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Spectra of certain operators and iterative functional equations

Abstract. We discuss spectral properties of the operator $F : \mathcal{D} \to F[\mathcal{D}]$, defined by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k}\, \varphi(2^k x).$$

$\mathcal{D}$ is the vector space of real functions φ such that the sum above converges for all $x \in \mathbb{R}$. The point spectrum and the eigenspaces of F and of its restriction to the vector space $\mathcal{U}$ of ultimately bounded functions are given. Moreover we compute the point spectrum and eigenspaces, the continuous spectrum and the residual spectrum of F, restricted to the Banach spaces $\mathcal{B}$ of bounded functions and $\mathcal{C}$ of bounded and continuous functions.

1. Background

We first consider the operator $F : \mathcal{D} \to F[\mathcal{D}]$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k}\, \varphi(2^k x), \tag{1}$$

where

$$\mathcal{D} = \{\varphi :\ \mathbb{R} \to \mathbb{R};\ F[\varphi] : \mathbb{R} \to \mathbb{R}\}.$$

So $\varphi \in \mathcal{D}$ iff the right hand side of (1) converges for every $x \in \mathbb{R}$.

There are several reasons to study this operator:

1. F generates continuous nowhere differentiable functions from very simple ones, like the Takagi function $F[d]$ from $d(x) = \operatorname{dist}(x, \mathbb{Z})$ or the Weierstrass function $F[c]$ from $c(x) = \cos 2\pi x$. See [4].
2. F plays a role in the stability theory of functional equations. See [1] and [2].
3. The study of operator theoretical properties of F exhibits interesting connections with the theory of iterative functional equations. See [3],

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[4] and [5]. For general facts concerning iterative functional equations see the monograph [7]. For the convenience of the reader, the solutions of some iterative functional equations connected to the operator F are constructed explicitely, although they could as well be deduced from [7].

The structure of $\mathcal{D}$ is investigated in [6]. In this paper we consider F and its restrictions to certain subspaces of $\mathcal{D}$, namely

$\mathcal{U} := \{\varphi : \mathbb{R} \to \mathbb{R};$ there are positive numbers $M(\varphi),\ \omega(\varphi)$ such that $|x| \geq \omega(\varphi)$ implies$|\varphi(x)| \leq M(\varphi)\}$,

$\mathcal{B} := \{\varphi : \mathbb{R} \to \mathbb{R};$ there is a positive number $M(\varphi)$ such that $|\varphi(x)| \leq M(\varphi)$ for every $x \in \mathbb{R}\}$,

$\mathcal{C} := \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi$ is bounded and continuous$\}$.

The following statements are in part proved in [3]-[6].

$F_1 := F : \mathcal{D} \to F[\mathcal{D}]$ is a vector space isomorphism,
$F_2 := F|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ is a vector space automorphism,
$F_3 := F|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a Banach space automorphism,
$F_4 := F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a Banach space automorphism.

An important tool for the proofs is the following fact which connects F with a first iterative functional equation.

PROPOSITION 1

a) *Assume that $\varphi \in \mathcal{D}$. Then $F[\varphi]$ satisfies the de Rham – type functional equation*

$$f(x) - \frac{1}{2} f(2x) = \varphi(x) \quad \textit{for every } x \in \mathbb{R}. \tag{2}$$

b) *Assume that $\varphi \in \mathcal{D}$ is given. Then equation (2) has at most one solution $f \in \mathcal{U}$, namely $f = F[\varphi]$.*

Proposition 1 is contained in [6]. Part b) reveals the importance of the subspace $\mathcal{U}$ of ultimately bounded functions. The statement is no longer true, if we allow unbounded solutions. So far, $F_2 : \mathcal{U} \to \mathcal{U}$ has not been examined in the literature.

For F_3 and F_4 the concepts of classical spectral theory for linear continuous operators $T \in L(X, X)$ on a Banach space X over $\mathbb{R}$ apply. We have the resolvent

$$\rho(T) := \{\lambda \in \mathbb{R};\ (\lambda I - T)^{-1} \in L(X, X)\}$$

$(I := \mathrm{id}_X)$ and the spectrum

$$\sigma(T) := \mathbb{R} \setminus \rho(T).$$

The spectrum can be partitioned into the point spectrum

$$\sigma_p(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ not injective}\},$$

the continuous spectrum

$$\sigma_c(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ injective, not surjective, } (\lambda I - T)(X) \text{ dense}\}$$

and the residual spectrum

$$\sigma_r(T) := \{\lambda \in \mathbb{R};\ \lambda I - T \text{ injective, not surjective, } (\lambda I - T)(X) \text{ not dense}\}.$$

Thus $\sigma_p(T)$ is the set of eigenvalues of T. We denote by $E\ (T, \lambda) := \{x \in X;\ Tx = \lambda x\}$ the eigenspace corresponding to $\lambda \in \sigma_p(T)$.

For F_1 and F_2 (where no topology is involved) the defining properties given above for the set $\sigma_p(F_\nu)$ of eigenvalues and the corresponding eigenspaces $E(F_\nu, \lambda)$ still make sense, whereas the continuous spectrum and the residual spectrum are not defined.

The aim of this paper is a complete description of $\sigma_p(F_\nu)$ and $E(F_\nu, \lambda)$ for $1 \leqslant \nu \leqslant 4$, which is given in Section 2 and a complete description of $\sigma_c(F_\nu)$ and $\sigma_r(F_\nu)$ for $3 \leqslant \nu \leqslant 4$, which is given in Section 3. Our systematic treatment extends some auxiliary results from [4] and [6]. In addition to (2) some other iterative functional equations will enter the scene.

2. Point spectra and eigenspaces of F_ν

Let F_ν, $1 \leqslant \nu \leqslant 4$, be defined as in Section 1 and, to unify notation, write $\mathcal{D}_\nu$ for the domain of F_ν, i.e. $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_2 = \mathcal{U}$, $\mathcal{D}_3 = \mathcal{B}$, $\mathcal{D}_4 = \mathcal{C}$.

Before dealing with the individual spectra and eigenspaces we collect some facts which are true for all F_ν.

Proposition 2

a) *We have* $0 \notin \sigma_p(F_\nu)$ *and* $1 \notin \sigma_p(F_\nu)$ *for* $1 \leqslant \nu \leqslant 4$.

b) *Fix* $\nu \in \{1, 2, 3, 4\}$, *and assume that* φ *belongs to the eigenspace* $E(F_\nu, \lambda)$. *Then* φ *satisfies the Schröder functional equation*

$$f(x) = \gamma(\lambda) f(2x) \quad \text{for every } x \in \mathbb{R}, \tag{3}$$

where

$$\gamma(\lambda) := \frac{1}{2}\frac{\lambda}{\lambda - 1}. \tag{3a}$$

c) $\lambda \leqslant \frac{1}{2}$ *implies* $\lambda \notin \sigma_p(F_\nu)$ *for* $1 \leqslant \nu \leqslant 4$.

Proof. a) and b). Assume that $\varphi \in E(F_\nu, \lambda)$ and $\varphi \neq \mathbf{o}$ (the zero function defined on $\mathbb{R}$) for some fixed $\nu \in \{1, 2, 3, 4\}$, i.e., $F_\nu[\varphi] = \lambda\varphi$. Then $\varphi \in \mathcal{D}_\nu$ and by Proposition 1 a)

$$F_\nu[\varphi](x) - \frac{1}{2}F_\nu[\varphi](2x) = \varphi(x)$$

for every $x \in \mathbb{R}$. Hence

$$(\lambda - 1)\varphi(x) = \frac{1}{2}\lambda\varphi(2x)$$

for every $x \in \mathbb{R}$. Since $\lambda = 1$ or $\lambda = 0$ would imply $\varphi = \mathbf{o}$, assertions a) and b) are proved.

c) Assume that $\varphi \in E(F_\nu, \lambda) \setminus \{\mathbf{o}\}$ for some fixed $\nu \in \{1, 2, 3, 4\}$. Then $\varphi \in \mathcal{D}_\nu$ and therefore $F_\nu[\varphi](x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$ converges for every $x \in \mathbb{R}$. By b) we have

$$\varphi(x) = \gamma\varphi(2x) = \gamma^m\varphi(2^m x) \tag{4}$$

for every $x \in \mathbb{R}$ and $m \in \mathbb{N}$, where $\gamma = \gamma(\lambda)$. We obtain $F_\nu[\varphi](x) = \sum_{k=0}^{\infty}(2\gamma)^{-k}\varphi(x)$ and because of $\varphi \neq \mathbf{o}$, $\sum_{k=0}^{\infty}(2\gamma)^{-k}$ must converge. Therefore necessarily $\left|\frac{1}{2\gamma}\right| < 1$, so that $\left|\frac{\lambda-1}{\lambda}\right| < 1$, i.e., $\lambda > \frac{1}{2}$.

Remark

The general solution $g : \mathbb{R} \to \mathbb{R}$ of the Schröder equation (3), which will be from now on referred to by (S_λ), is constructed as follows.

Choose any $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$ and extend it uniquely by (S_λ) to a function $g_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

Then extend g_1 to a function $g : \mathbb{R} \to \mathbb{R}$ by $g(0) := 0$ in case of $\lambda \neq 2$ and by $g(0) := \alpha \in \mathbb{R}$ arbitrarily in case of $\lambda = 2$.

Proposition 2 says that the eigenvalues of $F_\nu (1 \leqslant \nu \leqslant 4)$ are contained in the set

$$J = (\frac{1}{2}, \infty) \setminus \{1\}.$$

It turns out that in fact $\sigma_p(F_1)$ is the full set J. The point spectra $\sigma_p(F_\nu)$ then shrink in a remarkable way according to the shrinking of the domain $\mathcal{D}_\nu$ of F_ν, namely:

$$\sigma_p(F_2) = [\tfrac{2}{3}, 2] \setminus \{1\}, \quad \sigma_p(F_3) = \{\tfrac{2}{3}, 2\}, \quad \sigma_p(F_4) = \{2\}.$$

Moreover it turns out that all the eigenspaces $E(F_\nu, \lambda)$ for $\lambda \in \sigma_p(F_\nu)$ can be characterized as solution sets of equation (S_λ) under certain constraints. They are all of infinite dimension with exactly one exception: $E(F_4, 2)$. These facts are proved in the following

Proposition 3

We have

a) $\sigma_p(F_1) = (1/2, \infty) \setminus \{1\}$,

b) $\sigma_p(F_2) = [2/3, 2] \setminus \{1\}$,

c) $\sigma_p(F_3) = \{2/3, 2\}$,

d) $\sigma_p(F_4) = \{2\}$.

The eigenspaces

$$E(F_\nu, \lambda) = \{\varphi \in F_\nu;\ \varphi \ satisfies\ (S_\lambda)\}$$

are explicitely described (characterized) in the proof.

Proof. a) By Proposition 2, for any $\lambda \in \sigma_p(F_1)$ we have $\lambda \in J$.

Now let $\lambda \in J$. Take any nonzero solution $g : \mathbb{R} \to \mathbb{R}$ of the Schröder equation (S_λ) as described in Remark. Then g satisfies (4) with γ given by (3a), whence

$$\begin{aligned} F_1[g](x) &= \sum_{k=0}^{\infty} \frac{1}{2^k}\ g(2^k x) = \sum_{k=0}^{\infty} \frac{1}{(2\gamma)^k}\ g(x) = \frac{2\gamma}{2\gamma - 1}\ g(x) \\ &= \lambda g(x), \end{aligned}$$

which means that λ is an eigenvalue of F_1. Moreover, for $\lambda \in J$

$$E(F_1, \lambda) = \{\varphi \in \mathcal{D};\ \varphi \text{ satisfies } (S_\lambda)\} = \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ satisfies } (S_\lambda)\}.$$

The construction in Remark characterizes the elements of $E(F_1, \lambda)$ and clearly $\dim E(F_1, \lambda) = \infty$ for every $\lambda \in \sigma_p(F_1)$.

b) By a), for any $\lambda \in \sigma_p(F_2)$, we have $\lambda \in J$.

Now let $\lambda \in J$ and take, as in Remark, any bounded nonzero initial function $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$. Then its extension g_1 by $g(2x) = \frac{1}{\gamma} g(x)$ is ultimately bounded iff $\left|\frac{1}{\gamma}\right| = \left|2\frac{\lambda - 1}{\lambda}\right| \leqslant 1$, i.e., iff $\frac{2}{3} \leqslant \lambda \leqslant 2$.

As in a), this shows that any $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ is an eigenvalue of F_2. All elements of $E(F_2, \lambda)$ for $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ are generated from bounded $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$, because the extension g_1 of an unbounded g_0 would remain unbounded in the vicinity of $+\infty$ or $-\infty$, hence it does not belong to $\mathcal{U}$. Note that a function $g \in E(F_2, \lambda)$ is not necessarily bounded around zero. Clearly $\dim E(F_2, \lambda) = \infty$ for every $\lambda \in \sigma_p(F_2)$.

c) By b), for any $\lambda \in \sigma_p(F_3)$, we have $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$.

Now let $\lambda \in [\frac{2}{3}, 2] \setminus \{1\}$ and take, as in Remark, any bounded nonzero initial function $g_0 : (-2, -1] \cup [1, 2) \to \mathbb{R}$. Then its extension $g_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = \gamma g(2x)$ stays bounded iff $|\gamma| = 1$, i.e. iff $\lambda = \frac{2}{3}$ $(\gamma = -1)$ or $\lambda = 2$ $(\gamma = 1)$.

This shows as in a), that any $\lambda \in \{\frac{2}{3}, 2\}$ is an eigenvalue of F_3. As all elements of $E(F_3, \lambda)$ for $\lambda \in \{\frac{2}{3}, 2\}$ are bounded, the generating initial function has to be bounded as well.

Clearly $\dim E(F_3, \lambda) = \infty$ for $\lambda = \frac{2}{3}$ and for $\lambda = 2$.

d) By c), for any $\lambda \in \sigma_p(F_4)$, we have $\lambda \in \{\frac{2}{3}, 2\}$.

The elements of $E(F_4, \lambda)$ are the continuous and bounded solutions of (S_λ). The only continuous solution of $(S_{\frac{2}{3}}) : f(x) = -f(2x)$ is the function $\mathbf{o}$. Therefore $\lambda = \frac{2}{3}$ is not an eigenvalue of F_4. The only continuous solutions of $(S_2) : f(x) = f(2x)$ are the constant functions. So $\sigma_p(F_4) = \{2\}$ and $E(F_2, 2) = \text{span } \{\mathbf{1}\}$, where $\mathbf{1}(x) = 1$ for every $x \in \mathbb{R}$.

3. The spectra of F_3 and F_4 in the Banach space setting

We start with $F_3 \in L(\mathcal{B}, \mathcal{B})$. We have seen in Proposition 3 that $\sigma_p(F_3) = \{\frac{2}{3}, 2\}$. Therefore $\frac{2}{3}I - F_3$ and $2I - F_3$ are not injective. The remaining details on the spectrum of F_3 are given in

THEOREM 1

The point spectrum $\sigma_p(F_3)$ is $\{\frac{2}{3}, 2\}$. For every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, the operator $\lambda I - F_3$ is bijective. Consequently, the continuous spectrum $\sigma_c(F_3)$ and the residual spectrum $\sigma_r(F_3)$ are both empty.

Proof. For $\lambda \neq \frac{2}{3}$ and $\lambda \neq 2$ the operator $\lambda I - F_3$ is injective by Proposition 3 c). It remains to show that for any given $f \in \mathcal{B}$ and $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, the operator equation

$$(\lambda I - F_3)[\varphi] = f \tag{5}$$

has a solution $\varphi \in \mathcal{B}$. To do so, first assume that (5) has a solution $\varphi \in \mathcal{B}$. Then

$$f(x) = \lambda\varphi(x) - \left\{\varphi(x) + \frac{1}{2}\,\varphi(2x) + \frac{1}{2^2}\,\varphi(2^2x) + \cdots\right\},$$

$$\frac{1}{2}\,f\,(2x) = \frac{1}{2}\lambda\varphi(2x) - \left\{\frac{1}{2}\varphi(2x) + \frac{1}{2^2}\,\varphi(2^2x) + \cdots\right\},$$

hence

$$(\lambda - 1)\varphi(x) - \frac{1}{2}\,\lambda\varphi(2x) = f(x) - \frac{1}{2}\,f(2x). \tag{6}$$

For $\lambda = 1$, we define, according to (6),

$$\Phi(x) := f(x) - 2f\left(\frac{x}{2}\right). \tag{6a}$$

A simple calculation shows that this function Φ is in fact a bounded solution of (5). Excluding from now on the case $\lambda = 1$, we write equation (6) in the equivalent form

$$\varphi(x) - \frac{\lambda}{2(\lambda - 1)}\varphi(2x) = \frac{1}{\lambda - 1}\{f(x) - \frac{1}{2}f(2x)\}$$

or

$$\varphi(x) - \gamma\varphi(2x) = g(x) \tag{7}$$

with γ given by (3a) and

$$g(x) = \frac{1}{\lambda - 1}\left\{f(x) - \frac{1}{2}\, f(2x)\right\}.$$

Clearly $g \in \mathcal{B}$ and $|\gamma| \neq 1$ (as $\lambda \neq \frac{2}{3}$, $\lambda \neq 2$). Iteration of (7) gives

$$\varphi(x) = \gamma\varphi(2x) + g(x) = \cdots = \gamma^m\varphi(2^m x) + \sum_{k=0}^{m-1} \gamma^k g(2^k x)$$

for every $x \in \mathbb{R}$, $m \in \mathbb{N}$. Consequently, for $|\gamma| < 1$ there is at most one bounded solution Φ of (7), given by

$$\Phi(x) = \sum_{k=0}^{\infty} \gamma^k g(2^k x). \tag{8}$$

A direct substitution shows that (7) is satisfied with $\varphi = \Phi$. Going back further to (6), we see that

$$f(x) - \frac{1}{2}\, f(2x) = (\lambda - 1)\Phi(x) - \frac{1}{2}\,\lambda\Phi(2x) =: h(x).$$

As f and h are bounded, we have $f = F[h]$ by Proposition 1 b), hence

$$\begin{aligned} f(x) &= (\lambda - 1)F[\Phi](x) - \frac{1}{2}\,\lambda F[\Phi](2x) \\ &= (\lambda - 1)\left\{\Phi(x) + \frac{1}{2}\Phi(2x) + \frac{1}{2^2}\Phi(2^2x) + \cdots\right\} \\ &\quad - \frac{1}{2}\lambda\left\{\Phi(2x) + \frac{1}{2}\Phi(2^2x) + \frac{1}{2^2}\Phi(2^3x) + \cdots\right\} \\ &= \lambda\,\Phi(x) - F[\Phi](x). \end{aligned}$$

So in case $|\gamma| < 1$, i.e., $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, equation (5) has a solution $\Phi \in \mathcal{B}$, given by (8).

Now let $|\gamma| > 1$. We write equation (7) in the equivalent form

$$\varphi(x) = \frac{1}{\gamma}\,\varphi\Big(\frac{x}{2}\Big) - \frac{1}{\gamma}\,g\Big(\frac{x}{2}\Big). \tag{9}$$

Iteration of (9) gives

$$\varphi(x) = \left(\frac{1}{\gamma}\right)^n \varphi\left(\frac{x}{2^n}\right) - \sum_{k=1}^{n}\left(\frac{1}{\gamma}\right)^k g\left(\frac{x}{2^k}\right)$$

for every $x \in \mathbb{R}$, $n \in \mathbb{N}$. Because of $|\gamma| > 1$, there is at most one bounded solution Φ of (9), given by

$$\Phi(x) = -\sum_{k=1}^{\infty}\left(\frac{1}{\gamma}\right)^k g\left(\frac{x}{2^k}\right). \tag{10}$$

On the other hand, this function Φ satisfies equation (9):

$$\begin{aligned}\Phi(x) - \frac{1}{\gamma}\Phi\left(\frac{x}{2}\right) &= -\left\{\frac{1}{\gamma}g\left(\frac{x}{2}\right) + \frac{1}{\gamma^2}g\left(\frac{x}{2^2}\right) + \cdots\right\} \\ &\quad + \frac{1}{\gamma}\left\{\frac{1}{\gamma}g\left(\frac{x}{2^2}\right) + \frac{1}{\gamma^2}g\left(\frac{x}{2^3}\right) + \cdots\right\} \\ &= -\frac{1}{\gamma}\, g\left(\frac{x}{2}\right).\end{aligned}$$

Hence Φ satisfies equation (6) and, as in the case $|\gamma| < 1$, also equation (5).

The case $|\gamma| > 1$ corresponds to the remaining values $\lambda \in (\frac{2}{3}, 1) \cup (1, 2)$. Recall that $\lambda = 1$ has already been treated.

Finally, we discuss the spectrum of $F_4 \in L(\mathcal{C}, \mathcal{C})$.

By Proposition 3 d), the point spectrum of F_4 is just the singleton $\{2\}$. The eigenvalue $\frac{2}{3}$ of F_3 is no longer an eigenvalue of F_4. It turns out that $\frac{2}{3}$ belongs to the residual spectrum of F_4 and that the resolvent of F_4 coincides with the resolvent of F_3.

Theorem 2

The point spectrum and the residual spectrum of F_4 are singletons: $\sigma_p(F_4) = \{2\}$, $\sigma_r(F_4) = \{\frac{2}{3}\}$. For every $\lambda \in \mathbb{R} \setminus \{2/3, 2\}$ the operator $\lambda I - F_4$ is bijective. Consequently $\sigma_c(F_4)$ is empty.

Proof. For $\lambda \neq 2$, the operator $\lambda I - F_4$ is injective by Proposition 3 d). Next we show, that for any $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$ and for any given $f \in \mathcal{C}$, the operator equation

$$(\lambda I - F_4)[\varphi] = f \tag{11}$$

has a solution $\varphi \in \mathcal{C}$. To see this we argue exactly as in the proof of Theorem 1.

Let the value $\Phi(x)$ of the function $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by (6a) when $\lambda = 1$, by (8) when $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$ and by (10) when $\lambda \in (\frac{2}{3}, 1) \cup (1, 2)$. Then Φ is a bounded solution of equation (11), as we have seen in the proof of Theorem 1. Moreover, Φ is continuous, as $x \mapsto f(x) - 2f\left(\frac{x}{2}\right)$ is continuous and because the series (8) and (10) are uniformly convergent on $\mathbb{R}$ with continuous terms. So $\mathbb{R} \setminus \{\frac{2}{3}, 2\}$ belongs to the resolvent $\rho(F_4)$.

The only remaining case to be checked is $\lambda = \frac{2}{3}$ (not covered by the argument in the proof of Theorem 1). If equation (11) has a solution $\varphi \in \mathcal{C}$ for $\lambda = \frac{2}{3}$, then necessarily

$$\varphi(x) + \varphi(2x) = 3\left\{\frac{1}{2}f(2x) - f(x)\right\} = g(x). \tag{12}$$

This corresponds to equation (7), recall that $\lambda = \frac{2}{3}$ iff $\gamma = -1$. Note further, that $g = -3F_4^{-1}[f]$ or equivalently $f = -\frac{1}{3}F_4[g]$. Iteration of (12) gives

$$\varphi(x) = (-1)^n \varphi(2^n x) + \sum_{k=0}^{n-1} (-1)^k g(2^k x) \tag{13}$$

for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
Now take any $\tilde{g} \in \mathcal{C}$ such that $\tilde{g}(2^k) = (-1)^k$ and $\| \tilde{g} \| = \sup \{|\tilde{g}(t)|;\ t \in \mathbb{R}\} = 1$. Then for the corresponding $\tilde{\varphi}$ we would obtain by (13)

$$\begin{aligned}\tilde{\varphi}(1) &= (-1)^n \tilde{\varphi}(2^n) + \sum_{k=0}^{n-1} (-1)^k (-1)^k \\ &= (-1)^n \tilde{\varphi}(2^n) + n\end{aligned}$$

for every $n \in \mathbb{N}$, which is impossible for a bounded $\tilde{\varphi}$. Hence (12) has no solution $\varphi = \tilde{\varphi} \in \mathcal{C}$ for $g = \tilde{g}$.

Consequently, equation (11) with $\lambda = \frac{2}{3}$, i.e.

$$(\frac{2}{3} I - F_4)[\varphi] = f \tag{14}$$

has no solution $\varphi = \tilde{\varphi} \in \mathcal{C}$ for $f = \tilde{f} = -\frac{1}{3}F_4[\tilde{g}]$.

Now let $f^* \in \mathcal{C}$ such that $\| \tilde{f} - f^* \| \leqslant \frac{1}{9}$.

If (14) has a solution $\varphi = \varphi^* \in \mathcal{C}$ for $f = f^*$, then (12) and (13) are satisfied with $g = g^*$ and we have, using $\| F_4^{-1} \| = \frac{3}{2}$ (which is proved in [4])

$$\begin{aligned}\| \tilde{g} - g^* \| &= \| -3F_4^{-1}[\tilde{f}] + 3F_4^{-1}[f^*] \| \\ &= 3 \| F_4^{-1}[\tilde{f} - f^*] \| \\ &\leqslant 3 \| F_4^{-1} \| \| \tilde{f} - f^* \| \\ &= \frac{1}{2}.\end{aligned}$$

This implies $|\tilde{g}(2^k) - g^*(2^k)| = |(-1)^k - g^*(2^k)| \leqslant \frac{1}{2}$, hence $g^*(2^k) = (-1)^k \cdot \varepsilon_k$ with $\varepsilon_k \in [\frac{1}{2}, \frac{3}{2}]$. Now by (13)

$$\varphi^*(1) = (-1)^n \varphi^*(2^n) + \sum_{k=0}^{n-1} \varepsilon_k$$

for every $n \in \mathbb{N}$ which is again impossible for $\varphi^* \in \mathcal{C}$. This contradiction shows that no element of the closed ball with center $\tilde{f}$ and radius $\frac{1}{9}$ belongs to the image $(\frac{2}{3}I - F_4)[\mathcal{C}]$, so that this set is not dense in $\mathcal{C}$, hence $\frac{2}{3} \in \sigma_r(F_4)$.

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