 also that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q_m, R_1)$ holds.

LEMMA 3.9

*If there exists a point $R$ such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds we can find points $P$ in the subspace $S_{m-1}$ generated by $Q_0$, $Q_1$, $\ldots$, $Q_{m-1}$ such that the relation $\rho(Q_0, Q_1, \ldots, Q_m, P)$ holds.*

If $R$ is contained in one of the subspaces spanned by the faces of the simplex then using an affine map $h_i$ that permutes the points $Q_0, Q_1, \ldots, Q_m$ appropriately, we can find points $R_i = h_i(R)$ in each of these subspaces such that the relation $\rho(h_i(Q_0), h_i(Q_1), \ldots, h_i(Q_m), h_i(R))$ is satisfied. Since we may arbitrarily permute the points $h_i(Q_0), h_i(Q_1), \ldots, h_i(Q_m)$ without disturbing $\rho$ it follows that $\rho(Q_0, Q_1, \ldots, Q_m, R_i)$.

Let us now assume that $R$ is not contained in any one of these subspaces. Then if $\lambda_0, \lambda_1, \ldots, \lambda_m$ are its barycentric coordinates, we have $0 \neq \lambda_i$ for $i = 0, 1, \ldots, m$. We may assume $\lambda_m \neq -1$. (If $\lambda_m = -1$ then $\lambda_i \neq -1$ for a suitable index $i$ and we may interchange the points $Q_i$ and $Q_m$.) Let $T(R)$ be the point which has barycentric coordinates $\lambda_0, \lambda_1, \ldots, \lambda_m$ with respect to $Q_0, Q_1, \ldots, Q_{m-1}, R$. Then the barycentric coordinates of $T(R)$ with respect to $Q_0, Q_1, \ldots, Q_m$ are $\lambda_0(1+\lambda_m), \lambda_1(1+\lambda_m), \ldots, \lambda_m^2$ and $T(R)$ is distinct from $Q_0, Q_1, \ldots, Q_m$ and not contained in any of the subspaces spanned by the faces of the simplex with vertices $Q_0, Q_1, \ldots, Q_m$.

Moreover by Lemma 3.4 we get $\rho(Q_0, Q_1, \ldots, R, T(R))$. We are now going to show that either $\rho(Q_0, Q_1, \ldots, Q_m, T(R))$ is also true or there exists a point

$P$ in the subspace $S_{m-1} = [Q_0, Q_1, \dots, Q_{m-1}]$ such that $\rho(Q_0, Q_1, \dots, Q_m, P)$ is satisfied.

To this end consider an arbitrary $h \in G$. Assume the points $h(Q_0), h(Q_1), \dots, h(Q_m)$ are dependent. Then obviously

$$\dim[h(Q_0), h(Q_1), \dots, h(Q_m), h(T(R))] \leqslant m.$$

Hence assume that $h(Q_0), h(Q_1), \dots, h(Q_m)$ are independent. If $h(R)$ is not contained in the subspace $[h(Q_0), h(Q_1), \dots, h(Q_{m-1})]$ then $h(T(R))$ belongs to the subspace

$$[h(Q_0), h(Q_1), \dots, h(Q_{m-1}), h(R)] = [h(Q_0), h(Q_1), \dots, h(Q_{m-1}), h(Q_m)].$$

If this is true for all $h$ for which $h(Q_0)$, $h(Q_1)$, $\dots$, $h(Q_m)$ are independent the relation $\rho(Q_0, Q_1, \dots, Q_m, T(R))$ is satisfied.

Otherwise there exists $h$ such that $h(Q_0)$, $h(Q_1)$, $\dots$, $h(Q_m)$ are independent but $h(R) \in [h(Q_0), h(Q_1), \dots, h(Q_{m-1})]$. Then we may choose an affine mapping $a$ such that $ah(Q_i) = Q_i$, $i = 0, 1, \dots, m$ and $ah(R) = P$ belongs to $[Q_0, Q_1, \dots, Q_{m-1}]$. By Lemma 3.2 this implies that $\rho(Q_0, Q_1, \dots, Q_m, P)$. Note that we are finished in this case.

Hence we may assume that $\rho(Q_0, Q_1, \dots, Q_m, T(R))$ is satisfied and that we may pass from a point $R$ to $T(R)$ in the way just explained whenever convenient. This means that we may assume that $|\lambda_m| \neq 1$. For, if $\lambda_m = 1$ and all the other $\lambda_i$ are $\pm 1$ we may first replace $R$ by $T(R)$ which has $\lambda_0(1 + \lambda_m) = \pm 2$ as its first barycentric coordinate. Thus $|\lambda_i| \neq 1$ for some $i$ and we interchange $Q_i$ and $Q_m$. We may even assume that $|\lambda_m| < 1$. For if $|\lambda_m| > 1$ we interchange the points $Q_m$ and $R$ which is possible because of Lemma 3.5. Then

$$-\lambda Q_m = \lambda_0 Q_0 + \dots + \lambda_{m-1} Q_{m-1} - R$$

and so

$$Q_m = -\frac{\lambda_0}{\lambda_m} Q_0 - \dots - \frac{\lambda_{m-1}}{\lambda_m} Q_{m-1} + \frac{1}{\lambda_m} R.$$

With this last assumption let us now construct a sequence $(R_n)$ taking $R_0 = R$ and $R_{n+1} = T(R_n)$. If this sequence breaks off at some stage we have found a point $P$ in $[Q_0, Q_1, \dots, Q_{m-1}]$ such that $\rho(Q_0, Q_1, \dots, Q_m, P)$ is satisfied and we are finished. Hence we may assume that the sequence does not break off. It is easy to see by induction on $n$ that the barycentric coordinates of $R_n$ are

$$\mu_i = \lambda_i(1 + \lambda_m + \dots + \lambda_m^{2^n - 1}), \quad i = 0, 1, \dots, m-1; \qquad \mu_m = \lambda_m^{2^n}.$$

Since $|\lambda_m| < 1$ the sequence $(R_n)$ converges to a point $S$ with barycentric coordinates $\frac{\lambda_i}{1-\lambda_m}$ for $i = 0, 1, \dots, m-1$ and $0$ for $i = m$. This point $S$ lies within the face $Q_0, Q_1, \ldots, Q_{m-1}$ of the simplex $Q_0, Q_1, \ldots, Q_m$ and from the assumption $m > 1$ it follows that $S$ is distinct from $Q_0$, $Q_1$, ..., $Q_{m-1}$. Since its last coordinate is zero, it is also distinct from $Q_m$. Hence by Lemma 3.3 it follows that $\rho(Q_1, Q_2, \ldots, Q_m, S)$ is true. This proves Lemma 3.9.

LEMMA 3.10

*If there is a point $R$ contained in $S_{m-1}$ such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds then for any $R_1$ outside $S_{m-1}$ there are points $Q$ such that*

$$\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$$

*is true. Moreover, for a given point $R_1$ the set of possible points $Q$ is open and dense in $E_n$.*

Choose $R_1$ outside $S_{m-1}$ and choose $h \in G$ such that $h(Q_0) = Q_0$, $h(Q_1) = Q_1$, ..., $h(Q_{m-1}) = Q_{m-1}$, and $h(R) = R_1$. This is possible since $G$ is $(m+1)$-fold transitive.

Choose a point $P$ outside $S_{m-1}$ whose image $h(P)$ also does not belong to the subspace $S_{m-1} = [Q_0, Q_1, \ldots, Q_{m-1}]$. This means that $P$ must not belong to the two surfaces $[Q_0, Q_1, \ldots, Q_{m-1}]$ and $h^{-1}([Q_0, Q_1, \ldots, Q_{m-1}])$ of (topological) dimension $m - 1$.

Since $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds and $R \in S_{m-1}$ by Lemma 3.6 it follows that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$ does. Hence $\rho(Q_0, Q_1, \ldots, Q_{m-1}, h(P), R_1)$ is true because $h(R) = R_1$ and $h(Q_i) = Q_i$, $i = 0, 1, \ldots, m-1$. Thus we may take $Q = h(P)$. The subset from which we can choose $P$ is open and dense in $E_n$. Therefore the subset from which $Q = h(P)$ can be chosen is also open and dense.

We are now in a position to complete the proof of Theorem 2. For, from Lemma 3.9 it follows that the hypothesis of Lemma 3.10 can always be satisfied. Thus we may conclude that $m = n$, for otherwise we could find points $Q$ and $R_1$ such that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$ is true but $R_1 \notin [Q_0, Q_1, \ldots, Q_{m-1}, Q]$, a contradiction. Then G is $(n+1)$-fold transitive, because of Lemma 3.1.

REMARK 2

Note that the conclusion of Theorem 2 is not true if $f$ is not continuous. To see this, take a subfield $K$ of $\mathbb{R}$. Consider $E_n$ as a vector space over $K$ and let $f$ be a linear mapping of $E_n$ over $K$ which is not linear over the field $\mathbb{R}$. Such a mapping $f$ is necessarily discontinuous and $G = \langle f, A\rangle$ is a group of affine mappings of $E_n$ considered as an affine space over $K$ and hence not $(n+1)$-fold transitive.

The author does not know any example of a group $G = \langle f, A\rangle$ with $f$ continuous which admits non-trivial $k$-ary invariants for some $k > n+1$. In fact, the groups considered in Theorem 2 might well turn out to be $k$-fold transitive for all $k$. At the present time however, this seems an open problem.

In this context it is worth noting that the groups $G = \langle f, A \rangle$ are a special kind of Jordan groups and therefore much further information is available on them (see [1]).

## 4. A special class of mappings

Theorem 1 raises the following question: if $g$ is not an affine map maybe more could be said, e.g. that the group $\langle g, S \rangle$ is $(n+1)$-fold transitive. In this section we consider bijective mappings which are differentiable (in both directions) and which are themselves not affine but are such that the group $\langle g, S \rangle$ contains affine mappings not in $S$. In this case $\langle g, S \rangle$ must contain the whole affine group and hence is $(n+1)$-fold transitive.

THEOREM 3

*Let $g$ be as in Theorem 1. Assume further that $g$ is not affine but $\langle g, S \rangle$ contains some affine mapping not in $S$. Then $\langle g, S \rangle$ contains the affine group and is $(n+1)$-fold transitive on $E_n$.*

Since differentiable mappings are continuous this follows from the maximality of $S$ in $A$ and from Theorem 2 proved in the previous section. Note that the differentiability is not required in this theorem. It has been kept in the hypothesis to depart as little as possible from the context of Theorem 1.

We will now show by means of an example that the class of $C^1$ mappings considered in Theorem 3 is not empty. For this purpose we may look for a differentiable mapping $\sigma$ given by

$$\sigma(x, y) = (G(x, y), y)$$

which is itself not linear but whose iterative square $\sigma \circ \sigma$ is the linear mapping $(x, y) \to (x + uy, y)$ for some $u \neq 0$. This means that we are looking for a differentiable function $G(x, y)$ which satisfies the functional equation

$$G[G(x, y), y] = x + uy \tag{4.1}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying

$$g(x + 1) = g(x) + 1. \tag{4.1a}$$

We can get such a function from any periodic function $h$ with period 1 by setting $g(x) = x + h(x)$. We may choose $h$ in such a way that $-\frac{1}{3} \leqslant h'(x) \leqslant \frac{1}{2}$. Then $g'(x) = 1 + h'(x)$ will be positive within the bounds $\frac{2}{3} \leqslant g'(x) \leqslant \frac{3}{2}$ so that $g(x)$ is strictly increasing. We may also assume that 1 is the smallest period of the function $h$.

For each value of $y$ we modify $h(x)$ by a factor $\lambda(y)$ such that $0 \leqslant \lambda(y) \leqslant 1$ and $\lambda(y) \to 0$ as $y \to 0$. We may choose $\lambda(y) = 1 - e^{-y^2}$ for this purpose. Set $g_y(x) = x + \lambda(y)h(x)$.

This function still satisfies (4.1a) and $\frac{2}{3} \leqslant g'_y(x) \leqslant \frac{3}{2}$, so its inverse $g_y^{-1}$ exists. Let us now set

$$G(x,y) = \begin{cases} uyg_y^{-1}\left[g_y\left(\frac{x}{uy}\right) + \frac{1}{2}\right] & \text{when } y \neq 0, \\ x & \text{otherwise.} \end{cases}$$

An easy computation shows that $G$ satisfies equation (4.1). Therefore the (iterative) square of the mapping $\sigma$ is the linear mapping given by $\sigma \circ \sigma(x,y) = (x + uy, y)$. Hence $\sigma$ must be bijective.

It remains to check that $\sigma$ is differentiable. It is easily seen that the inverse function $g_y^{-1}$ also satisfies (4.1a) and so it can be written in the form

$$g_y^{-1}(x) = x + \bar{h}(x,y)$$

where $\bar{h}(x,y)$ is periodic in $x$ with period 1. From the bounds for $g'_y(x)$ we get $-\frac{1}{3} \leqslant \frac{\partial \bar{h}(x,y)}{\partial x} \leqslant \frac{1}{2}$. For the sake of symmetry let us write $h(x,y)$ instead of $\lambda(y)h(x)$. From $x = g_y(g_y^{-1}(x))$ we obtain

$$h(x + \bar{h}(x,y), y) + \bar{h}(x,y) = 0. \tag{4.2}$$

The value $\bar{h}(x,y)$ of the function $\bar{h}$ is uniquely determined by this relation for each pair $(x,y)$ and from the implicit function theorem it follows that the function $\bar{h}$ has continuous partial derivatives if $h$ has. In fact, we may use the above relation to compute the partial derivatives of $\bar{h}$ with respect to $x$ and $y$

$$\frac{\partial \bar{h}(x,y)}{\partial x} = -\frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\left(1 + \frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\right)^{-1} \tag{4.3$a$}$$

$$\frac{\partial \bar{h}(x,y)}{\partial y} = -\frac{\partial h}{\partial y}\left(x + \bar{h}(x,y), y\right)\left(1 + \frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\right)^{-1} \tag{4.3$b$}$$

It follows from these formulae that since $\frac{\partial h}{\partial x}(x,y) = o(y)$ and $\frac{\partial h}{\partial y}(x,y) = O(y)$ as $y \to 0$, independently of the first argument $x$, the same remains true for $\frac{\partial \bar{h}}{\partial x}(x,y)$ and $\frac{\partial \bar{h}}{\partial y}(x,y)$.

We may now rewrite the expression for $G(x,y)$ in the case $y \neq 0$ as follows

$$G(x,y) = x + uyh(\xi, y) + \frac{uy}{2} + uy\bar{h}(\eta, y), \tag{4.4}$$

where

$$\xi = \frac{x}{uy}, \quad \eta = \xi + h(\xi, y) + \frac{1}{2}.$$

When $y \neq 0$ we may calculate the partial derivatives $G_x$ and $G_y$ in a straightforward manner. Thus

$$G_x(x,y) = \left(1 + \frac{\partial h}{\partial x}(\xi, y)\right)\left(1 + \frac{\partial \bar{h}}{\partial x}(\eta, y)\right). \tag{4.5}$$

It follows that the function $G_x$ defined by the above expression is continuous (for $y \neq 0$) and that $G_x(x,y) \to 1$ as $y \to 0$ regardless of the first argument. Thus $G_x$ is continuous everywhere. For $G_y$ we obtain in the case $y \neq 0$

$$\begin{aligned} G_y(x,y) = {} & \frac{u}{2} + u\left[h(\xi,y) + \bar{h}(\eta,y)\right] \\ & + uy\left[\frac{\partial h}{\partial y}(\xi,y) + \frac{\partial \bar{h}}{\partial y}(\eta,y) + \frac{\partial \bar{h}}{\partial x}(\eta,y)\frac{\partial h}{\partial y}(\xi,y)\right] \\ & - \frac{x}{y}\left[\frac{\partial h}{\partial x}(\xi,y) + \frac{\partial \bar{h}}{\partial x}(\eta,y)\left(1 + \frac{\partial h}{\partial x}(\xi,y)\right)\right]. \end{aligned}$$

It is easy to see that the function $G_y$ is continuous when $y \neq 0$. For $y = 0$ it follows from (4.4) that $G_y(x,0) = \frac{u}{2}$. Moreover since, as remarked above, $\frac{1}{y}\frac{\partial h}{\partial x}$ and $\frac{1}{y}\frac{\partial \bar{h}}{\partial x}$ tend to zero as $y \to 0$ while $\frac{\partial h}{\partial y}$ and $\frac{\partial \bar{h}}{\partial y}$ remain bounded it follows that $G_y(\zeta,y) \to \frac{u}{2}$ as $y \to 0$ regardless of the first argument. Thus $G_y$ is continuous everywhere and we have proved that $\sigma$ is differentiable.

For the proof that the inverse of $\sigma$ is also differentiable we only need to show that the Jacobian of $\sigma$ never vanishes. (Note that this does not follow from the bijectivity of $\sigma$ which has been established above.) But the Jacobian of $\sigma$ at $(x,y)$ is equal to $G_x(x,y)$ and from (4.5) it follows that $G_x(x,y) \neq 0$ for $y \neq 0$. For $y = 0$ this is still true since $G_x = 1$ in this case.

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László Székelyhidi

# On a paper of T.M.K. Davison

**Abstract.** In his paper [3] the author shows that Chebyshev polynomials of the first kind show up in relation with d'Alembert's functional equation. Here we point out a similar property of Chebyshev polynomials concerning the square norm equation and we exhibit that the reason is due to close relations with hypergroups.

## 1. Introduction

In [3] the author presents the following theorem.

THEOREM 1
*Let $R$ be a commutative ring and let $f : \mathbb{Z} \to R$ a function with $f(0) = 1$. Then*

$$f(m+n) + f(m-n) = 2f(m)f(n)$$

*holds for all $m, n$ in $\mathbb{Z}$ if and only if*

$$f(n) = T_{|n|}(f(1))$$

*holds for all $n$ in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.*

In this paper we show that this theorem is a consequence of the fact that d'Alembert's equation has a close connection to the exponentials of a special polynomial hypergroup, the Chebyshev hypergroup. In addition, we show a similar connection between the square-norm equation, the additive functions of the Chebyshev hypergroup and the derivatives of the Chebyshev polynomials of the first kind (simply: Chebyshev polynomials).

In this work concerning hypergroups and polynomial hypergroups we refer to [1], [2], [5]. Concerning exponential functions and additive functions on polynomial hypergroups see also [7], [8], [9], [10].

AMS (2000) Subject Classification: .

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## 2. Discrete polynomial hypergroups

An important special class of hypergroups is closely related to orthogonal polynomials.

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be real sequences with the following properties: $c_n > 0$, $b_n \geqslant 0$, $a_{n+1} \geqslant 0$ for all $n$ in $\mathbb{N}$, moreover $a_0 = 0$, and $a_n + b_n + c_n = 1$ for all $n$ in $\mathbb{N}$. We define the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = a_nP_{n-1}(x) + b_nP_n(x) + c_nP_{n+1}(x)$$

for all $n \geqslant 1$ and $x$ in $\mathbb{R}$. The following theorem holds.

Theorem 2

*If the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies the above conditions, then there exist constants $c(n,m,k)$ for all $n,m,k$ in $\mathbb{N}$ such that*

$$P_nP_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k$$

*holds for all $n,m$ in $\mathbb{N}$.*

*Proof.* By the theorem of Favard (see [4], [6]) the conditions on the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ imply that there exists a probability measure $\mu$ on $[-1,1]$ such that $(P_n)_{n\in\mathbb{N}}$ forms an orthogonal system on $[-1,1]$ with respect to $\mu$. As $P_n$ has degree $n$, we have

$$P_nP_m = \sum_{k=0}^{n+m} c(n,m,k)P_k$$

for all $n,m$ in $\mathbb{N}$, where

$$c(n,m,k) = \frac{\int_{[-1,1]} P_kP_nP_m\,d\mu}{\int_{[-1,1]} P_k^2\,d\mu}$$

holds for all $n,m,k$ in $\mathbb{N}$. The orthogonality of $(P_n)_{n\in\mathbb{N}}$ with respect to $\mu$ implies $c(n,m,k) = 0$ for $k > n+m$ or $n > m+k$ or $m > n+k$. Hence our statement is proved.

The formula in the theorem is called *linearization formula*, and the coefficients $c(n,m,k)$ are called *linearization coefficients*. The recursive formula for the sequence $(P_n)_{n\in\mathbb{N}}$ implies $P_n(1) = 1$ for all $n$ in $\mathbb{N}$, hence we have

$$\sum_{k=|n-m|}^{n+m} c(n,m,k) = 1$$

for all $n$ in $\mathbb{N}$. It may or may not happen that $c(n,m,k) \geqslant 0$ for all $n,m,k$

in $\mathbb{N}$. If it happens then we can define a hypergroup structure on $\mathbb{N}$ by the following rule:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)\delta_k$$

for all $n, m$ in $\mathbb{N}$, with involution as the identity mapping and with $e$ as 0. The resulting commutative hypergroup is called *the discrete polynomial hypergroup associated with the sequence* $(P_n)_{n\in\mathbb{N}}$.

As an example we consider the hypergroup associated with the Chebyshev polynomials. The corresponding recurrence relation is

$$xT_n(x) = \frac{1}{2}\big(T_{n+1}(x) + T_{n-1}(x)\big)$$

for all $n \geqslant 1$ and $x$ in $\mathbb{R}$. By induction it follows that

$$T_m(x)T_n(x) = \frac{1}{2}\big(T_{n+m}(x) + T_{|n-m|}(x)\big)$$

holds for all $m, n$ in $\mathbb{N}$ and $x$ in $\mathbb{R}$. Hence in this case the linearization coefficients are nonnegative, and the resulting hypergroup associated with the Chebyshev polynomials is the *Chebyshev hypergroup*.

On any discrete polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ and for any function $f : \mathbb{N} \to \mathbb{C}$ one defines

$$f(m * n) = \sum_{k=|m-n|}^{m+n} c(n,m,k)f(k)$$

for all $n, m$ in $\mathbb{N}$. Using this concept we call the function $f : \mathbb{N} \to \mathbb{C}$ an *exponential*, if $f(0) = 1$ and

$$f(m * n) = f(m)f(n)$$

for all $m, n$ in $\mathbb{N}$, and *additive*, if

$$f(m * n) = f(m) + f(n)$$

holds for all $m, n$ in $\mathbb{N}$. Observe that we can give a meaning to $f(m * n)$ also in the case if $f$ has its range in a commutative ring with unit, assuming that the coefficients $c(n, m, k)$ are nonnegative integers. This happens in the case of the Chebyshev hypergroup. Another special property of the Chebyshev polynomials of the first kind is that their coefficients are integers which makes it possible to define them in the obvious way on an arbitrary commutative ring $R$ with unit. In this case $T_0(x) = e$, the unit of the ring, $T_1$ is the identity mapping on $R$ and for $T_{n-1}$, $T_n$ and $T_{n+1}$ the above recursive relation holds.

It follows that for any complex valued function $f$ defined on the Chebyshev hypergroup and having values in the commutative ring $R$ with unit $e$ we have

$$f(m*n) = \frac{1}{2}\big(f(m+n) + f(|m-n|)\big)$$

for all $m, n$ in $\mathbb{N}$. Hence $R$-valued exponentials on the Chebyshev hypergroup are characterized by $f(0) = e$ together with the functional equation

$$\varphi(m+n) + \varphi(|m-n|) = 2\varphi(m)\varphi(n),$$

and $R$-valued additive functions on the Chebyshev hypergroup are characterized by the functional equation

$$a(m+n) + a(|m-n|) = 2a(m) + 2a(n).$$

Here we recall the following two theorems which motivate our observations.

THEOREM 3
*Let $K = (\mathbb{N}, *)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $\varphi : \mathbb{N} \to \mathbb{C}$ is an exponential on $K$ if and only if there exists a complex number $z$ such that*

$$\varphi(n) = P_n(z)$$

*holds for all $n$ in $\mathbb{N}$.*

THEOREM 4
*Let $K = (\mathbb{N}, *)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $a : \mathbb{N} \to \mathbb{C}$ is an additive function on $K$ if and only if there exists a complex number $c$ such that*

$$a(n) = cP_n'(1)$$

*holds for all $n$ in $\mathbb{N}$.*

For the proofs of these theorems see e.g. [2], [7], [8], [9], [10]. We can easily extend these results to $R$-valued functions on the Chebyshev hypergroup.

THEOREM 5
*Let $R$ be a commutative ring with unit. The function $\varphi : \mathbb{N} \to R$ is an exponential with respect to the Chebyshev hypergroup if and only if there exists an element $r$ in $R$ such that*

$$\varphi(n) = T_n(r)$$

*holds for all $n$ in $\mathbb{N}$.*

*Proof.* Denote by $e$ the unit of $R$. Let $r$ be a fixed element of $R$ and let $\varphi(n) = T_n(r)$ for all $n$ in $\mathbb{N}$. By the linearization formula for the Chebyshev polynomials of the first kind we have

$$\varphi(m*n) = \frac{1}{2}\big(\varphi(m+n) + \varphi(|m-n|)\big)$$

$$
\begin{aligned}
&= \frac{1}{2}\big(T_{m+n}(r) + T_{|m-n|}(r)\big) = T_m(r)T_n(r) \\
&= \varphi(m)\varphi(n)
\end{aligned}
$$

for all $m, n$ in $\mathbb{N}$, that is, $\varphi$ is an exponential. Conversely, let $\varphi : \mathbb{N} \to R$ be an exponential and we define $\psi(n) = T_n(\varphi(1))$ for all $n$ in $\mathbb{N}$. Then $\psi(0) = e = \varphi(0)$ and $\psi(1) = \varphi(1)$. On the other hand, for any positive integer $n$ we have

$$
\begin{aligned}
\psi(n * 1) &= \frac{1}{2}\big(\psi(n+1) + \psi(n-1)\big) \\
&= \frac{1}{2}\big(T_{n+1}(\varphi(1)) + T_{n-1}(\varphi(1))\big) = T_n(\varphi(1))T_1(\varphi(1)) \\
&= \psi(n)\varphi(1),
\end{aligned}
$$

which means

$$\psi(n+1) = 2\psi(n)\varphi(1) - \psi(n-1).$$

Similarly, it follows for all positive integer $n$ that

$$\varphi(n+1) = 2\varphi(n)\varphi(1) - \varphi(n-1).$$

From this we infer that $\varphi = \psi$ and the theorem is proved.

THEOREM 6

*Let $R$ be a commutative ring with unit. The function* $: \mathbb{N} \to R$ *is additive with respect to the Chebyshev hypergroup if and only if there exists an element $c$ in $R$ such that*

$$a(n) = cT_n'(1)$$

*holds for all $n$ in* $\mathbb{N}$.

*Proof.* First of all we remark that $T_n'(1)$ is an integer for all $n$ in $\mathbb{N}$. This follows from the equation

$$T_m'(1) + T_n'(1) = \frac{1}{2}\big(T_{m+n}'(1) + T_{|m-n|}'(1)\big)$$

which can be obtained from the linearization formula by differentiation and then substituting $x = 1$. Denote by $e$ the unit of $R$. Let $c$ be a fixed element of $R$ and let $a(n) = cT_n'(1)$ for all $n$ in $\mathbb{N}$. The above formula shows that $a$ is additive. Conversely, let $a : \mathbb{N} \to R$ be additive and we define $\psi(n) = a(1)T_n'(1)$ for all $n$ in $\mathbb{N}$. Then $\psi(0) = 0 = a(0)$ and $\psi(1) = a(1)$. On the other hand, for any positive integer $n$ we have

$$
\begin{aligned}
\psi(n * 1) &= \frac{1}{2}\big(\psi(n+1) + \psi(n-1)\big) \\
&= \frac{1}{2}a(1)\big(T_{n+1}'(1) + T_{n-1}'(1)\big) = a(1)\big(T_n'(1) + T_1'(1)\big) \\
&= \psi(n) + a(1),
\end{aligned}
$$

which means that

$$\psi(n+1) = 2\psi(n) - \psi(n-1) + 2a(1).$$

Similarly, it follows for all positive integer $n$ that

$$a(n+1) = 2a(n) - a(n-1) + 2a(1).$$

From this we infer that $a = \psi$ and the theorem is proved.

## 3. Exponential and additive functions on the Chebyshev hypergroup

Let $R$ be a commutative rung with unit $e$. As we have seen, a function $\varphi : \mathbb{N} \to R$ is an exponential on the Chebyshev hypergroup if and only if it satisfies $\varphi(0) = e$ and

$$\varphi(m+n) + \varphi(|m-n|) = 2\varphi(m)\varphi(n)$$

for all $m, n$ in $\mathbb{N}$, which is a slight modification of d'Alembert's functional equation. Similarly, a function $a : \mathbb{N} \to R$ is an additive function on the Chebyshev hypergroup if and only if it satisfies

$$a(m+n) + a(|m-n|) = 2a(m) + 2a(n)$$

for all $m, n$ in $\mathbb{N}$, which is a slight modification of the square-norm functional equation. Consequently, according to the above results we have the following result, which is Theorem 1 above from [3].

THEOREM 7

*Let $R$ be a commutative ring with unit $e$. Let $f : \mathbb{Z} \to R$ be a function with $f(0) = e$. Then*

$$f(m+n) + f(m-n) = 2f(m)f(n)$$

*holds for all $m, n$ in $\mathbb{Z}$ if and only if*

$$f(n) = T_{|n|}(f(1))$$

*holds for all $n$ in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.*

*Proof.* It is obvious that $f$ is even, hence it satisfies

$$f(m+n) + f(|m-n|) = 2f(m)f(n)$$

for all $m, n$ in $\mathbb{N}$. This means, that $f$ is an $R$-valued exponential of the Chebyshev hypergroup, and by Theorem 5 we have

$$f(n) = T_n(r)$$

for all $n$ in $\mathbb{N}$ with some complex $r$ in $R$. For $n = 1$ we have $r = f(1)$ and the theorem is proved.

The case of additive functions can be treated similarly.

THEOREM 8
*Let $R$ be a commutative ring with unit. Let $f : \mathbb{Z} \to R$ be a function. Then*

$$f(m+n) + f(m-n) = 2f(m) + 2f(n)$$

*holds for all $m, n$ in $\mathbb{Z}$ if and only if*

$$f(n) = f(1)T'_{|n|}(1)$$

*holds for all $n$ in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.*

*Proof.* It is obvious that $f(0) = 0$, which implies that $f$ is even, hence it satisfies

$$f(m+n) + f(|m-n|) = 2f(m) + 2f(n)$$

for all $m, n$ in $\mathbb{N}$. This means, that $f$ is an additive function on the Chebyshev hypergroup, and by Theorem 6 we have

$$f(n) = cT'_n(1)$$

for all $n$ in $\mathbb{N}$ with some $c$ in $R$. For $n = 1$ we have $c = f(1)$ and the theorem is proved.

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Report of Meeting

# 8th International Conference on Functional Equations and Inequalities, Złockie, September 10-15, 2001

The Eighth International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University in Kraków since 1984, was held from September 10 to September 15, 2001, in the hotel "Geovita" at Złockie. Circumstances independent of the organizers of both meetings caused that the Conference took place only three weeks later than the annual, 39th Symposium on Functional Equations held at Sandbjerg (Denmark, August 12-18, 2001).

A support of the Polish State Committee for Scientific Research (KBN) is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Michał Śliwa, the Rector Magnificus of the Pedagogical University in Kraków, who greeted the participants, thanked the organizers and wished a fruitful and nice stay in this beatiful region of Poland.

There were 56 participants who came from 8 countries: Austria (1), Germany (3), France (1), Hungary (8), Slovenia (1), The U.S.A. (1), Yugoslavia (1), and from Poland: Bielsko-Biała (1), Gdańsk (1), Gliwice (1), Katowice (11), Kraków (22), Rzeszów (3), Warszawa (1).

During 18 sessions 47 talks were delivered, mainly on functional equations in several variables and their stability (also for conditional equations), functional equations steming from independence of random matrices, special Banach space operators, iterative functional equations, iteration theory, equations and inclusions for multivalued functions and on convex functions. The following particular topics were also dealt with: generalized derivatives, ODEs in metric spaces and PDEs solved via recurrences, functional equations on quasigroups, hypergroups and other general structures, geometry of matrices, iterative functional inequalities. There were 7 contributions to problems-and-remarks sessions.

The organizing Committee was chaired by Professors Dobiesław Brydak and Bogdan Choczewski. Dr. Jacek Chmieliński acted as a scientific secretary. Miss Ewa Dudek, Miss Janina Wiercioch and Mr Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference office at Złockie.

At the same week the Annual Meeting of the Polish Mathematical Society was held in Nowy Sącz, the capital of the region. For the first time in the history of these meetings functional equations were included in the programme. Professor Roman Ger gave an invited lecture entitled *Równania funkcyjne – zarys rozwoju i aktualny stan badań* on Wednesday, September 12. Since this afternoon there were no sessions, Polish participants were able to attend Prof. Ger's talk.

Deeply moved by the tragic events in the United States of America, the participants expressed their sympathy and solidarity with the American nation by means of forming a hand-to-hand Solidarity Chain after Professor Thomas Riedel's (Louisville, KY) talk at Wednesday, September 12 and by commemorative suspending at noon the session on Friday, September 14.

The Conference was closed by Professor Dobiesław Brydak. The 9th ICFEI is planned to be held in September, 2003.

The Chairmen want to cordially thank the participants for their coming, presenting valuable contributions and creating the unique atmosphere of friendship and solidarity. They express the best thanks to the members of the whole office staff at Złockie for their effective and dedicated work and helpful assistance, and to the managers of the hotel "Geovita" for their hospitality and quality of services.

The abstracts of talks are printed in the alphabetical order, and the problems and remarks presented chronologically. The careful and efficient work of Dr. J. Chmieliński on completing the material and preparing (together with Mr W. Wilk) the present report for printing is acknowledged with thanks.

*Bogdan Choczewski*

## Abstracts of Talks

**Roman Badora** *On almost periodic spherical functions*

We consider the functional equation

$$M_k(f(x+ky)) = g(x)h(y), \quad x, y \in G,$$

where $G$ is a topological Abelian group, $K$ a subgroup of its automorphisms, $M$ stands for the invariant mean on the space of almost periodic functions defined on $K$ and $f, g, h : G \to \mathbb{C}$ are unknown functions. We show that almost periodic solutions of this equation can be expressed by means of characters of the group $G$. Our considerations are motivated by the result of H. Stetkær on continuous solutions of the equation

$$\int_K f(x+ky)\,d\mu(k) = g(x)h(y), \quad x,y \in G,$$

where $G$ and $K$ are compact.

**Anna Bahyrycz** *On the indicator plurality function*

Joint work with Zenon Moszner.

A solution of the conditional functional equation:

$$f(x)\cdot f(y) \neq \underline{0} \Longrightarrow f(x+y) = f(x)\cdot f(y),$$

for which there exists a number $r \in \mathbb{R}(1) \setminus \{1\}$ such that:

$$f(rx) = f(x),$$

where $f : \mathbb{R}(n) := [0,\infty)^n \setminus \{\underline{0}\} \to \mathbb{R}(n)$, $\underline{0} := (0,\ldots,0) \in \mathbb{R}^n$ and $x+y := (x_1+y_1,\ldots,x_n+y_n)$, $x\cdot y := (x_1y_1,\ldots,x_ny_n)$, $rx := (rx_1,\ldots,rx_n)$, for $x = (x_1,\ldots,x_n) \in \mathbb{R}(n)$, $y = (y_1,\ldots,y_n) \in \mathbb{R}(n)$, is called an *indicator plurality function.*

We show that this function $f$, if $r$ is a transcedental number, must have its values in the set $0(n) := \{0,1\}^n \setminus \{\underline{0}\}$ if $n \leqslant 2$, and can have the values off the set $0(n)$ if $n > 2$.

The question under which assumptions the indicator plurality function must have its values in the set $0(n)$ has been posed by Z. Moszner.

**Karol Baron** *Orthogonality and additivity modulo a discrete subgroup*

Let $E$ be a real inner product space of dimension at least 2, $G$ a topological Abelian group, and $K$ a discrete subgroup of $G$. Following [1] and [2] we consider functions $f : E \to G$ continuous at least at one point and such that

$$f(x+y) - f(x) - f(y) \in K \quad \text{for all orthogonal } x,y \in E.$$

It turns out that (with no additional assumption) there exist continuous additive functions $a : \mathbb{R} \to G$ and $A : E \to G$ such that

$$f(x) - a\left(\|x\|^2\right) - A(x) \in K \quad \text{for every } x \in E.$$

[1] K. Baron and J. Rätz, *Orthogonality and additivity modulo a subgroup*, Aequationes Math. **46** (1993), 11-18.

[2] J. Brzdęk, *On orthogonally exponential and orthogonally additive mappings*, Proc. Amer. Math. Soc. **125** (1997), 2127-2132.

**Lech Bartłomiejczyk** *A characterization of operations invariant under bijections*

Joint work with Józef Drewniak.

We characterize $n$-ary operations $F : I^n \to I$ on the unit interval such that

$$F(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(F(x_1, \dots, x_n))$$

holds for every increasing bijection $\varphi : I \to I$ and for every $(x_1, \dots, x_n) \in I^n$.

**Bogdan Batko** *On the stability of alternative Cauchy equations*

Let us consider a conditional (alternative) Cauchy equation

$$(x, y) \in \mathcal{Z} \implies f(x + y) = f(x) + f(y),$$

where $\mathcal{Z}$ is a set. We deal with the stability of conditional Cauchy equations with the condition (the set $\mathcal{Z}$) dependent on the unknown function $f$. Our main stability results concern Mikusiński's equation

$$f(x + y) \neq 0 \implies f(x + y) = f(x) + f(y),$$

as the fundamental example of this kind of equations.

**Zoltán Boros** *$\mathbb{Q}$-derivatives of Jensen-convex functions*

Definitions and properties of radial $\mathbb{Q}$-derivatives of Jensen-convex functions are presented. Some of these properties provide a characterization of Jensen-convex functions among real valued functions defined on an open interval.

**Nicole Brillouët-Belluot** *The ACP method for solving some composite functional equations*

We show how important for solving some composite functional equations is the ACP method based on the well-known theorem of J. Aczél (given also by R. Craigen and Z. Páles) which gives the representation of a continuous cancellative associative operation on a real interval. In particular, we find with this method all continuous solutions of the following functional equation

$$f(af(x)f(y) + b(f(x)y + f(y)x) + cxy) = f(x)f(y) \quad (x, y \in \mathbb{R})$$

which generalizes both Ebanks functional equation and Baxter functional equation.

**Dobiesław Brydak** *On a linear functional inequality*

Regular solutions $\varphi$ of the inhomogeneous iterative functional inequality

$$\varphi(f(x)) \leqslant g(x)\varphi(x) + h(x)$$

are discussed in a neighbourhood of the attractive fixed point of the given function $f$. Results presented are related to those found in [1], cf. also [2], Section 12.5, pp. 488-490.

[1] D. Brydak, B. Choczewski, *On nonhomogeneous linear functional inequality*, Zeszyty Naukowe Uniw. Jagiellońskiego **441**, Prace Mat. **18** (1977), 89-92.

[2] M. Kuczma, B. Choczewski, R. Ger, *Iterative Functional Equations*, Encyclopedia of Math. and Its Appl. vol. **32**, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney, 1990.

**Bogdan Choczewski** *On a functional equation of Wilson type*

Joint work with Zbigniew Powązka.

The study is motivated by E. Wachnicki's paper [1] dealing with an integral mean value thoerem. The equation reads

$$af(x)+bf(y)=f(ax+by)\,g(y-x),\quad x,y\in X, \tag{1}$$

where $a$, $b$ are some positive reals, $f:X\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ are unknown functions and $X$ is either the real line or positive or negative half-line.

We find solutions $(f,g)$ of equation (1) on $X$ mainly in the case where $f$ is locally integrable and $g$ is continuous at the origin. In particular, among solutions exponential funtions show up. As a tool we use, among others, simple Schröder equations

$$f(px)=q\,f(x),$$

with suitable constants $p$ and $q$.

[1] E. Wachnicki, *Sur un développement de la valeur moyenne*, Rocznik Nauk.-Dydakt. Wyż. Szk. Ped. Kraków. Prace Mat. **14** (1997), 35-48.

**Jacek Chudziak** *Functional equation of the Gołąb-Schinzel type*

We deal with the equation

$$f(x\varphi(f(y))+y\psi(f(x)))=f(x)f(y),\quad x,y\in\mathbb{R}, \tag{$*$}$$

where $f:\mathbb{R}\to\mathbb{R}$ and $\varphi,\psi:\mathbb{R}\to\mathbb{R}$ are unknown functions.

The equation $(*)$ is a generalization of the well known Gołąb-Schinzel equation.

**Krzysztof Ciepliński** *General construction of non-dense disjoint iteration groups on the circle*

Let $\mathcal{F}=\{F^v:S^1\to S^1,\ v\in V\}$ be a disjoint iteration group on the unit circle $S^1$, that is a family of homeomorphisms such that

$$F^{v_1}\circ F^{v_2}=F^{v_1+v_2},\quad v_1,v_2\in V,$$

and each $F^v$ either is the identity mapping or has no fixed point ($(V,+)$ is a 2-divisible nontrivial Abelian group). Denote by $L_{\mathcal{F}}$ the set of all cluster points of $\{F^v(z),\ v\in V\}$ for $z\in S^1$. We give a general construction of disjoint iteration groups for which $\emptyset\neq L_{\mathcal{F}}\neq S^1$.

**Stefan Czerwik** *On the stability of the quadratic functional equation in some special function spaces*

Joint work with Krzysztof Dłutek.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation. Some aspects of the stability problems for this equation in some function spaces will be discussed.

**Joachim Domsta** *On $f$-slow variability of solutions to linear equations*

Let us assume that

$$f : X \to X, \quad g : X \to \mathbb{R}_+ := (0, \infty), \quad \text{where } X \neq \emptyset.$$

DEFINITION

We say that function $\Psi : X \to \mathbb{R}_+$ is slowly varying on the orbits of $f$ (briefly: $f$-slowly varying) whenever for every pair $(x, y) \in X^2$ the limit

$$\Psi_f(x|y) := \lim_{n\to\infty} \frac{\Psi(f^n(x))}{\Psi(f^n(y))}$$

equals 1.

The main results concerns the solutions of the following equation

$$\Psi(f(x)) = g(x) \cdot \Psi(x), \quad x \in X. \tag{1}$$

If $\Psi(y) \neq 0$ then we have the following system of equations implied by (1),

$$\frac{\Psi(x)}{\Psi(y)} = \frac{\Psi\left(f^n(x)\right)}{\Psi\left(f^n(y)\right)} \cdot \gamma_{f,g;n}(x|y), \quad \text{for } n \in \mathbb{N},$$

where

$$\gamma_{f,g;n}(x|y) := \prod_{j=0}^{n-1} \frac{g(f^j(y))}{g(f^j(x))}, \quad \text{for } x, y \in X.$$

Therefore, the limit $\Psi_f(x|y)$ exists in $\mathbb{R} \setminus \{0\}$ iff the limit

$$\gamma_{f,g}(x|y) := \lim_{n\to\infty} \gamma_{f,g;n}(x|y)$$

exists in $\mathbb{R} \setminus \{0\}$ and $\Psi(x) \cdot \Psi(y) \neq 0$. If this holds, then

$$\frac{\Psi(x)}{\Psi(y)} = \Psi_f(x|y) \cdot \gamma_{f,g}(x|y).$$

THEOREM

*The equation* (1) *possesses an $f$-slowly varying solution if and only if the principial function $\gamma_{f,g}(\cdot|y)$ exists in the class of positive functions on $X$*

*for some/every $y \in X$ and $\lim_{n\to\infty} g(f^n(x)) = 1$ for all $x \in X$. Then all $f$-slowly varying solutions are of constant non-zero sign and they are represented as follows,*

$$\Psi(x) = \Psi(y) \cdot \gamma_{f,g}(x|y), \quad \text{for all } x, y \in X.$$

Corollary

*The function $\Pi(x) := \Gamma(x+1)$, $x > 0$, is the unique solution of the problem*

$$\Pi(x+1) = (x+1) \cdot \Pi(x), \quad \text{for } x > 0, \ \text{and } \Pi(1) = 1.$$

*for which $\tilde{\Pi}(x) := e^{x-1} \cdot x^{-x} \cdot \Pi(x)$, $x > 0$, is $(x+1)$-slowly varying.*

**Tibor Farkas** *On functions additive with respect to algorithms*

We prove that for an arbitrary interval filling sequence there exist two algorithms such that the additivity of a function with respect to them implies its linearity. In contrast to some known results, we prove the linearity of the function without requiring any special properties for the interval filling sequence and any regularity properties for the function.

**Roman Ger** *Rational associative operations and the corresponding addition formulas*

Joint work with Katarzyna Domańska.

Our goal is to present a method of solving Cauchy type functional equations of the form

$$f(x+y) = F(f(x), f(y))$$

(addition formulas) where the given binary operation $F$ defined on a subset of the real plane $\mathbb{R}^2$ is rational and associative.

The key tools we are using are two representation theorems for rational associative mappings on the plane due to A. Chéritat (1999) and some results on Cauchy type equations assumed almost everywhere.

The solutions are found in the class of functions with "small" counterimages of singletons (i.e. belonging to a proper linearly invariant set ideal in the domain space).

**Attila Gilányi** *On a uniqueness problem for homogeneous utility representations*

Joint work with C.T. Ng.

We investigate a uniqueness question, posed in [1], for homogeneous utility representations. In order to answer it, we solve the functional equation

$$F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]$$

under slightly different conditions as it was done in [4] (cf. also [5], [2] and [3]).

[1] J. Aczél, R.D. Luce, C.T. Ng, *Separability, segregation, and homogeneity in a theory of utility*, manuscript.

[2] J. Aczél, R. Ger, A. Járai, *Solution of a functional equation arising from utility that is both separable and additive*, Proc. Amer. Math. Soc. **127** (1999), 2923-2929.

[3] J. Aczél, Gy. Maksa, Zs. Páles, *Solution of a functional equation arising in an axiomatization of the utility of binary gambles*, Proc. Amer. Math. Soc. **129** (2001), 483-493.

[4] J. Aczél, Gy. Maksa, C.T. Ng, Zs. Páles, *A functional equation arising from ranked additive and separable utility*, Proc. Amer. Math. Soc. **129** (2001), 989-998.

[5] A. Lundberg, *On the functional equation* $f(\lambda(x)+g(y)) = \mu(x)+h(x+y)$, Aequationes Math. **16** (1977), 21-30.

**Máté Györy** *Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality*

**Attila Házy** *On approximately Jensen-convex functions*

Joint work with Zsolt Páles.

A function $f : D \to \mathbb{R}$ is called $(\varepsilon, \delta)$-midconvex if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\left(f(x)+f(y)\right)+\delta+\varepsilon\mid x-y\mid .$$

Our main result shows that if $f$ is locally bounded from above and $(\varepsilon, \delta)$-midconvex, then $f$ satisfies the following convexity inequality

$$f\left(\lambda x+(1-\lambda)y\right) \leqslant \lambda f(x)+(1-\lambda)f(y)+2\delta+2\varepsilon\varphi(\lambda)\mid x-y\mid$$

for every $x, y \in D$ and $\lambda \in [0,1]$, where $\varphi$ is defined by

$$\varphi(\lambda) = \begin{cases} -2\lambda\log_2\lambda, & 0 \leqslant \lambda \leqslant \frac{1}{2}, \\ -2(1-\lambda)\log_2(1-\lambda), & \frac{1}{2} \leqslant \lambda \leqslant 1. \end{cases}$$

The case $\varepsilon = 0$ of the result reduces to that of Nikodem and Ng from 1993.

**Witold Jarczyk** *Reversibility on the circle*

A homeomorphism of a topological space is said to be *continuously reversible* if it is a composition of two continuous involutions or, equivalently, the homeomorphism and its inverse function are conjugated by a continuous involution. In the talk continuous reversibility of homeomorphisms of the circle is studied, especially those with no periodic points. This is a continuation of a research made by the author for the real case.

**Hans-Heinrich Kairies** *Images and pre-images of a Banach space operator*

The operator $F$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

is a continuous automorphism on the Banach space $B$ of real bounded functions and on several of its closed subspaces. $F$ is known to generate continuous nowhere differentiable [cnd] functions from simple elements of $B$ (Weierstrass, Takagi). But neither the image $F[\mathcal{N}]$ nor the pre-image $F^{-1}[\mathcal{N}]$ of the set $\mathcal{N}$ of cnd functions from $B$ is know at present. We describe some subsets of $F[\mathcal{N}]$ and of $F^{-1}[\mathcal{N}]$ as well as images and pre-images of other function sets.

**Barbara Koclęga-Kulpa** *On a conditional functional equation in normed spaces*

We deal with the functional equation

$$\|f(x \cdot y)\| = \|f(x) + f(y)\| \tag{1}$$

considered by Roman Ger for function $f$ mapping a given group into a strictly convex space (see [1]). We consider equation (1) for function $f$ defined on the algebra $M_n(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of all real or complex $n \times n$ matrices. Then equation (1) looks as follows

$$A \cdot B \neq [0] \Longrightarrow \|f(A \cdot B)\| = \|f(A) + f(B)\|,$$

where $A, B \in M_n(\mathbb{K})$ and $[0]$ stands for the zero matrix.

[1] R. Ger, *On a characterization of strictly convex spaces*, Atti Accad. Sci. Torino Cl. Sci Fis. Mat. Natur. **127** (1993), 131-138.

**Zygfryd Kominek** *On the quasisymmetry quotient*

We will consider the equation of the form

$$\frac{f(x+h) - f(x)}{f(x) - f(x-h)} = \varphi(h)\psi(x),$$

where $f, \psi : \mathbb{R} \to \mathbb{R}$, and $\varphi : (0, \infty) \to \mathbb{R}$ are unknown functions.

**Aleksandar Krapež** *Fully debalanced functional equations on almost quasigroups*

A quasigroup may be characterized by the property that the (left or right) translation by any element is permutation. If we allow that some translations may be constant functions, we get so called *almost quasigroups*. The quasigroups are characterized as isotopes of loops. Similarly we can characterize almost quasigroups as isotopes of either loops or loops with external zero added.

A special class of linear functional equations on almost quasigroups is considered. They are called *fully debalanced* and characterized by a single appearance of variables (both object and functional) on the left hand side of the given equation $u = b$, while the right hand side $b$ is a constant.

The general solution of such functional equation is described in terms of the structure of the tree of the term $u$.

**Zbigniew Leśniak** *On the structure of equivalence classes of a relation for a free mapping*

We present some results concerning an equivalence relation for a given free mapping $f$ of the plane. We discuss the problem if the equivalence classes of the relation are invariant under $f$ and if the restriction of $f$ to each of them is conjugate to a translation.

**Lajos Molnár** *Ortho-order automorphisms of Hilbert space effect algebras*

Joint work with Zsolt Páles.

Let $H$ be a (real or complex) Hilbert space. The effect algebra of $H$ is the operator-interval $[0, I]$ of all positive (selfadjoint, bounded linear) operators on $H$ which are bounded by the identity $I$. Effect algebras play very important role in the mathematical foundations of quantum mechanics (see, for example, [1]). It is well-known that if the dimension of $H$ is at least 3, then the $\perp$-order automorphisms of $[0, I]$ (which are the bijective transformations of the effect algebra that preserve the order $\leqslant$ in both directions and also preserve a kind of orthocomplementation $\perp$: $E \to I - E$) are implemented by unitary or antiunitary operators on $H$. In fact, the proof is usually based on the fundamental theorem of projective geometry which holds true only in spaces of dimension not less than 3. Because of the importance of effect algebras, it is a natural problem to clarify the situation in the 2-dimensional case. In fact, Cassinelli, De Vito, Lahti and Levrero faced this question in their paper [2]. Moreover, in their recent work [3], Lahti, Mączyński and Ylinen showed that if the considered automorphism is induced via the functional calculus by a Borel function of the interval $[0, 1]$, then it is necessarily the identity. The aim of this talk is to present the complete solution of the problem. Namely, we have the following result.

Theorem

*Let $H$ be a 2-dimensional (real or complex) Hilbert space and let $[0, I]$ be the effect algebra of $H$. Let $\phi : [0, I] \to [0, I]$ be a bijective transformation with the property that*

$$E \leqslant F \iff \phi(E) \leqslant \phi(F) \quad \text{and} \quad \phi(I - E) = I - \phi(E)$$

*holds for every $E, F \in [0, I]$. Then there exists an either unitary or antiunitary operator $U$ on $H$ such that*

$$\phi(E) = UEU^* \quad (E \in [0, I]).$$

[1] G. Ludwig, *Foundations of Quantum Mechanics, Vol. I*, Springer Verlag, 1983.

[2] G. Cassinelli, E. De Vito, P. Lahti and A. Lavrero, *A theorem of Ludwig revisited*, Found. Phys. **30** (2000), 1755-1761.

[3] P. Lahti, M. Mączyński, K. Ylinen, *A note on order and orthocomplementation preserving automorphisms of the set of effect operators on a Hilbert space*, Lett. Math. Phys. **55** (2001), 43-51.

[4] L. Molnár, Zs. Páles, *⊥-order automorphisms of Hilbert space effect algebras: The two-dimensional case*, J. Math. Phys. **42** (2001), 1907-1912.

**Zenon Moszner** *La fonction d'indice et la fonction exponentielle*

On donne une laison entre la fonction d'indice, c. à d. la solution de l'équation conditionnelle

$$f(c) \cdot f(d) \neq \underline{0} \Longrightarrow f(c+d) = f(c) \cdot f(d),$$

ou $f : \mathbb{R}(p) \to \mathbb{R}(p)$, $\mathbb{R}(p) = (0, +\infty)^p \setminus \{\underline{0}\}$, $\underline{0} = (0, \dots, 0) \in \mathbb{R}^p$, $c = (c_1, \dots, c_p) \in \mathbb{R}(p)$, $d = (d_1, \dots, d_p) \in \mathbb{R}(p)$, $c + d = (c_1 + d_1, \dots, c_p + d_p)$, $c \cdot d = (c_1 d_1, \dots, c_p d_p)$, et la fonction exponentielle, c. à d. la solution $f : \mathbb{R}(p) \to \mathbb{R}(p)$ de l'équation

$$f(c+d) = f(c) \cdot f(d).$$

**Kazimierz Nikodem** *On $K$-$\lambda$-convex set-valued functions*

Let $K$ be a convex cone in a vector space $Y$, $I \subset \mathbb{R}$ be an interval and $\lambda : I^2 \to (0,1)$ be a given function.

A set-valued function $F : I \to \mathrm{n}(Y)$ is called *$K$-$\lambda$-convex* if

$$\lambda(x,y)F(x) + (1 - \lambda(x,y))F(y) \subset F(\lambda(x,y)x + (1 - \lambda(x,y))y) + K$$

for all $x, y \in I$.

$F$ is $K$-convex if

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K$$

for all $x, y \in I$ and $t \in [0,1]$.

The following set-valued generalization of a result proved recently by Zs. Páles [1] holds:

THEOREM

*Let $K$ be a closed convex cone in a real locally convex space $Y$, $I \subset \mathbb{R}$ be an open interval and $\lambda : I^2 \to (0,1)$ be a function continuous in each variable. If a set-valued function $F : I \to \mathrm{c}(Y)$ is $K$-$\lambda$-convex and locally $K$-upper bounded at every point, then it is $K$-convex.*

[1] Zs. Páles, *Bernstein-Doetsch-type results for general functional inequalities*, Rocznik Nauk. Dydakt. Akad. Pedagog. w Krakowie, Prace Matematyczne **17** (2000), 197-206.

**Zsolt Páles** *Approximately convex functions*

A real valued function $f$ defined on a real interval $I$ is called $(\varepsilon, \delta)$-convex if it satisfies

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta$$

for every $x, y \in I$, $t \in [0, 1]$.

The main results offer various characterizations for $(\varepsilon, \delta)$-convexity. One of them states that $f$ is $(\varepsilon, \delta)$-convex for some positive $\varepsilon$ and $\delta$ if and only if $f$ can be decomposed into the sum of a convex function, a function with bounded supremum norm, and a function with bounded Lipschitz-modulus. In the special case $\varepsilon = 0$, the results reduce to that of Hyers, Ulam, and Green obtained in 1952 concerning the so called $\delta$-convexity.

[1] J.W. Green, *Approximately convex functions*, Duke Math. J. **19** (1952), 499-504.

[2] D.H. Hyers, S.M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821-828.

[3] Zs. Páles, *On approximately convex functions*, Proc. Amer. Math. Soc., to appear.

**Tomasz Powierża** *Higher order set-valued iterative roots of bijections*

Let $f$ be a self-mapping of a non-empty set $X$ and $r \geqslant 2$ be a positive integer. We say that a function $G : X \to 2^X$ is a *set-valued iterative root of order* $r$ of $f$ if

$$f(x) \in G^r(x) \quad \text{for } x \in X,$$

where

$$G^0(x) := \{x\},$$
$$G^{n+1}(x) := \bigcup_{y \in G^n(x)} G(y)$$

for $x \in X$ and $n \in \mathbb{N}_0$.

We present a simple construction of a class of set-valued iterative roots of bijections. These roots have at most two-element values and coincide with normal iterative root if they exist. Moreover, every iterative root can be obtained using this construction.

**Maria Ewa Pliś** *Summability of formal solutions to Laplace type PDE's*

We consider a nonlinear differential equation

$$P(D)u = \alpha u^m,$$

where $P$ is a polynomial of two variables, $m \in \mathbf{N}$, $m \geqslant 2$. We construct a formal solution

$$u(x) = T[e^{-xz}], \tag{$\star$}$$

for some Laplace distribution $T$. Applying $P(D)$ to $u$ in the form $(\star)$ we arrive at the convolution equation

$$P(z)T = \alpha T^{*m}.$$

We find a solution of this equation in the form of formal series

$$T = \sum_{k=0}^{\infty} T_k$$

of Laplace distributions $T_k$. We show that assuming some properties of the set $\operatorname{Char} P$ we get some Gevrey class of such solutions.

**Thomas Riedel** *Some results relating to Flett's mean value theorem*

Joint work with M. Dao and P. K. Sahoo.

This talk consists of three results, which are connected with Flett's mean value theorem and use similar methods of proof. First we show that Flett's theorem as well as its generalization by Trahan and by Riedel and Sahoo still hold for functions, which are approximately differentiable. Second, we show that for a large class of functions, the Flett mean value point is stable in the sense of Hyers and Ulam. Finally we give another mean value theorem, similar in type to Flett's, namely

*If $f$ is differentiable on $[a,b]$, then there is an $\eta \in (a,b)$ such that*

$$\frac{1}{\eta - a}\left[f'(\eta) - \frac{f(\eta) - f(a)}{\eta - a}\right] + \frac{1}{\eta - b}\left[f'(\eta) - \frac{f(\eta) - f(b)}{\eta - b}\right] = \frac{f'(b) - f'(a)}{b - a}.$$

We conclude with a functional equation derived from this.

**Maciej Sablik** *A method of solving equations characterizing polynomial functions*

We present a lemma which generalizes some ealier results of W. H. Wilson and L. Székelyhidi and can be used as a tool in solving several equations characterizing polynomial functions, in particular some of those stemming from mean value theorems, and supposed to hold for functions defined in Abelian groups.

We also quote another lemma proved by I. Pawlikowska who has generalized both our result as well as a lemma of Z. Daróczy and Gy. Maksa thus getting a method of characterizing polynomial functions as solutions of equations assumed to hold on convex subsets of linear spaces. The talk is illustrated by a number of examples.

**Adolf Schleiermacher** *Some consequences of a theorem of Liouville*

Let $S$ denote the automorphism group of the Euclidean space $E$ of dimension $n$ comprising Euclidean motions as well as similarities. Let $h$ be a $C^1$ diffeomorphism of $E$ not contained in $S$. We are interested in the group $G$ generated by $h$ and $S$ acting on $E$ and in its invariants, i.e. real valued functions satisfying $f(g(P_0), g(P_1), \ldots, g(P_m)) = f(P_0, P_1, \ldots, P_m)$ for all $g \in G$ and $(P_0, P_1, \ldots, P_m) \in D \subseteq E^{m+1}$. Using Liouville's theorem on conformal mappings in space we shall prove: Under the action of $G$ on $E^{n+1}$ every orbit containing a non-degenerate $n$-simplex is dense in $E^{n+1}$. As a corollary we obtain: For $m \leqslant n$ any continuous $(m+1)$-place invariant of $G$ whose domain of definition $D$ contains a non-degenerate $m$-simplex is constant.

It may be conjectured that if $h$ is not an affine mapping, then the group $G$ as defined above is $(n+1)$-fold transitive. We shall show that this is in fact true for a particular class of such groups.

**Stanisław Siudut** *Some remarks on measurable groups*

We give an example of a complete measurable group $(G, \Sigma, \lambda)$ such that

1. $\lambda(G) = \infty$,

2. $\lambda$ is not invariant under symmetry with respect to zero.

**Andrzej Smajdor** *Regular set-valued iteration semigroups and a set-valued differential problem*

The connection between differentiable iteration semigroups of continuous linear set-valued functions and set-valued solutions of some linear ordinary differential equations is considered.

**Wilhelmina Smajdor** *Entire solutions of a functional equation of Pexider type*

All entire solutions of the functional equation

$$|f(x+y)| + |g(x-y)| = |h(x+\overline{y})| + |k(x-\overline{y})|$$

are determined.

**Paweł Solarz** *On iterative roots of a homeomorphism of the circle with an irrational rotation number*

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with the positive orientation, $F : S^1 \to S^1$ be a homeomorphism with an irrational rotation number $\alpha(F)$, and $L_F(z)$ the set of all cluster points of $\{F^n(z), \ n \in \mathbb{Z}\}$ for $z \in S^1$. Let $S^1 \setminus L_F \neq \emptyset$. Suppose that the iterative kernel of $F$, $K_F := \varphi[S^1 \setminus L_F]$, has the property $(\sqrt[n]{s})_m K_F = K_F$, where

$$\left(\sqrt[n]{s}\right)_m = e^{2\pi i \frac{1}{n}(\alpha(F)+m)} \quad \text{for } m \in \{0, \dots, n-1\}, \ n \in \mathbb{N}.$$

In this case $F$ has infinitely many iterative roots, i.e., continuous solution of the equation $G^n = F$ depends on an arbitrary function.

**Joanna Szczawińska** *On Lipschitz midconcave multifunctions*

We give a version of the Bernstein-Doetsch Theorem for multifunctions with bounded and convex values. If a $J$-concave set-valued function is lower bounded on a ball then it is locally Lipschitzian.

**László Székelyhidi** *Exponential polynomials on polynomial hypergroups*

Joint work with Ágota Orosz.

Hypergroups have been investigated since the pioneer works of C. F. Dunkl, R. I. Jewett and R. Spector. The presence of translation operators makes it possible to investigate some classical functional equations on hypergroups. Here we present results concerning additive, exponential and polynomial functions on polynomial hypergroups.

**Tomasz Szostok** *On some conditional functional equations*

We are looking for an unconditional functional equation which would preserve the properties of a given conditional equation. We consider the well known orthogonal Cauchy equation

$$x \perp y \Longrightarrow f(x+y) = f(x) + f(y)$$

and the Ptolemaic equation

$$x \perp y \Longrightarrow f(x)^2 + f(y)^2 = f(x+y)f(x-y)$$

which was recently considered by Margherita Fochi. For both of these equations such unconditional equations are found.

**Peter Šemrl** *Geometry of matrices*

Two matrices are said to be *adjacent* if the rank of their difference is one. Hua's fundamental theorems of geometry of matrices characterize bijective maps preserving adjacency in both directions on various spaces of matrices. We present some recent results on such maps.

**Jacek Tabor** *Differential equations in metric spaces*

We generalize the notion of an autonomous differential equation to a complete metric space $X$. Instead of the vector field we assume that we are given a function $F : X \to C(\mathbb{R}_+, X)$ (the space of continuous functions from $\mathbb{R}_+$ to $X$ such that $F(x)[0] = x$ for every $x \in X$).

We say that a continuous function $u : [0, T) \to X$ is a solution to the differential equation

$$\frac{du}{dt} = F(u)$$

if it is locally tangent to $F$, that is if

$$\lim_{h\to 0^+} \frac{d(u(t+h), F(u(t))[h])}{h} = 0 \quad \text{for } t \in [0, T).$$

Under some natural assumption on $F$ we prove analogues of the classical Peano and Picard existence results.

**Józef Tabor** *On the Hyers operator*

Joint work with Jacek Tabor.

Let $X$ be a metric space, and let $K \subset X$. Assume that for every $x \in X$ there exists a unique best approximation from $K$ denoted by $\Pi_k(x)$. Properties of the mapping $\Pi_k$ are investigated. As a corollary we obtain a partial solution to the problem of Z. Moszner concerning continuity of the Hyers operator.

**Peter Volkmann** *A characterization of the Dinghas derivative*

The generalized $n$-th derivative

$$D^n f(\alpha) = \lim_{\substack{x \leqslant \alpha \leqslant y \\ y - x \downarrow 0}} \left(\frac{n}{y-x}\right)^n \Delta^n_{\frac{y-x}{n}} f(x)$$

(Alexander Dinghas 1966) of $f : \mathbb{R} \to \mathbb{R}$ at $\alpha \in \mathbb{R}$ is characterized by

$$f(\alpha + t) = g(t) + D^n f(\alpha) \cdot \frac{t^n}{n!} + o(t^n) \quad (t \to 0),$$

where $g$ is a polynomial function of degree at most $n-1$, i.e. $\Delta^n_x g(y) = 0$ $(x, y \in \mathbb{R})$. For continuous $f$ this result is known from my doctoral thesis at Freie Universität Berlin 1971.

**Anna Wach-Michalik** *Convexity and functional equations connected with Euler's Beta function*

The function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\beta(x) = B(x, x)$ (where $B$ is Euler's Beta function) is a particular solution of the functional equation:

$$f(x+1) = \frac{x}{2(2x+1)} f(x) \quad \text{and} \quad f(1) = 1. \tag{1}$$

We have proved a theorem, similar to that due to H. Bohr and J. Mollerup.

THEOREM

*If* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a solution of* (1), log-*convex on* $(\gamma, +\infty)$ *for some* $\gamma \geqslant 0$, *then* $f = \beta$.

We also examine the set

$M_\beta = \{g : \mathbb{R}_+ \to \mathbb{R} :$ for every $f$ satysfying (1) if $g \circ f$ is convex then $f = \beta\}$.

**Janusz Walorski** *On the existence of continuous and smooth solutions of the Schröder equation*

We consider continuous and smooth solutions of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

and the linear equation

$$\varphi(f(x)) = g(x)\varphi(x) + F(x)$$

using the method motivated by [1] and [2].

[1] G. Belitskii, V. Tkachenko, *On solvability of linear difference equations in smooth and real analytic vector functions of several variables*, Integr. Equat. Oper. Th. **18** (1994), 123-129.

[2] J. Morawiec, J. Walorski, *On the existence of smooth solutions of linear functional equations*, Integr. Equat. Oper. Th. **39** (2001), 222-228.

**Jacek Wesołowski** *Functional equations related to independence properties of random matrices*

Problems of characterizations of probability measures can often be reduced to functional equations. For instance the celebrated Lukacs [5] theorem on characterizing the gamma distribution by independence of the sum and the quotient of two independent non-degenerate positive random variables, if existence of densities is assumed, reduces to the problem of solution of the following equation:

$$f_1(u)f_2(v) = vf_3(uv)f_4((1-u)v), \quad u \in (0,1),\ v \in (0,\infty),$$

where $f_i$, $i = 1, 2, 3, 4$, are integrable non-negative functions.

Matrix variate versions of this result are concerned with, so called, Wishart distribution, which is a probability measure on the cone $\mathcal{V}_+$ of positive definite symmetric, say $n \times n$, matrices, defined by the density

$$\gamma_{p,a}(dy) = \frac{(\det a)^p}{\Gamma_n(p)}(\det y)^{p-\frac{n+1}{2}} \exp\left(-(a,y)\right)I_{\mathcal{V}_+}(y)\,dy, \quad y \in \mathcal{V}_+,$$

where $\Gamma_n$ is the $n$-variate Gamma function, $a \in \mathcal{V}_+$ and $p > \frac{n-1}{2}$. Such characterizations of the Wishart distribution were studied, for instance, in Casalis and Letac [2] and Letac and Massam [3] by the Laplace transform technique. Their setting was based on certain invariance property of the "quotient" of random matrices. So, in a sense, there was a strong belief that this invariance property is somehow deeply rooted in the problem under study (it is satisfied trivially in the univariate case). However the approach exploiting densities, developed in Bobecka and Wesołowski [1], reveals that no invariance property is needed in order to characterize the Wishart distribution by the Lukacs type independence property. Two basic steps in the proof are connected with solutions of the following equations:

$$a(x) = g(yxy) - g(y(e-x)y), \quad x \in \mathcal{V}_+, \ y \in \mathcal{D} = \{z \in \mathcal{V}_+ : \ e - z \in \mathcal{V}_+\},$$

$$a_1(x) + a_2(y) = g(yxy) + g(y(e-x)y), \quad x \in \mathcal{V}_+, \ y \in \mathcal{D}.$$

They are solved under some smoothness conditions imposed on functions present in the equations.

Observe that these equations look somewhat similar to the multiplicative Cauchy equation in the cone $\mathcal{V}_+$

$$f(x)f(y) = f\left(x^{\frac{1}{2}}yx^{\frac{1}{2}}\right), \quad x, y \in \mathcal{V}_+,$$

where $f : \mathcal{V}_+ \to (0,\infty)$. It appears that, at least, under differentiability assumption, the only solution of this equation has the form $f(x) = (\det x)^\lambda$, where $\lambda$ is a real number.

Similar equations related to characterizations of matrix variate GIG and gamma distributions by, so called Matsumoto-Yor independence property, will be also presented. This part of the talk will follow Letac and Wesołowski [4] and Wesołowski [6].

Finally, a new independence property for matrix variate beta distributions will be shown. Here the respective characterization question leads to an open question: the functional equation for densities is still under study.

[1] K. Bobecka, J. Wesołowski, *The Lukacs-Olkin-Rubin theorem without invariance of the "quotient"*, Preprint Mar. 01, 2001.

[2] M. Casalis, G. Letac, *The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones*, Ann. Statist. **24** (1996), 763-786.

[3] G. Letac, H. Massam, *Quadratic and inverse regressions for Wishart distributions*, Ann. Statist. **26** (1998), 573-595.

[4] G. Letac, J. Wesołowski, *An independence property for the product of GIG and gamma laws*, Ann. Probab. **28** (2000), 1371-1383.

[5] E. Lukacs, *A characterization of the gamma distribution*, Ann. Math. Statist. **26** (1955), 319-324.

[6] J. Wesołowski, *The Matsumoto-Yor independence property for GIG and Gamma laws, revisited*, Math. Proc. Cambridge Philos. Soc. (2001), to appear.

## Problems and Remarks

**1. Problem.** (On approximate sandwich theorems)
Let $S$ be an abelian semigroup. Assume that $p, q : S \to \mathbb{R}$ are functions such that $q \leqslant p$ and $p$ is $\varepsilon$-subadditive (i.e. $p+\varepsilon$ is subadditive), $q$ is $\delta$-superadditive (i.e. $q-\delta$ is superadditive) for some nonnegative $\varepsilon, \delta$.

Then, by a well-known sandwich theorem, there exists an additive function $\psi : S \to \mathbb{R}$ such that

$$q - \delta \leqslant \psi \leqslant p + \varepsilon. \tag{1}$$

Define

$$\varphi(x) = \begin{cases} q(x) & \text{if } \psi(x) < q(x), \\ \psi(x) & \text{if } q(x) \leqslant \psi(x) \leqslant p(x), \\ p(x) & \text{if } p(x) < \psi(x). \end{cases}$$

Obviously, $q \leqslant \varphi \leqslant p$ and, due to (1), $\psi - \varepsilon \leqslant \varphi \leqslant \psi + \delta$. Thus, for $x, y \in S$, we have

$$\varphi(x+y) \leqslant \psi(x+y) + \delta \leqslant \psi(x) + \psi(y) + \delta \leqslant \varphi(x) + \varphi(y) + (2\varepsilon + \delta).$$

Similarly,

$$\varphi(x+y) \geqslant \varphi(x) + \varphi(y) - (2\delta + \varepsilon).$$

Therefore $\varphi$ is $(2\varepsilon + \delta)$-subadditive and $(2\delta + \varepsilon)$-superadditive.

Does there exist a function $f$, $q \leqslant f \leqslant p$, such that $f$ is $\varepsilon$-subadditive and $\delta$-superadditive?

*Zsolt Páles*

**2. Problem and Remarks.** (Communicated by J. Aczél and presented by Zenon Moszner.)
*Genèse*: Soit $x$ le nombre des travailleurs d'une fabrique, $y$ le capital de roulement, $z = F(x,y)$ la production à temps $t_0$ et $x', y', z' = F(x', y') : \mathbb{R}_+^2 \to \mathbb{R}_+$ les mêmes valeurs à temps $t$. Nous supposons que $x' = \Phi(x,y,t) : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$, $y' = \Psi(x,y,t) : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$ et que

(a) pour chaque $t_1$ et $t_2$ il existe un $t_3$ tel que

$$\Phi[\Phi(x,y,t_1), \Psi(x,y,t_1), t_2] = \Phi(x,y,t_3),$$
$$\Psi[\Phi(x,y,t_1), \Psi(x,y,t_1), t_2] = \Psi(x,y,t_3),$$

(b) pour chaque $t$ il existe $\bar{t}$ tel que

$$\Phi[\Phi(x,y,t), \Psi(x,y,t), \bar{t}\,] = x,$$
$$\Psi[\Phi(x,y,t), \Psi(x,y,t), \bar{t}\,] = y,$$

(c) il existe $t_0$ pour lequel

$$\Phi(x, y, t_0) = x \quad \text{et} \quad \Psi(x, y, t_0) = y.$$

Le problème est suivant: Existe-il pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ continue, non-décroissante par rapport à chaque variable et quasi-concave des fonctions (non-banales) $\Phi, \Psi : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$ continues, satisfaisantes aux conditions (a), (b), (c) et une fonction $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ continue telles que

$$H(F(x, y)t) = F(\Phi(x, y, t), \Psi(x, y, t))? \tag{E}$$

En interprétation économique: la production à temps $t$ ne dépend-elle que de la production $F(x, y)$ à temps $t_0$ et du temps $t$?

Remarques.

1. Pour la fonctions (banales) $\Phi(x, y, t) = x$, $\Psi(x, y, t) = y$ et $H(z, t) = z$ l'équation (E) a lieu pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$.
2. La condition (c) est une conséquence de (a) et (b) et les conditions (a) et (b) désignent que la famille $B = \{(\Phi(x, y, t), \Psi(x, y, t))\}_{t \in \mathbb{R}}$ est un groupe des bijections de $\mathbb{R}_+^2$ sur $\mathbb{R}_+^2$ par rapport à la superposition.
3. L'existence de la fonction $H$ est équivalente à la condition:

   $$\begin{aligned} &F(x_1, y_1) = F(x_2, y_2) \Longrightarrow \\ &\forall\, t \in \mathbb{R} : F(\Phi(x_1, y_1, t), \Psi(x_1, y_1, t)) = F(\Phi(x_2, y_2, t), \Psi(x_2, y_2, t)) \end{aligned}$$

   et cette implication désigne que la famille des niveaux de $F$ (level sets) est invariante par rapport à la famille $B$, c. à d. chaque niveau est transformé par chaque bijection de $B$ à un niveau (le même ou l'autre).
4. Le problème de Mitchell est done équivalent an suivant:

   – existe-il pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ continue, non-décroissante et quasi-concave un groupe continu des bijections continues de $\mathbb{R}_+^2$ telles que la famille des niveaux de $F$ est invariante par rapport à ces bijections.

   La réponse ne dépend done que des niveaux de $F$. Par exemple les orbites de $B$ sont invariantes par rapport à $B$.

*T. Mitchell (Southern Illinois University, Carbondale)*

**3. Remark.** (On S. Rolewicz's problem)
This is an information on the progress in proving the following

Conjecture (S. Rolewicz, private communication)
*The only even, nonnegative and differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the inequality*

$$f(t) - f(s) - f'(s)(t-s) \geq f(t-s), \quad t, s \in \mathbb{R},$$

*are given by the formula* $f(t) = Ct^2$, $t \in \mathbb{R}$, $C \geq 0$.

This was first proved by the speaker in the case where $f$ is twice differentiable in a neighborhood of the origin and it satisfies an initial condition, cf. [1]. Then we worked together with Z. Kominek proceeding the same way and trying to remove additional assumptions. Finally, Z. Kominek succeeded to prove the conjecture as it stands, by using density of a set of reals. (Added in proofs: cf. [2].) Another proof, shorter than ours (and avoiding evenness of $f$), was invented by R. Girgensohn when he made acquainted with our result. The paper of three authors *Solution of Rolewicz's Problem* has been accepted for publication online in *SIAM Problems & Solutions* (http://www.siam.org/journals/problems/01-005.htm.), together with a separate formulation of the problem.

[1] B. Choczewski, *Note on a functional-differential inequality.* In: *Functional Equations - Results and Advances*, ed. by Z. Daróczy and Zs. Páles. Dedicated to the Millenium of The Hungarian State. Kluwer Academic Publishers, Boston/Dordrecht/London, 2001, 21-24.

[2] B. Choczewski, Z. Kominek, *A proof of S. Rolewicz's conjecture*, this volume.

*Bogdan Choczewski*

**4. Problem.** (On $(\varepsilon, \delta)$-Wright-convex functions)
Let $I$ be an open interval and let $\varepsilon, \delta \geqslant 0$. If $A : \mathbb{R} \to \mathbb{R}$ is an additive function, $h : I \to \mathbb{R}$ is bounded function with $\|h\| \leqslant \delta$, $l : I \to \mathbb{R}$ is Lipschitz with $\mathrm{Lip}\,(l) \leqslant \varepsilon$, $g : I \to \mathbb{R}$ is a convex function, then the function

$$f = A + g + h + l \tag{1}$$

satisfies the following inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y) + 4\delta + 4\varepsilon t(1-t)|x-y|. \tag{2}$$

If $f : I \to \mathbb{R}$ is any function that satisfies (2), then does $f$ enjoy the decomposition (1) where $A$ is additive, $g$ is convex, $h$ is bounded and $l$ is Lipschitz? If $\varepsilon = 0$ then the affirmative answer was found by J. Mrowiec.

*Zsolt Páles*

**5. Remark.** (Solution to the Problem 1. on approximate sandwich theorems)

THEOREM

*Let $S$ be an abelian semigroup, $\varepsilon, \delta \geqslant 0$ and let $p : S \to \mathbb{R}$ be an $\varepsilon$-subadditive, $q : S \to \mathbb{R}$ be a $\delta$-superadditive function such that $q \leqslant p$. Then there exists an $\varepsilon$-subadditive and $\delta$-superadditive function $\varphi$ such that $q \leqslant \varphi \leqslant p$.*

*Proof.* By Zorn's lemma, we can find a minimal (with respect to the pointwise ordering) $\varepsilon$-subadditive function $p_0$ such that $q \leqslant p_0 \leqslant p$. Similarly, we can find a maximal $\delta$-superadditive function $q_0$ such that $q \leqslant q_0 \leqslant p_0 \leqslant p$. We show that $q_0 = p_0$, hence $\varphi = q_0 = p_0$ will be the separating function in question.

Assume that there exists $u \in S$ such that $q_0(u) < p_0(u)$. Choose $c$ so that $q_0(u) < c < p_0(u)$. Define $\overline{p} : S \to \mathbb{R}$ by

$$\overline{p}(t) = \inf \left\{ kc + \sum_{i=1}^{n} p_0(x_i) + (k+n-1)\varepsilon : \ k, n \geqslant 0, \ ku + \sum_{i=1}^{n} x_i = t \right\}$$

Then (with $k = 0$, $n = 1$, $x_1 = t$) we get $\overline{p}(t) \leqslant p_0(t)$ and (with $k = 1$, $n = 0$, $t = c$) we obtain $\overline{p}(u) \leqslant c < p_0(u)$. It is not difficult to see that $\overline{p}$ is also $\varepsilon$-subadditive. Since $\overline{p} \leqslant p_0$, $\overline{p}(u) < p_0(u)$, by the minimality of $p_0$ we get that $q_0 \not\leqslant \overline{p}$, that is there exists $t$ such that $\overline{p}(t) < q_0(t)$. Thus, there are $k, n \geqslant 0$, $x_1, \dots, x_n \in S$ such that

$$kc + \sum_{i=1}^{n} p_0(x_i) + (k+n-1)\varepsilon < q_0 \left( ku + \sum_{i=1}^{n} x_i \right). \tag{1}$$

Here $k = 0$ is impossible because

$$p_0 \left( \sum_{i=1}^{n} x_i \right) \leqslant \sum_{i=1}^{n} p_0(x_i) + (n-1)\varepsilon < q_0 \left( \sum_{i=1}^{n} x_i \right)$$

is an obvious contradiction.

A similar argument shows (interchange the roles of $p_0, q_0$ and $\varepsilon, \delta$) that there exist $l \geqslant 1$, $m \geqslant 0$, $y_1, \dots, y_m \in S$ such that

$$p_0 \left( lu + \sum_{j=1}^{m} y_j \right) < lc + \sum_{j=1}^{m} q_0(y_j) - (l+m-1)\delta. \tag{2}$$

Multiplying (1) by $l$, (2) by $k$ and adding up these two inequalities, we get

$$\begin{aligned} l \sum_{i=1}^{n} p_0(x_i) + l(k+n-1)\varepsilon + kp_0 \left( lu + \sum_{j=1}^{m} y_j \right) \\ < k \sum_{j=1}^{m} q_0(y_j) - k(l+m-1)\delta + lq_0 \left( ku + \sum_{i=1}^{n} x_i \right). \end{aligned}$$

By the $\varepsilon$-subadditivity of $p_0$ and the $\delta$-superadditivity of $q_0$, this yields

$$p_0\left(klu + l\sum_{i=1}^{n} x_i + k\sum_{j=1}^{m} y_j\right) + (k-1)(l-1)\varepsilon$$
$$< q_0\left(klu + l\sum_{i=1}^{n} x_i + k\sum_{j=1}^{m} y_j\right) - (k-1)(l-1)\delta$$

which contradicts $q_0 \leqslant p_0$, $\varepsilon, \delta \geqslant 0$. Thus the proof is complete.

*László Székelyhidi and Zsolt Páles*

**6. Remark.** (On functional equations connected with an embedding problem). In 1997 L. Reich posed the problem which can be shortly described as follows: under what assumptions the well-known linear functional equation can be "embeded" into a "continuous time" equation prolonging in natural way its iterative process [2]. The fundamental role in solving this problem play solutions of three functional equations in several variables

$$F(s+t,x) = F(t,F(s,x)), \tag{T}$$
$$G(s+t,x) = G(s,x)G(t,F(s,x)), \tag{G}$$
$$H(s+t,x) = G(t,F(s,x))H(s,x) + H(t,F(s,x)), \tag{H}$$

defined, in general, on the product of a commutative semigroup and a set $X$, with values in $X$, a given field $\mathbb{K}$ and a linear space $Z$ over $\mathbb{K}$, respectively.

We present new results on solutions of equations (T) and (G) defined on the product of the halfline of positive reals and an arbitrary closed interval. Presented results are contained in the paper [1].

Let $F : (0,\infty) \times [a,b] \to [a,b]$, where $-\infty \leqslant a < b \leqslant \infty$, be a continuous iteration semigroup such that the following conditions hold

(i) the function $F(1,\cdot)$ has no fixed points in $(a,b)$,

(ii) there is a point $c \in [a,b]$ such that the functions $F(1,\cdot)|_{[a,c]}$ and $F(1,\cdot)|_{[c,b]}$ are, respectively, strictly increasing and constant, if $F(1,x) > x$ for $x \in (a,b)$ and, respectively, constant and strictly increasing, if $F(1,x) < x$ for $x \in (a,b)$.

In the comprehensive paper [3] by M.C. Zdun one can find the collection of theorems completly describing continuous iteration semigroups, but now we present the result which is a reformulation and simplification of some Zdun's theorems.

Theorem A

*Let* $-\infty \leqslant a < b \leqslant \infty$ *and let* $F : (0,\infty) \times [a,b] \to [a,b]$ *be a function satisfying conditions* (i) *and* (ii).

*The function* $F$ *is a continuous iteration semigroup if and only if there* *exists a continuous strictly monotonic function $\alpha$ mapping $[a, b]$ onto an interval $[p, q] \subset [-\infty, \infty]$ such that*

$$F(t, x) = \alpha^{-1}(\min\{\alpha(x) + t, q\}) \tag{1}$$

*for every $t \in (0, \infty)$ and $x \in [a, b]$.*

The theorem describing general solutions of equations (G) in the case when $F$ is a continuous iteration semigroup is the following.

THEOREM B

*Let $-\infty \leqslant a < b \leqslant \infty$, $F : (0, \infty) \times [a, b] \to [a, b]$ be a continuous iteration semigroup satisfying conditions* (i) *and* (ii) *and let $\alpha$ be a continuous strictly monotonic function mapping an interval $[a, b]$ onto $[p, q] \subset [-\infty, \infty]$ such that the function $F$ is of the form* (1)*. Assume that $(Y, \cdot)$ is a commutative group.*

*The function $G : (0, \infty) \times [a, b] \to Y$ is a solution of equation* (G) *if and only if there exists a function $\phi : \alpha^{-1}([p, q] \cap \mathbb{R}) \to Y$ and solutions $m_1, m_2 : (0, \infty) \to Y$ of Cauchy's equation*

$$m(s + t) = m(s)m(t) \tag{M}$$

*such that for every $t \in (0, \infty)$ and $x \in [a, b]$*

$$G(t, x) = \begin{cases} m_1(t), & \text{if } \alpha(x) + t = p, \\ \frac{\phi(F(t,x))}{\phi(x)} & \text{if } p < \alpha(x) + t \leqslant q \text{ and } x \neq b, \\ \frac{\phi(F(t,x))}{\phi(x)} m_2(\alpha(x) + t - q), & \text{if } \alpha(x) + t > q \text{ and } x \neq b, \\ m_2(t), & \text{if } x = b, \end{cases} \tag{2}$$

*whenever $\alpha$ is increasing.*

The form of solutions of (G) in the case when the generator $\alpha$ is decreasing is similar.

Remark that the theorem on the form of general solution of equation (H) in the case when $F$ is a continuous iteration semigroup, $G$ solves (G) and both are defined on the product of $(0, \infty)$ and an arbitrary compact interval is now prepared by the present author.

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**7. Problems.** (Functional equations related to bijections preserving independence)

*1. Real positive random variables.* Consider functions: $\gamma_{p,b} : (0,\infty) \to (0,\infty)$, $\beta_{p,q} : (0,1) \to (0,\infty)$ and $\mu_{p,a,b} : (0,\infty) \to (0,\infty)$ defined by the formulas

$$\begin{aligned}\gamma_{p,b}(x) &= C_1 x^{p-1} e^{-bx},\\ \beta_{p,q}(x) &= C_2 x^{p-1}(1-x)^{q-1},\\ \mu_{p,a,b}(x) &= C_3 x^{-p-1} e^{-ax-\frac{b}{x}},\end{aligned}$$

where $a$, $b$, $p$, $q$ are positive constants and $C_1$, $C_2$, $C_3$ are normalizing constants making the respective functions integrable to 1 over their domains. Observe that they are, respectively, probability density functions of the gamma, beta and generalized inverse Gaussian distribution (which will be denoted by the same symbols). Consider also three bijections $\psi_1 : (0,\infty)^2 \to (0,1) \times (0,\infty)$, $\psi_2 : (0,\infty)^2 \to (0,\infty)^2$, $\psi_3 : (0,1)^2 \to (0,1)^2$ defined by

$$\begin{aligned}\psi_1(x,y) &= \left(\frac{x}{x+y}, x+y\right),\\ \psi_2(x,y) &= \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y}\right),\\ \psi_3(x,y) &= \left(\frac{1-y}{1-xy}, 1-xy\right).\end{aligned}$$

Then it can be easily checked that these maps preserve independence in the following sence:

1° if the random vector $(X,Y)$ has the distribution $\gamma_{p,b} \otimes \gamma_{q,b}$ ("$\otimes$" stands for the product measure), i.e. the component random variables are independent (we write it as: $(X,Y) \sim \gamma_{p,b} \otimes \gamma_{q,b}$) then $(U,V) = \psi_1(X,Y) \sim \beta_{p,q} \otimes \gamma_{p+q,b}$.

2° If $(X,Y) \sim \mu_{p,a,b} \otimes \gamma_{p,q}$ then $(U,V) = \psi_2(X,Y) \sim \mu_{p,b,a} \otimes \gamma_{p,b}$.

3° If $(X,Y) \sim \beta_{p,q} \otimes \beta_{p+q,r}$ then $(U,V) = \psi_3(X,Y) \sim \beta_{r,q} \otimes \beta_{r+p,p}$.

Consider now converse problems, i.e. assume that $(X,Y)$ is a random vector with independent components (and a suitable range) and $(U,V) = \psi_i(X,Y)$ for a given fixed $i \in \{1,2,3\}$ has also independent components. The basic question reads: does this property of preserving independence under given $\psi_i$ characterize respective distributions? Such a question leads to the following functional equations (if existence of densities is assumed): For $i = 1$ and $(u,v) \in (0,1) \times (0,\infty)$ almost everywhere

$$f_U(u) f_V(v) = v f_X(uv) f_Y((1-u)v);$$

for $i = 2$ and $(u, v) \in (0, \infty)^2$ almost everywhere

$$f_U(u) f_V(v) = \frac{1}{u^2(u+v)^2} f_X\left(\frac{1}{u+v}\right) f_Y\left(\frac{1}{u} - \frac{1}{u+v}\right);$$

for $i = 3$ and $(u, v) \in (0, 1)^2$ almost everywhere

$$f_U(u) f_V(v) = \frac{v}{1-uv} f_X\left(\frac{1-v}{1-uv}\right) f_Y(1-uv);$$

where $f_U$, $f_V$, $f_X$, $f_Y$ are probability density functions, i.e. are non-negative and integrable to 1 over their domains. In each of these three cases the solutions are known to be of the form as in the direct results (see 1°-3° above) under additional technical assumptions that $f_X$ and $f_Y$ are strictly positive and their logarithms are locally integrable functions. The question lies in removing these restrictions.

*2. Matrix variate random variables.* Denote by $\mathcal{V}_+$ the cone of symmetric, real, positive definite $n \times n$ matrices, which is a subset of the Euclidean space $\mathcal{V}$ of symmetric, real $n \times n$ matrices with the inner product $(a, b) = \operatorname{trace}(ab)$. Fix the Lebesgue measure on $\mathcal{V}$ by assigning the unit mass to the unit cube. Then the matrix variate gamma, beta and generalized inverse Gaussian distributions are defined by their densities of the form:

$$\begin{aligned}
\gamma_{p,b}(x) &= C_1(\det x)^{p-\frac{n+1}{2}} \exp\left(-(a, x)\right), \quad x \in \mathcal{V}_+; \\
\beta_{p,q}(x) &= C_2(\det x)^{p-\frac{n+1}{2}} (\det(e-x))^{q-\frac{n+1}{2}}, \ x \in \mathcal{D} = \{x \in \mathcal{V}_+ : e - x \in \mathcal{V}_+\}; \\
\mu_{p,a,b}(x) &= C_3(\det x)^{-p-\frac{n+1}{2}} \exp\left(-(a, x) - (b, x^{-1})\right), \quad x \in \mathcal{V}_+,
\end{aligned}$$

where $a, b \in \mathcal{V}_+$, $p, q > \frac{n-1}{2}$, and $C_1$, $C_2$, $C_3$ are normalizing constants ($e$ denotes the identity matrix).

Again consider three bijections: $\psi_1 : \mathcal{V}_+^2 \to \mathcal{D} \times \mathcal{V}_+$, $\psi_2 : \mathcal{V}_+^2 \to \mathcal{V}_+^2$, $\psi_3 : \mathcal{D}^2 \to \mathcal{D}^2$, defined by

$$\begin{aligned}
\psi_1(x, y) &= \left((x+y)^{-\frac{1}{2}} x (x+y)^{-\frac{1}{2}}, x + y\right), \\
\psi_2(x, y) &= \left((x+y)^{-1}, x^{-1} - (x+y)^{-1}\right), \\
\psi_3(x, y) &= \left(\left(e - y^{\frac{1}{2}} x y^{\frac{1}{2}}\right)^{-\frac{1}{2}} (e-y) \left(e - y^{\frac{1}{2}} x y^{\frac{1}{2}}\right)^{-\frac{1}{2}}, e - y^{\frac{1}{2}} x y^{\frac{1}{2}}\right).
\end{aligned}$$

Then, by computing respective jacobians, it can be checked that the statements 1°-3° of the preceding section hold true (with the meaning of all the symbols as defined in this section). With the converse results the situation is much more complicated than in the univariate case. First of all assume that $(X, Y)$ and $(U, V) = \psi_i(X, Y)$ for a given fixed $i \in \{1, 2, 3\}$ have independent components. Then for $i = 1$ we obtain the following equation for the densities:

$$f_U(u) f_V(v) = (\det v)^{\frac{n+1}{2}} f_X\left(v^{\frac{1}{2}} u v^{\frac{1}{2}}\right) f_Y\left(v^{\frac{1}{2}} (e-u) v^{\frac{1}{2}}\right)$$

for $(u,v) \in \mathcal{D} \times \mathcal{V}_+$ almost everywhere. This equation was solved under quite restrictive conditions that $f_X$ and $f_Y$ are strictly positive and twice differentiable on their domains. The proof was based on solutions of other two functional equations:

$$a(x) = g(yxy) - g(y(e-x)y), \quad (x,y) \in \mathcal{D} \times \mathcal{V}_+$$

solved under the differentiability condition assumed for $g$ and

$$a_1(x) + a_2(y) = g(yxy) + g(y(e-x)y), \quad (x,y) \in \mathcal{D} \times \mathcal{V}_+ ,$$

solved under the assumption that $g$ is twice differentiable. For all the above three equations the open problem lies in reducing the apriori smoothness conditions imposed on the unknown functions. Also a related multiplicative Cauchy functional equation in $\mathcal{V}_+$ of the form

$$f(x)f(y) = f\left(y^{\frac{1}{2}}xy^{\frac{1}{2}}\right), \quad (x,y) \in \mathcal{V}_+^2 ,$$

where $f : \mathcal{V}_+ \to (0,\infty)$ was solved under the differentiability assumption imposed on $f$. Here again the question lies in removing the smoothness restriction.

For $i = 2$ the equation for densities reads

$$f_U(u)f_V(v) = \frac{f_X((u+v)^{-1})f_Y(u^{-1} - (u+v)^{-1})}{(\det u \det(u+v))^{n+1}}$$

for $(u,v) \in \mathcal{V}_+^2$ almost everywhere, which was solved under the restriction that $f_X$ and $f_Y$ are strictly positive differentiable functions. Again the question here lies in removing these artificial assumptions.

For $i = 3$ the equation for densities is of the form

$$f_U(u)f_V(v) = \left(\frac{\det v}{\det y(u,v)}\right)^{\frac{n+1}{2}} f_Y(y(u,v))f_X\left(y^{-\frac{1}{2}}(u,v)(e-v)y^{-\frac{1}{2}}(u,v)\right)$$

for $(u,v) \in \mathcal{D}^2$ almost everywhere, where $y(u,v) = e - v^{\frac{1}{2}}uv^{\frac{1}{2}}$. However in this case even a solution under additional assumption smoothness conditions is not known. Observe that if densities $f_X$ and $f_Y$ are strictly positive then the equation can be reduced to

$$g_1(t) + g_2(w) = g_3(w(e-t^{-2})w) + g_4(t(e-w^{-2})t)$$

for $(t,w) \in (e + \mathcal{V}_+)^2$ almost ewerywhere.

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