Annales Academiae Paedagogicae Cracoviensis
Folia 16 Studia Mathematica III (2003)

Grzegorz Bryll, Robert Sochacki

A method of comparison of Post-complete propositional logics

Abstract. In this paper we consider a necessary and sufficient condition for every propositional calculus to be Post–complete. We will show, how this condition may be used to compare consequence operations for these calculi.

Let S denote the set of all well formed formula of a fixed language of a propositional calculus. By the symbol (R, X) we mean any propositional calculus, where R is a set of rules and X is a subset of S. The set of all formulae belonging to the set X or derivable from the set X by means of the rules R will be denoted by $Cn(R, X)$. The consequence operation Cn is finistic. The notion of Post–completeness is defined as follows:

DEFINITION 1
$(R, X) \in Cpl \iff \forall_{\alpha \notin Cn(R,X)}[Cn(R, X \cup \{\alpha\}) = S]$.

Let us recall definitions of two sets of rules. With any pair (R, X) there can be connected two sets of rules of inference:

DEFINITION 2
$r \in Perm(R, X) \iff \forall_{\Pi \subseteq S} \forall_{\alpha \in S}[\langle \Pi, \alpha \rangle \in r \wedge \Pi \subseteq Cn(R, X) \implies \alpha \in Cn(R, X)]$.

DEFINITION 3
$r \in Der(R, X) \iff \forall_{\Pi \subseteq S} \forall_{\alpha \in S}[\langle \Pi, \alpha \rangle \in r \implies \alpha \in Cn(R, X \cup \Pi)]$.

The set $Der(R, X)$ contains all rules derivable from R and X. $Perm(R, X)$ is the set of all rules permissible in the system (R, X).

It can be shown (see [2, 3]) that:

LEMMA 1
a) $r \in Perm(R, X) \iff Cn(R \cup \{r\}, X) \subseteq Cn(R, X)$,
b) $Der(R, X) \subseteq Perm(R, X)$,
c) $(R, X) \in Cpl \iff Perm(R, X) \subseteq Der(R, X)$.

For any logical matrix $\mathfrak{M}$ the symbols $V(\mathfrak{M})$ and $E(\mathfrak{M})$ denote a set of all unfailing rules and a set of all valid formulae (tautologies) in the matrix $\mathfrak{M}$ respectively. We define the set $V(\mathfrak{M})$ in the following way:

Definition 4
$r \in V(\mathfrak{M}) \Longleftrightarrow \forall_{\Pi \subseteq S} \forall_{\alpha \in S}[\langle \Pi, \alpha \rangle \in r \ \wedge \ \Pi \subseteq E(\mathfrak{M}) \Longrightarrow \alpha \in E(\mathfrak{M})]$.

The next lemma will be useful (see [2]):

Lemma 2
$X \subseteq E(\mathfrak{M}) \wedge R \subseteq V(\mathfrak{M}) \Longrightarrow Cn(R, X) \subseteq E(\mathfrak{M})$.

Lemma 2 states that all formulae derivable from the set $X \subseteq E(\mathfrak{M})$ by means of the rules R are valid in the matrix $\mathfrak{M}$. The completeness theorem for the calculus (R, X) with respect to the matrix $\mathfrak{M}$ may be formulated as follows:

Theorem 1
$\emptyset \neq X \subseteq E(\mathfrak{M}) \Longrightarrow [V(\mathfrak{M}) = Perm(R, X) \Longleftrightarrow E(\mathfrak{M}) = Cn(R, X)]$.[1]

Given a pair (R, X) we define the operation C:

Definition 5
$C_{(R,X)}(Y) = Cn(R, X \cup Y)$.

It is easy to prove that the function $C_{(R,X)}$ is a consequence with the properties:

Lemma 3
a) $C_{(R,X)}(\emptyset) = C_{(R,X)}(X)$,
b) $Y \subseteq C_{(R,X)}(\emptyset) \Longrightarrow C_{(R,X)}(Y) = C_{(R,X)}(\emptyset)$.

Lemma 4

$$(R, X) \in Cpl \Longrightarrow C_{(R,X)}(Y) = \begin{cases} C_{(R,X)}(\emptyset), & if \ \ Y \subseteq C_{(R,X)}(\emptyset), \\ S, & if \ \sim Y \subseteq C_{(R,X)}(\emptyset). \end{cases}$$

Lemma 5

$$\left(C_{(R,X)}(Y) = \begin{cases} C_{(R,X)}(\emptyset), & if \ \ Y \subseteq C_{(R,X)}(\emptyset), \\ S, & if \ \sim Y \subseteq C_{(R,X)}(\emptyset) \end{cases} \right) \Longrightarrow (R, X) \in Cpl.$$

The easy proofs of these lemmas are left to the reader. From Lemmas 4 and 5 it follows that the completeness notion may be characterized in the following way:

[1] If the calculus (R, X) fulfils this condition, then the matrix $\mathfrak{M}$ is said to be weakly adequate to (R, X).

THEOREM 2

$$(R,X) \in Cpl \Longleftrightarrow C_{(R,X)}(Y) = \begin{cases} C_{(R,X)}(\emptyset), & if\ \ Y \subseteq C_{(R,X)}(\emptyset), \\ S, & if\ \ \sim Y \subseteq C_{(R,X)}(\emptyset). \end{cases}$$

THEOREM 3

$$X \subseteq E(\mathfrak{M}) \wedge R \subseteq V(\mathfrak{M}) \wedge (R,X) \in Cpl \Longrightarrow C_{(R,X)}(\emptyset) = E(\mathfrak{M}).$$

Proof. Assume that $X \subseteq E(\mathfrak{M})$, $R \subseteq V(\mathfrak{M})$ and $(R,X) \in Cpl$. Thus, by Lemma 2 and Definition 5 the inclusion holds:

$$C_{(R,X)}(\emptyset) \subseteq E(\mathfrak{M}). \tag{1}$$

For the converse inclusion we assume that $\alpha \in E(\mathfrak{M})$ and on the contrary that $\alpha \notin C_{(R,X)}(\emptyset)$. From Definitions 1 and 5 and by assumption $(R,X) \in Cpl$ we conclude that:

$$C_{(R,X)}(\{\alpha\}) = S. \tag{2}$$

Since $X \cup \{\alpha\} \subseteq E(\mathfrak{M})$ and $R \subseteq V(\mathfrak{M})$, then on the basis of Lemma 2 and Definition 5 we obtain $C_{(R,X)}(\{\alpha\}) \subseteq E(\mathfrak{M})$. Because $E(\mathfrak{M}) \neq S$, thus $C_{(R,X)}(\{\alpha\}) \neq S$, which contradicts (2).

Let us consider two propositional calculi (R_1,X_1) and (R_2,X_2), where $X_1 \cup X_2 \subseteq S$ and two consequence functions C_1, C_2 which are defined below:

DEFINITION 6

For every $Y \subseteq S$*:*
a) $C_1(Y) = Cn(R_1, X_1 \cup Y)$,
b) $C_2(Y) = Cn(R_2, X_2 \cup Y)$.

We will prove the next Theorem:

THEOREM 4

$$(R_1,X_1),(R_2,X_2) \in Cpl \wedge R_1 \cup R_2 \subseteq V(\mathfrak{M}) \wedge X_1 \cup X_2 \subseteq E(\mathfrak{M}) \Longrightarrow C_1 = C_2.$$

Proof. From assumptions, Definitions 6 and 5 and Theorem 3, it follows that:

$$C_1(\emptyset) = C_2(\emptyset) = E(\mathfrak{M}). \tag{3}$$

According to the assumption $(R_1,X_1),(R_2,X_2) \in Cpl$, Definitions 6 and 5 and Theorem 2 we have two equalities:

$$C_1(Y) = \begin{cases} C_1(\emptyset), & if\ \ Y \subseteq C_1(\emptyset), \\ S, & if\ \ \sim Y \subseteq C_1(\emptyset). \end{cases} \tag{4}$$

$$C_2(Y) = \begin{cases} C_2(\emptyset), & if\ \ Y \subseteq C_2(\emptyset), \\ S, & if\ \ \sim Y \subseteq C_2(\emptyset). \end{cases} \tag{5}$$

Let us consider the following two cases:

I. $Y \subseteq C_1(\emptyset)$. Then from (3) we have $Y \subseteq C_2(\emptyset)$. We conclude from (4) and (5) that $C_1(Y) = C_1(\emptyset)$ and $C_2(Y) = C_2(\emptyset)$, hence $C_1(Y) = C_2(Y)$.

II. $\sim (Y \subseteq C_1(\emptyset))$. By (3) it follows that $\sim (Y \subseteq C_2(\emptyset))$. Using (4) and (5) we obtain $C_1(Y) = S = C_2(Y)$, hence $C_1(Y) = C_2(Y)$.

In both cases it has been shown that for any $Y \subseteq S$ we have $C_1(Y) = C_2(Y)$, thus $C_1 = C_2$, which completes the proof.

This theorem states connection between some propositional calculi and consequences determined by them.

Sometimes, the notion of Post–completeness is defined in a weaker version. Let us denote by $Sb(X)$ the set of all substitutions of formulae belonging to the set X.

DEFINITION 7
$(R, X) \in Cpl* \iff \forall_{\alpha \notin Cn(R,X)}[Cn(R, X \cup Sb(\{\alpha\})) = S]$.

It is visible that $Cpl \subseteq Cpl*$. The theorem concerning weakly completeness may be written in the form (see Theorem 3):

THEOREM 5
$X \subseteq E(\mathfrak{M}) \wedge R \subseteq V(\mathfrak{M}) \wedge (R, X) \in Cpl* \Longrightarrow C_{(R,X)}(\emptyset) = E(\mathfrak{M})$.

Propositional calculi $Z_1 = (R_1, X_1)$ and $Z_2 = (R_2, X_2)$ may be compared in another way: Z_1 and Z_2 are equivalent if and only if $Z_1 \leqslant Z_2$ and $Z_2 \leqslant Z_1$, where the relation $\leqslant$ is defined as follows:

$$Z_2 \leqslant Z_1 \iff [\forall_{r \in R_1}(r \in Der(R_2, X_2)) \wedge X_1 \subseteq Cn(R_2, X_2)].$$

EXAMPLES:
Let us consider the disjunctional–negational language and the following rules of inference:

$$r_0: \frac{\beta \in Sb(\{\alpha\}) \quad \alpha}{\beta} \qquad r_1: \frac{AN\alpha\beta \quad \alpha}{\beta} \qquad r_2: \frac{A\alpha\beta \quad N\alpha}{\beta}$$

We will compare the following propositional calculi:

I. The Łukasiewicz system (R_1, X_1) (see [1]), where $R_1 = \{r_0, r_1\}$ and $X_1 = \{A1_L, A2_L, A3_L\}$. The axioms have the form:

A1$_L$. ANANApqrANpr,

A2$_L$. ANANApqrANqr,

$A3_L$. ANANprANANqrANApqr.

Adopting the definition $C\alpha\beta = AN\alpha\beta$ and the rule of definitional replacement these axioms may be rewritten in the form:

$A1^*_L$. CCApqrCpr,

$A2^*_L$. CCApqrCqr,

$A3^*_L$. CCprCCqrCApqr.

In the remainder of this paper the abbreviation $C = AN$ will be adopted.

II. The Rasiowa system (R_1, X_2) (see [4]), where $X_2 = \{A1_R, A2_R, A3_R\}$,

$A1_R$. CAppp,

$A2_R$. CpApq,

$A3_R$. CCprCAqpArq.

III. The Whitehead–Russell system (R_1, X_3) (see [6]), where $X_3 = \{A1_W, A2_W, A3_W, A4_W, A5_W\}$,

$A1_W$. CAppp,

$A2_W$. CqApq,

$A3_W$. CApqAqp,

$A4_W$. CApAqrAqApr,

$A5_W$. CCqrCApqApr.

P. Bernays has proved, that axiom $A4_W$ is independent of $A1_W$, $A2_W$, $A3_W$, $A5_W$ (see [6]).

IV. The Reichbach system (R_1, X_4) (see [5]), where $X_4 = \{A1_M, A2_M, A3_M\}$,

$A1_M$. ApCpq,

$A2_M$. CApqCCqpp,

$A3_M$. CApqCCqrApr.

V. The Reichbach system (R_2, X_5) (see [5]), where $R_2 = \{r_0, r_2\}$ and $X_5 = \{A1_N, A2_N, A3_N, A4_N, A5_N\}$,

$A1_N$. NNApCpq,

$A2_N$. NNCApqCCqpp,

$A3_N$. NNCApqCCqrApr,

$A4_N$. Cpp,

$A5_N$. CApqApNNq.

The above systems are Post–complete. Let us consider the 2-valued matrix $\mathfrak{M} = (\{0,1\},\{1\},\{a,n\})$, where $a(x,y) = \max(x,y)$ and $n(x) = 1 - x$. It is easy to prove that $R_1 \cup R_2 \subseteq V(\mathfrak{M})$ and $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \subseteq E(\mathfrak{M})$.

Let: $C_1(Y) = Cn(R_1, X_1 \cup Y)$,
$C_2(Y) = Cn(R_1, X_2 \cup Y)$,
$C_3(Y) = Cn(R_1, X_3 \cup Y)$,
$C_4(Y) = Cn(R_1, X_4 \cup Y)$,
$C_5(Y) = Cn(R_2, X_5 \cup Y)$.

On the basis of Theorem 4 it follows that $C_1 = C_2 = C_3 = C_4 = C_5$.

References

[1] J. Łukasiewicz, *Eein Vollständigkeitsbeweis des zweiwertigen Aussagenkalküls*, Comptes rendus de la Société des Sciences et des Lettres de Varsovie **24** (1931), 153–183.

[2] W. A. Pogorzelski, *Klasyczny rachunek zdań*, Wyd. 3, PWN, Warszawa 1975.

[3] W. A. Pogorzelski, P. Wojtylak, *Elements of the theory of completeness in propositional logic*, Silesian University, Katowice 1982.

[4] H. Rasiowa, *Sur certain systeme d'axiomes du calcul des propositions*, Norsk Mathematisk Tidsskrift **31** (1949), 1–3.

[5] J. Reichbach, *Über den auf Alternative und Negation aufgebauten Aussagenkalkül*, Studia Logica **1** (1953), 13–18.

[6] A. N. Whitehead, B. Russell, *Principia mathematica*, Cambridge 1910.

Grzegorz Bryll
Instytut Matematyki i Informatyki
Wyzsza Szkoła Pedagogiczna
Armii Krajowej 13/15,
42-200 Częstochowa
Poland

Robert Sochacki
Instytut Matematyki i Informatyki
Uniwersytet Opolski
Oleska 48,
45-052 Opole
Poland
E-mail: sochacki@math.uni.opole.pl