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Some consequences of a theorem of Liouville

Abstract. Let E_n denote the n-dimensional Euclidean space and S the group of Euclidean similarities. It is shown that the group $\langle g, S\rangle$ generated by S and a single diffeomorphism g outside S has an orbit which is dense in $(E_n)^{n+1}$.

1. Introduction

By the theorem referred to in the title we mean Liouville's theorem on conformal mappings in space, that is, mappings which preserve all angles between smooth curves. It may be stated as follows.

Liouville's Theorem

Any sufficiently smooth conformal mapping between connected open regions of a Euclidean space of dimension at least three is induced by a Möbius transformation acting on the whole space together with a point at infinity.

In textbooks on differential geometry and related subjects this theorem is usually proved for differentiable mappings of class C^3 (see e.g. [3], p. 140, [4], vol. I, p. 373, or [12], vol. III, p. 310 and vol. IV, p. 13). However in 1958, Philip Hartman has proved it for C^1 mappings (see [6]) and more recently Yu. Rešetnyak has proved an even stronger theorem in which he makes no differentiability assumptions at all (see [8], [9]).

Let E_n denote the n-dimensional Euclidean space and S its group of automorphisms comprising Euclidean motions as well as similarities. Thus with respect to Cartesian coordinates S consists of all mappings of the form

$$X \mapsto \lambda X \cdot M + V, \quad 0 \neq \lambda \in \mathbb{R}, \quad M \cdot M^{\top} = I \quad (M \text{ an orthogonal matrix}).$$

A matrix of the form λM where M is orthogonal will be called *quasi-orthogonal.* For an arbitrary number m let us denote by $(E_n)^{m+1}$ the set $E_n \times E_n \times \ldots \times E_n$ where the factor appears $m+1$ times. As S acts on E_n it also acts on the sets $(E_n)^{m+1}$.

For an $(m+1)$-tuple $(P_0, P_1, \ldots, P_m) \in (E_n)^{m+1}$ we shall denote by $[P_0, P_1, \ldots, P_m]$ the affine subspace spanned by these points. An $(m+1)$-

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tuple of points not contained in any subspace of dimension less than m will be called *independent* or will be referred to as a non-degenerate m-simplex or just as an *m-simplex* for brevity.

THEOREM 1

Let $m \leqslant n$ and let g denote a differentiable C^1 mapping of E_n onto itself which has a differentiable inverse. If $g \notin S$ then the group $G = \langle g, S \rangle$ generated by g and the group S of Euclidean similarities has an orbit Ω which is dense in $(E_n)^{m+1}$. Precisely speaking, any orbit containing a non-degenerate m-simplex is dense in $(E_n)^{m+1}$.

The main tool in the proof is Liouville's theorem in the case where the mapping is defined in the whole of E_n, combined with the fact that the Euclidean group S is maximal within the affine group (see [5], [7], or [11]). Although Liouville's theorem is not valid for planar regions in general, its analogue for the case when the region considered is the whole plane is still true. Therefore Theorem 1 holds also for $n = 2$.

Let G be a group acting on E_n and $\mathcal{D}$ a subset of $(E_n)^{m+1}$ which is invariant under the action of G induced on $(E_n)^{m+1}$. We consider functions $f : \mathcal{D} \to \mathbb{R}$ where $\mathbb{R}$ denotes the field of real numbers satisfying the following functional equation:

$$f(g(P_0), g(P_1), \ldots, g(P_m)) = f(P_0, P_1, \ldots, P_m), \quad \text{for all } g \in G \qquad \text{(i.1)}$$

Any function f satisfying (i.1) with respect to a given group G is called an invariant with respect to the group G. An invariant with respect to the group S of Euclidean similarities will be called a *Euclidean invariant*. As an immediate consequence of the theorem above we obtain:

COROLLARY 1

Let $m \leqslant n$. Let the bijective mapping $g : E_n \to E_n$ and its inverse be of class C^1 and assume (i.1) *holds for a continuous Euclidean invariant f and for $G = \langle S, g \rangle$. Assume further that f is defined on a set which contains a non-degenerate m-simplex. Then either $g \in S$ or else f is constant.*

Corollary 1 is a slight improvement of theorem 2 of [10], p. 107, which was proved without any reference to transitivity properties of the group G that is, independently of Theorem 1. Let us call an invariant f trivial if it assumes distinct values only on tuples $(P_0, P_1, \ldots, P_m)$ and $(Q_0, Q_1, \ldots, Q_m)$ which are not mapped onto each other by any bijection since they can be distinguished by means of the identity relation. Evidently non-trivial $(m+1)$-ary invariants exist with respect to a group G if and only if the group is not $(m+1)$-fold transitive. The example of the affine group shows that the theorem and the corollary cannot be improved in a certain sense. For, by the maximality of S within the affine group, the affine group is generated by any affine mapping not in S together with S. Also the affine group is transitive on m-simplexes but not on arbitrary $(m+1)$-tuples. Hence non-trivial $(n+1)$-ary invariants exist for the affine group so that we cannot omit the continuity in the corollary. But the situation may change if we require additionally that g is not an affine mapping. It is not known to the author whether there exist differentiable mappings g other than affine ones such that $G = \langle g, S\rangle$ is not $(n+1)$-fold transitive.

Also the assumption that $m \leqslant n$ cannot be omitted. Here again the affine group provides us with a counterexample. Consider an r-simplex $P_0, P_1, \dots, P_r$. Recall that each point X of the subspace $[P_0, P_1, \dots, P_r]$ can be written in a unique way as

$$X = \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_r P_r$$

where $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$ and λ_0, λ_1, ..., λ_r are called the *barycentric coordinates of* X with respect to P_0, P_1, ..., P_r. As a function of P_0, P_1, ..., P_r, and X the i-th barycentric coordinate $\lambda_i = \lambda_i(P_0, P_1, \dots, P_r, X)$ is a continuous affine invariant which is not constant. Let us now take $r = n$. From the fact that the point X is uniquely determined by its barycentric coordinates $\lambda_0, \dots, \lambda_n$ it follows easily that the affine group has no dense orbits on $(E_n)^{n+2}$.

Note however, that the invariants $\lambda_i = \lambda_i(P_0, P_1, \dots, P_r, X)$ are no counter-examples against Corollary 1 since they do not contain any $(r+1)$-simplexes in their domain of definition. Hence the condition of the existence of a simplex in the domain of definition is also essential and cannot be omitted from the hypotheses of the corollary.

2. Proof of Theorem 1

The following Lemmas 2.1-2.3 will serve as preliminary steps towards the proof.

We start with an elementary fact about the Euclidean group, namely that there are no groups in between the Euclidean group and the affine group. Thus if $S \subset H \subseteq A$ where H is a subgroup, then $H = A$. Denote by $GL_n(\mathbb{R})$ the group of all real $n \times n$ matrices with non-vanishing determinant and by $\mathbb{R}^* O_n(\mathbb{R})$ the group of all quasi-orthogonal matrices. Then it is easily seen that the above assertion is equivalent with the following:

Lemma 2.1

Let $H_n(\mathbb{R})$ be a subgroup of $GL_n(\mathbb{R})$ properly containing the group $\mathbb{R}^* O_n(\mathbb{R})$. Then $H_n(\mathbb{R}) = GL_n(\mathbb{R})$.

We consider a bijective mapping $g : E \to E$ such that g and its inverse are of class C^1. As in the theorem let $G = \langle g, S\rangle$ and assume $g \notin S$. We express g in terms of coordinates in the form

$$g(X) = (u_1(x_1, x_2, \dots, x_n), u_2(x_1, x_2, \dots, x_n), \dots, u_n(x_1, x_2, \dots, x_n)).$$

Then with respect to an arbitrary point $P_0 = (x_{01}, x_{02}, \ldots, x_{0n})$ we have (with $P = (x_1, x_2, \ldots, x_n)$)

$$u_i(P) = u_i(P_0) + \sum_1^n \frac{\partial u_i}{\partial x_j}(x_j - x_{0j}) + o_i(P)$$

where

$$\lim_{P \to P_0} \frac{o_i(P)}{|P - P_0|} = 0,$$

where $|P - P_0|$ stands for the Euclidean distance between P and P_0. Let D denote the Jacobian matrix $D = \left(\frac{\partial u_i}{\partial x_j}\right)$. Then the above equations can be rewritten as

$$g(P) = g(P_0) + (P - P_0) \cdot D^\top + o(P), \tag{1}$$

where $o(P)$ is a vector valued function such that $\frac{o(P)}{|P-P_0|} \to 0$ as $|P - P_0| \to 0$.

When $n \geqslant 3$ from Liouville's theorem it follows:

Lemma 2.2

There exists a point P where D is not quasi-orthogonal.

Indeed, assume the contrary: the Jacobian matrix D of g is quasi-orthogonal at each point P.

An arbitrary C^1-mapping preserves angles between smooth curves going through a point P if, and only if, its Jacobian matrix D at P is quasi-orthogonal. Hence g preserves all angles between smooth curves, i.e. g is a conformal mapping. When $n \geqslant 3$ it follows by Liouville's theorem that g is induced by a Möbius transformation. Let us recall that Möbius transformations are bijective mappings of the set $E_n \cup \{\infty\}$ onto itself which can be composed from inversions at spheres or hyperplanes (see [2]). Alternatively, they can be characterized as maps preserving the system of point sets which are either given by spheres of E_n or by hyperplanes of E_n together with the point ∞. Since g is defined on the whole space E_n and maps E_n onto itself we may identify g with the Möbius transformation in question, which fixes the point ∞. It is well-known that any Möbius transformation fixing ∞, as a mapping of E_n must belong to S. Thus $g \in S$ contrary to our assumption that $g \notin S$.

The case $n = 2$ needs some special attention since there exist conformal mappings of planar regions which are not induced by Möbius transformations. As is well-known however, it follows from the theorem of Casorati-Weierstraß that a conformal and one-to-one mapping of the entire complex plane is necessarily of one of the forms $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$. Since such mappings belong to S it follows that Lemma 2.2 holds also when $n = 2$.

If $g(P) \neq P$ consider the mapping $h : E_n \to E_n$ given by

$$h(X) = g(X) - g(P) + P.$$

This mapping belongs to $G = \langle g, S \rangle$, fixes the point P and has the same Jacobian matrix as g. Let G_P denote the subgroup of G fixing the point P. The Jacobian matrices of elements of G_P taken at P form a group containing the matrix D and also all quasi-orthogonal matrices. It follows that this group is the full linear group $GL_n(\mathbb{R})$. If Q is another point then G_Q is conjugate to G_P by a translation t taking P to Q, i.e. $G_Q = tG_Pt^{-1}$. This implies that the Jacobian matrices of elements of G_Q also exhaust the full linear group.

LEMMA 2.3

Let P be an arbitrary point. Then for each invertible matrix M there exists an element h contained in the stabilizer G_P having M as its Jacobian matrix.

Let Δ_1 and Δ_2 be n-simplexes with the vertices $P_0, P_1, \ldots, P_n$ and $Q_0, Q_1, \ldots, Q_n$, respectively. In order to prove the theorem it suffices to show that for a given $\varepsilon > 0$ there exists an n-simplex Δ_3 in the orbit of Δ_1 with vertices $P'_0, P'_1, \ldots, P'_n$ such that $|Q_i - P'_i| < \varepsilon$, $i = 0, 1, \ldots, n$. Since G contains arbitrary translations we may assume here that $P_0 = Q_0 = O$ where O denotes the origin. By Lemma 2.3 there exists a mapping $h \in G_O$ with the Jacobian matrix A satisfying $P_i \cdot A^\top = Q_i$, $i = 1, \ldots, n$. For h formula (1) becomes

$$h(P) = P \cdot A^\top + o(P), \quad P \to O.$$

Let $d = \max |P_i|$, $i = 1, \ldots, n$. Choose δ such that $|o(P)| < \frac{\varepsilon}{d}|P|$ if $|P| < \delta$. Choose λ such that $\lambda|P_i| \leqslant \delta$. Then $|o(\lambda P_i)| < \frac{\varepsilon}{d}\lambda|P_i| \leqslant \varepsilon\lambda$. Let $f : E_n \to E_n$ denote the similarity, $f(X) = \lambda X$. Then $f^{-1}hf \in G$ and $f^{-1}(h(f(P_i)) = Q_i + Y_i$ where $|Y_i| = \lambda^{-1}|o(\lambda P_i)| < \varepsilon$. This proves the theorem.

REMARK 1

Theorem 1 can easily be extended to arbitrary differentiable mappings. To do this one has to use the stronger result of Yu. Rešetnyak rather than Liouville's theorem for C^1 mappings. All that needs to be done here is to show that Rešetnyak's condition of conformality (see [8]) is satisfied by a differentiable mapping at a point P provided the Jacobian matrix of the mapping is quasi-orthogonal at P. Then Rešetnyak's result will ensure that a differentiable mapping which has quasi-orthogonal Jacobian matrix everywhere is a Möbius transformation. Hence Lemma 2.2 can be proved in a similar way as we have done it for C^1 mappings. The rest of the proof goes unchanged.

3. Continuous extensions of the affine group

In this section we consider extensions of the group A of all affine mappings $X \mapsto X \cdot M + B$ of E_n by a continuous transformation f, i.e., a bijective mapping $f : E_n \to E_n$ which is continuous in both directions.

THEOREM 2

For any continuous transformation f of E_n either $f \in A$ or the group $G = \langle f, A \rangle$ is $(n+1)$-fold transitive on E_n.

Proof. The proof will be achieved through the following elementary Lemmas 3.1 - 3.10 below.

Assume that for $1 \leqslant m \leqslant n$ there exist mutually distinct points Q_0, $Q_1, \dots,$ Q_m, R such that $[Q_0, Q_1, \dots, Q_m]$ has dimension m and contains R and that none of the transforms of the tuple $(Q_0, Q_1, \dots, Q_m, R)$ spans a subspace of higher dimension. For brevity let us write

$$\rho(Q_0, Q_1, \dots, Q_m, R)$$

if this is true. By this definition we may permute the points $Q_0, Q_1, \dots, Q_m$ arbitrarily without disturbing the relation ρ. It should be clear that there exist relations of this type which are not empty. To see this we need only take $m = n$.

LEMMA 3.1

If m is the smallest number for which a non-empty relation ρ exists then G is $(m+1)$-fold transitive.

Let $Q_0, Q_1, \dots, Q_{m-1}, Q_m$ be distinct points which span a subspace of dimension less or equal $m - 1$. Since the subgroup A of G is transitive on m-simplexes it suffices to show that there is an $h \in G$ such that $h(Q_0), h(Q_1), \dots,$ $h(Q_{m-1}), h(Q_m)$ is an m-simplex. Choose a maximal independent subset among these points. Since the ordering plays no role here, we may assume that $Q_0, Q_1, \dots, Q_r$, $r < m$, are the points of this subset. Then since m was assumed minimal there exists an $h_1 \in G$ such that

$$h_1(Q_0), h_1(Q_1), \dots, h_1(Q_r), h_1(Q_{r+1})$$

are independent. If $r+2 \leqslant m$ and $h_1(Q_0), h_1(Q_1), \dots, h_1(Q_{r+1}), h_1(Q_{r+2})$ are dependent we may find $h_2 \in G$ such that

$$h_2[h_1(Q_0)], h_2[h_1(Q_1)], \dots, h_2[h_1(Q_{r+1})], h_2[h_1(Q_{r+2})]$$

are independent. Continuing in this way we find the required mapping $h \in G$.

LEMMA 3.2

Let $h \in G$. Then

$$\rho(Q_0, Q_1, \dots, Q_m, R)$$

implies

$$\rho(h(Q_0), h(Q_1), \dots, h(Q_m), h(R))$$

provided that the points $h(Q_0), h(Q_1), \dots, h(Q_m)$ are independent.

This follows immediately from the definition of the relation ρ. Since all affine mappings are contained in G it follows that if the tuples $(Q_0, Q_1, \dots, Q_m, R)$ and $(Q'_0, Q'_1, \dots, Q'_m, R')$ are affinely equivalent then the assertions

$$\rho(Q_0, Q_1, \dots, Q_m, R) \quad \text{and} \quad \rho(Q'_0, Q'_1, \dots, Q'_m, R')$$

are equivalent.

LEMMA 3.3

Assume that $\rho(Q_0, Q_1, \dots, Q_m, R_\nu)$ holds for all elements of a sequence (R_ν) converging to a point R distinct from $Q_0, Q_1, \dots, Q_m$. Then

$$\rho(Q_0, Q_1, \dots, Q_m, R)$$

holds as well.

Let h be an arbitrary element of G. Denote by L_ν the subspace spanned by the points $h(Q_0), h(Q_1), \dots, h(Q_m), h(R_\nu)$ and by L the subspace spanned by the points $h(Q_0), h(Q_1), \dots, h(Q_m), h(R)$. Let $m_1 = m_1(h)$ be the largest from among those numbers $\dim L_\nu$ that occur inifinitely often in the sequence and choose a subsequence R_λ such that $\dim L_\lambda = m_1$ for all λ. Note that $m_1 \leqslant m$ because of the assumption that $\rho(Q_0, Q_1, \dots, Q_m, R_\nu)$ is true.

If $h(R)$ depends on the points $h(Q_0), h(Q_1), \dots, h(Q_m)$ then obviously $\dim L = \dim[h(Q_0), h(Q_1), \dots, h(Q_m)] \leqslant \dim L_\lambda = m_1$.

If $h(R)$ is independent of the remaining points $h(Q_0), h(Q_1), \dots, h(Q_m)$ then the same is true of $h(R_\nu)$ at least for all ν larger than a certain number N. Then $\dim(L) = \dim(L_\lambda) = m_1$ for all $\lambda > N$.

Since $m_1(h) \leqslant m$ for all $h \in G$ it follows that $\rho(Q_1, Q_2, \dots, Q_m, R)$ is true.

From now on we shall work with barycentric coordinates.

LEMMA 3.4

If $S = (Q_0, Q_1, \dots, Q_m)$ and $S' = (Q'_0, Q'_1, \dots, Q'_m)$ span subspaces of dimension m and if R, R' are points such that the barycentric coordinates of R with respect to S coincide with the barycentric coordinates of R' with respect to S' then $\rho(Q_0, Q_1, \dots, Q_m, R)$ is equivalent with $\rho(Q'_0, Q'_1, \dots, Q'_m, R')$.

This follows from the fact that the tuples

$$(Q_0, Q_1, \dots, Q_m, R) \quad \text{and} \quad (Q'_0, Q'_1, \dots, Q'_m, R')$$

are affinely equivalent if and only if the barycentric coordinates of R with respect to $Q_0, Q_1, \dots, Q_m$ are the same as those of R' with respect to $Q'_0, Q'_1, \dots, Q'_m$.

LEMMA 3.5

If $\rho(Q_0, Q_1, \dots, Q_m, R)$ is true and R is not contained in the subspace spanned by $Q_0, Q_1, \dots, Q_{m-1}$ then $\rho(Q_0, Q_1, \dots, Q_{m-1}, R, Q_m)$ is also true.

This is true since $(Q_0, Q_1, \ldots, Q_{m-1}, R)$ is still an m-simplex if $(Q_0, Q_1, \ldots, Q_m)$ is, and since $Q_m \in [Q_0, \ldots, Q_{m-1}, R]$ if $R \in [Q_0, \ldots, Q_m]$. Apart from that for $\rho(Q_0, Q_1, \ldots, Q_{m-1}, R, Q_m)$ to be true we only require that the dimension of the subspace spanned by $h(Q_0)$, $h(Q_1), \ldots,$ $h(Q_m)$, $h(R)$ never exceeds m as h runs through G. This follows from the assumption that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true.

Lemma 3.6

If $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true and R is contained in the subspace S_{m-1} spanned by $Q_0, Q_1, \ldots, Q_{m-1}$ then $\rho(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$ is true for any point P outside the subspace S_{m-1}.

This follows from the fact that the tuple $(Q_0, Q_1, \ldots, Q_{m-1}, Q_m, R)$ is affinely equivalent with $(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$.

Let us now assume that m is minimal so that G is $(m+1)$-fold transitive.

Lemma 3.7

If $m = 1$ then $f \in A$.

If $m = 1$ then there exist points Q_0, Q_1, R such that $\rho(Q_0, Q_1, R)$ holds, i.e., the points Q_0, Q_1, R are distinct and collinear, and $g(Q_0)$, $g(Q_1)$, $g(R)$ are collinear for all $g \in G$. Let L denote the line going through Q_0, Q_1, and R. We thus have a set $M_0 = \{Q_0, Q_1, R\}$ of collinear points such that for each $g \in G$ the set $g(M_0)$ will again be collinear. By interchanging the points Q_0, Q_1, R if necessary, we may assume that R belongs to the segment Q_0Q_1.

Let $R = \sigma Q_0 + \tau Q_1$ where $\sigma + \tau = 1$ and σ, τ are the barycentric coordinates of R with respect to Q_0 and Q_1. If $\tau S = -\sigma Q_0 + Q_1$ then $Q_1 = \sigma Q_0 + \tau S$ whence $\rho(Q_0, S, Q_1)$. Similarly if $\sigma T = Q_0 - \tau Q_1$ then $Q_0 = \sigma T + \tau Q_1$ whence we have $\rho(T, Q_0, Q_1)$. The points S and T lie outside the segment Q_0Q_1 and on different sides of it. We are now going to enlarge the set M_0 successively.

First we may add the points S, T to get $M_1 = \{T, Q_0, R, Q_1, S\}$ which still has the property that $g(M_1)$ is a set of collinear points for all $g \in G$. Secondly, for all pairs of consecutive points C_1, C_2 in such a set we may add the point $C = \sigma C_1 + \tau C_2$. Combining these steps alternately we arrive at a sequence $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_\nu \subseteq \ldots$ of sets such that $g(M_\nu)$ consists of collinear points for all $g \in G$ and $M = \bigcup M_\nu$ is dense on the line L. Consequently, also the points of $g(M)$ are collinear for all $g \in G$. Since the set M is dense on L and the mappings g are continuous, it follows that the line L is mapped onto another line by every $g \in G$. The same must be true for any other line L_1 since on L_1 we can construct a set of points M_1 analogous to the set M on L.

Thus each $g \in G$ preserves lines and hence it is a semi-affine mapping of E_n. Since the field of real numbers has no automorphisms other than the identity, it follows that G consists entirely of affine mappings, i.e., $G = A$ and consequently $f \in A$.

We shall now assume that $m > 1$ which is only possible if $G \neq A$, i.e., $f \notin A$.

Lemma 3.8

If there exists a point R contained in S_{m-1} such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ is true then $\rho(Q_0, Q_1, \ldots, Q_m, X)$ is true for every point $X \in S_{m-1}$ which is distinct from $Q_0, Q_1, \ldots, Q_{m-1}$.

We can find a mapping $h \in G$ which fixes $Q_0, Q_1, \ldots, Q_{m-1}$ and maps R to another arbitrarily chosen point R_1 inside S_{m-1} which is distinct from $Q_0, Q_1, \ldots, Q_{m-1}$. Since h is a homeomorphism it cannot transform all points of $E_n \setminus S_{m-1}$ into S_{m-1}. Let Q be a point which remains outside, i.e., $h(Q) \in E_n \setminus S_{m-1}$. Choose an affine map a which fixes S_{m-1} pointwise and maps $h(Q)$ to Q. Then

$$ah(R) = R_1,\ ah(Q_0) = Q_0,\ ah(Q_1) = Q_1, \ldots,\ ah(Q_{m-1}) = Q_{m-1},$$

and

$$ah(Q) = Q.$$

By Lemmas 3.2 and 3.6 it follows that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$ is true and again by Lemma 3.6 also that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q_m, R_1)$ holds.

Lemma 3.9

If there exists a point R such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds we can find points P in the subspace S_{m-1} generated by Q_0, Q_1, …, Q_{m-1} such that the relation $\rho(Q_0, Q_1, \ldots, Q_m, P)$ holds.

If R is contained in one of the subspaces spanned by the faces of the simplex then using an affine map h_i that permutes the points $Q_0, Q_1, \ldots, Q_m$ appropriately, we can find points $R_i = h_i(R)$ in each of these subspaces such that the relation $\rho(h_i(Q_0), h_i(Q_1), \ldots, h_i(Q_m), h_i(R))$ is satisfied. Since we may arbitrarily permute the points $h_i(Q_0), h_i(Q_1), \ldots, h_i(Q_m)$ without disturbing ρ it follows that $\rho(Q_0, Q_1, \ldots, Q_m, R_i)$.

Let us now assume that R is not contained in any one of these subspaces. Then if $\lambda_0, \lambda_1, \ldots, \lambda_m$ are its barycentric coordinates, we have $0 \neq \lambda_i$ for $i = 0, 1, \ldots, m$. We may assume $\lambda_m \neq -1$. (If $\lambda_m = -1$ then $\lambda_i \neq -1$ for a suitable index i and we may interchange the points Q_i and Q_m.) Let $T(R)$ be the point which has barycentric coordinates $\lambda_0, \lambda_1, \ldots, \lambda_m$ with respect to $Q_0, Q_1, \ldots, Q_{m-1}, R$. Then the barycentric coordinates of $T(R)$ with respect to $Q_0, Q_1, \ldots, Q_m$ are $\lambda_0(1+\lambda_m), \lambda_1(1+\lambda_m), \ldots, \lambda_m^2$ and $T(R)$ is distinct from $Q_0, Q_1, \ldots, Q_m$ and not contained in any of the subspaces spanned by the faces of the simplex with vertices $Q_0, Q_1, \ldots, Q_m$.

Moreover by Lemma 3.4 we get $\rho(Q_0, Q_1, \ldots, R, T(R))$. We are now going to show that either $\rho(Q_0, Q_1, \ldots, Q_m, T(R))$ is also true or there exists a point

P in the subspace $S_{m-1} = [Q_0, Q_1, \ldots, Q_{m-1}]$ such that $\rho(Q_0, Q_1, \ldots, Q_m, P)$ is satisfied.

To this end consider an arbitrary $h \in G$. Assume the points $h(Q_0), h(Q_1), \ldots, h(Q_m)$ are dependent. Then obviously

$$\dim[h(Q_0), h(Q_1), \ldots, h(Q_m), h(T(R))] \leqslant m.$$

Hence assume that $h(Q_0), h(Q_1), \ldots, h(Q_m)$ are independent. If $h(R)$ is not contained in the subspace $[h(Q_0), h(Q_1), \ldots, h(Q_{m-1})]$ then $h(T(R))$ belongs to the subspace

$$[h(Q_0), h(Q_1), \ldots, h(Q_{m-1}), h(R)] = [h(Q_0), h(Q_1), \ldots, h(Q_{m-1}), h(Q_m)].$$

If this is true for all h for which $h(Q_0)$, $h(Q_1)$, $\ldots$, $h(Q_m)$ are independent the relation $\rho(Q_0, Q_1, \ldots, Q_m, T(R))$ is satisfied.

Otherwise there exists h such that $h(Q_0)$, $h(Q_1)$, $\ldots$, $h(Q_m)$ are independent but $h(R) \in [h(Q_0), h(Q_1), \ldots, h(Q_{m-1})]$. Then we may choose an affine mapping a such that $ah(Q_i) = Q_i$, $i = 0, 1, \ldots, m$ and $ah(R) = P$ belongs to $[Q_0, Q_1, \ldots, Q_{m-1}]$. By Lemma 3.2 this implies that $\rho(Q_0, Q_1, \ldots, Q_m, P)$. Note that we are finished in this case.

Hence we may assume that $\rho(Q_0, Q_1, \ldots, Q_m, T(R))$ is satisfied and that we may pass from a point R to $T(R)$ in the way just explained whenever convenient. This means that we may assume that $|\lambda_m| \neq 1$. For, if $\lambda_m = 1$ and all the other λ_i are ± 1 we may first replace R by $T(R)$ which has $\lambda_0(1 + \lambda_m) = \pm 2$ as its first barycentric coordinate. Thus $|\lambda_i| \neq 1$ for some i and we interchange Q_i and Q_m. We may even assume that $|\lambda_m| < 1$. For if $|\lambda_m| > 1$ we interchange the points Q_m and R which is possible because of Lemma 3.5. Then

$$-\lambda Q_m = \lambda_0 Q_0 + \cdots + \lambda_{m-1} Q_{m-1} - R$$

and so

$$Q_m = -\frac{\lambda_0}{\lambda_m} Q_0 - \cdots - \frac{\lambda_{m-1}}{\lambda_m} Q_{m-1} + \frac{1}{\lambda_m} R.$$

With this last assumption let us now construct a sequence (R_n) taking $R_0 = R$ and $R_{n+1} = T(R_n)$. If this sequence breaks off at some stage we have found a point P in $[Q_0, Q_1, \ldots, Q_{m-1}]$ such that $\rho(Q_0, Q_1, \ldots, Q_m, P)$ is satisfied and we are finished. Hence we may assume that the sequence does not break off. It is easy to see by induction on n that the barycentric coordinates of R_n are

$$\mu_i = \lambda_i(1 + \lambda_m + \cdots + \lambda_m^{2^n - 1}), \quad i = 0, 1, \ldots, m-1; \qquad \mu_m = \lambda_m^{2^n}.$$

Since $|\lambda_m| < 1$ the sequence (R_n) converges to a point S with barycentric coordinates $\frac{\lambda_i}{1-\lambda_m}$ for $i = 0, 1, \ldots, m-1$ and 0 for $i = m$. This point S lies within the face $Q_0, Q_1, \ldots, Q_{m-1}$ of the simplex $Q_0, Q_1, \ldots, Q_m$ and from the assumption $m > 1$ it follows that S is distinct from Q_0, Q_1, …, Q_{m-1}. Since its last coordinate is zero, it is also distinct from Q_m. Hence by Lemma 3.3 it follows that $\rho(Q_1, Q_2, \ldots, Q_m, S)$ is true. This proves Lemma 3.9.

LEMMA 3.10

If there is a point R contained in S_{m-1} such that $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds then for any R_1 outside S_{m-1} there are points Q such that

$$\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$$

is true. Moreover, for a given point R_1 the set of possible points Q is open and dense in E_n.

Choose R_1 outside S_{m-1} and choose $h \in G$ such that $h(Q_0) = Q_0$, $h(Q_1) = Q_1$, …, $h(Q_{m-1}) = Q_{m-1}$, and $h(R) = R_1$. This is possible since G is $(m+1)$-fold transitive.

Choose a point P outside S_{m-1} whose image $h(P)$ also does not belong to the subspace $S_{m-1} = [Q_0, Q_1, \ldots, Q_{m-1}]$. This means that P must not belong to the two surfaces $[Q_0, Q_1, \ldots, Q_{m-1}]$ and $h^{-1}([Q_0, Q_1, \ldots, Q_{m-1}])$ of (topological) dimension $m - 1$.

Since $\rho(Q_0, Q_1, \ldots, Q_m, R)$ holds and $R \in S_{m-1}$ by Lemma 3.6 it follows that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, P, R)$ does. Hence $\rho(Q_0, Q_1, \ldots, Q_{m-1}, h(P), R_1)$ is true because $h(R) = R_1$ and $h(Q_i) = Q_i$, $i = 0, 1, \ldots, m-1$. Thus we may take $Q = h(P)$. The subset from which we can choose P is open and dense in E_n. Therefore the subset from which $Q = h(P)$ can be chosen is also open and dense.

We are now in a position to complete the proof of Theorem 2. For, from Lemma 3.9 it follows that the hypothesis of Lemma 3.10 can always be satisfied. Thus we may conclude that $m = n$, for otherwise we could find points Q and R_1 such that $\rho(Q_0, Q_1, \ldots, Q_{m-1}, Q, R_1)$ is true but $R_1 \notin [Q_0, Q_1, \ldots, Q_{m-1}, Q]$, a contradiction. Then G is $(n+1)$-fold transitive, because of Lemma 3.1.

REMARK 2

Note that the conclusion of Theorem 2 is not true if f is not continuous. To see this, take a subfield K of $\mathbb{R}$. Consider E_n as a vector space over K and let f be a linear mapping of E_n over K which is not linear over the field $\mathbb{R}$. Such a mapping f is necessarily discontinuous and $G = \langle f, A \rangle$ is a group of affine mappings of E_n considered as an affine space over K and hence not $(n+1)$-fold transitive.

The author does not know any example of a group $G = \langle f, A \rangle$ with f continuous which admits non-trivial k-ary invariants for some $k > n + 1$. In fact, the groups considered in Theorem 2 might well turn out to be k-fold transitive for all k. At the present time however, this seems an open problem.

In this context it is worth noting that the groups $G = \langle f, A\rangle$ are a special kind of Jordan groups and therefore much further information is available on them (see [1]).

4. A special class of mappings

Theorem 1 raises the following question: if g is not an affine map maybe more could be said, e.g. that the group $\langle g, S\rangle$ is $(n+1)$-fold transitive. In this section we consider bijective mappings which are differentiable (in both directions) and which are themselves not affine but are such that the group $\langle g, S\rangle$ contains affine mappings not in S. In this case $\langle g, S\rangle$ must contain the whole affine group and hence is $(n+1)$-fold transitive.

THEOREM 3
Let g be as in Theorem 1. Assume further that g is not affine but $\langle g, S\rangle$ contains some affine mapping not in S. Then $\langle g, S\rangle$ contains the affine group and is $(n+1)$-fold transitive on E_n.

Since differentiable mappings are continuous this follows from the maximality of S in A and from Theorem 2 proved in the previous section. Note that the differentiability is not required in this theorem. It has been kept in the hypothesis to depart as little as possible from the context of Theorem 1.

We will now show by means of an example that the class of C^1 mappings considered in Theorem 3 is not empty. For this purpose we may look for a differentiable mapping σ given by

$$\sigma(x, y) = (G(x, y), y)$$

which is itself not linear but whose iterative square $\sigma \circ \sigma$ is the linear mapping $(x, y) \to (x + uy, y)$ for some $u \neq 0$. This means that we are looking for a differentiable function $G(x, y)$ which satisfies the functional equation

$$G[G(x, y), y] = x + uy \tag{4.1}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying

$$g(x + 1) = g(x) + 1. \tag{4.1a}$$

We can get such a function from any periodic function h with period 1 by setting $g(x) = x + h(x)$. We may choose h in such a way that $-\frac{1}{3} \leqslant h'(x) \leqslant \frac{1}{2}$. Then $g'(x) = 1 + h'(x)$ will be positive within the bounds $\frac{2}{3} \leqslant g'(x) \leqslant \frac{3}{2}$ so that $g(x)$ is strictly increasing. We may also assume that 1 is the smallest period of the function h.

For each value of y we modify $h(x)$ by a factor $\lambda(y)$ such that $0 \leqslant \lambda(y) \leqslant 1$ and $\lambda(y) \to 0$ as $y \to 0$. We may choose $\lambda(y) = 1 - e^{-y^2}$ for this purpose. Set $g_y(x) = x + \lambda(y)h(x)$.

This function still satisfies (4.1a) and $\frac{2}{3} \leqslant g_y'(x) \leqslant \frac{3}{2}$, so its inverse g_y^{-1} exists. Let us now set

$$G(x,y) = \begin{cases} uyg_y^{-1}\left[g_y\left(\frac{x}{uy}\right) + \frac{1}{2}\right] & \text{when } y \neq 0, \\ x & \text{otherwise.} \end{cases}$$

An easy computation shows that G satisfies equation (4.1). Therefore the (iterative) square of the mapping σ is the linear mapping given by $\sigma \circ \sigma(x,y) = (x + uy, y)$. Hence σ must be bijective.

It remains to check that σ is differentiable. It is easily seen that the inverse function g_y^{-1} also satisfies (4.1a) and so it can be written in the form

$$g_y^{-1}(x) = x + \bar{h}(x,y)$$

where $\bar{h}(x,y)$ is periodic in x with period 1. From the bounds for $g_y'(x)$ we get $-\frac{1}{3} \leqslant \frac{\partial \bar{h}(x,y)}{\partial x} \leqslant \frac{1}{2}$. For the sake of symmetry let us write $h(x,y)$ instead of $\lambda(y)h(x)$. From $x = g_y(g_y^{-1}(x))$ we obtain

$$h(x + \bar{h}(x,y), y) + \bar{h}(x,y) = 0. \tag{4.2}$$

The value $\bar{h}(x,y)$ of the function $\bar{h}$ is uniquely determined by this relation for each pair (x,y) and from the implicit function theorem it follows that the function $\bar{h}$ has continuous partial derivatives if h has. In fact, we may use the above relation to compute the partial derivatives of $\bar{h}$ with respect to x and y

$$\frac{\partial \bar{h}(x,y)}{\partial x} = -\frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\left(1 + \frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\right)^{-1} \tag{4.3a}$$

$$\frac{\partial \bar{h}(x,y)}{\partial y} = -\frac{\partial h}{\partial y}\left(x + \bar{h}(x,y), y\right)\left(1 + \frac{\partial h}{\partial x}\left(x + \bar{h}(x,y), y\right)\right)^{-1} \tag{4.3b}$$

It follows from these formulae that since $\frac{\partial h}{\partial x}(x,y) = o(y)$ and $\frac{\partial h}{\partial y}(x,y) = O(y)$ as $y \to 0$, independently of the first argument x, the same remains true for $\frac{\partial \bar{h}}{\partial x}(x,y)$ and $\frac{\partial \bar{h}}{\partial y}(x,y)$.

We may now rewrite the expression for $G(x,y)$ in the case $y \neq 0$ as follows

$$G(x,y) = x + uyh\left(\xi, y\right) + \frac{uy}{2} + uy\bar{h}\left(\eta, y\right), \tag{4.4}$$

where

$$\xi = \frac{x}{uy}, \quad \eta = \xi + h(\xi, y) + \frac{1}{2}.$$

When $y \neq 0$ we may calculate the partial derivatives G_x and G_y in a straightforward manner. Thus

$$G_x(x,y) = \left(1 + \frac{\partial h}{\partial x}\left(\xi, y\right)\right)\left(1 + \frac{\partial \bar{h}}{\partial x}\left(\eta, y\right)\right). \tag{4.5}$$

It follows that the function G_x defined by the above expression is continuous (for $y \neq 0$) and that $G_x(x,y) \to 1$ as $y \to 0$ regardless of the first argument. Thus G_x is continuous everywhere. For G_y we obtain in the case $y \neq 0$

$$\begin{aligned} G_y(x,y) = & \frac{u}{2} + u\left[h(\xi,y) + \bar{h}(\eta,y)\right] \\ & + uy\left[\frac{\partial h}{\partial y}(\xi,y) + \frac{\partial \bar{h}}{\partial y}(\eta,y) + \frac{\partial \bar{h}}{\partial x}(\eta,y)\frac{\partial h}{\partial y}(\xi,y)\right] \\ & - \frac{x}{y}\left[\frac{\partial h}{\partial x}(\xi,y) + \frac{\partial \bar{h}}{\partial x}(\eta,y)\left(1 + \frac{\partial h}{\partial x}(\xi,y)\right)\right]. \end{aligned}$$

It is easy to see that the function G_y is continuous when $y \neq 0$. For $y = 0$ it follows from (4.4) that $G_y(x,0) = \frac{u}{2}$. Moreover since, as remarked above, $\frac{1}{y}\frac{\partial h}{\partial x}$ and $\frac{1}{y}\frac{\partial \bar{h}}{\partial x}$ tend to zero as $y \to 0$ while $\frac{\partial h}{\partial y}$ and $\frac{\partial \bar{h}}{\partial y}$ remain bounded it follows that $G_y(\zeta,y) \to \frac{u}{2}$ as $y \to 0$ regardless of the first argument. Thus G_y is continuous everywhere and we have proved that σ is differentiable.

For the proof that the inverse of σ is also differentiable we only need to show that the Jacobian of σ never vanishes. (Note that this does not follow from the bijectivity of σ which has been established above.) But the Jacobian of σ at (x,y) is equal to $G_x(x,y)$ and from (4.5) it follows that $G_x(x,y) \neq 0$ for $y \neq 0$. For $y = 0$ this is still true since $G_x = 1$ in this case.

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