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POWER SET-problems with understanding the idea

Abstract. The article concerns some symbols introduced in the first year of mathematics studies and problems of understanding them by students. We will show it in the idea of power set. A sequence of didactic activities proposed to students will be presented, around the of associative and commutative properties of multiplication in the power set of natural numbers.

Also, I will propose a specific procedure of introducing the "power set" idea in the course involving this subject, which will exclude any problems encountered previously

The reason of writing the present article was a problem concerning the idea of power set, which appeared during the so called additional classes on algebra for mathematics students of the first year of mathematics teacher study. The number of students in these classes is quite large since it includes about 50-60% of the whole group. This fact makes things look different because their difficulties cannot be accidental but they ought to be a phenomenon deeply rooted in many years' practice of mathematics teaching in different kinds of schools.

During the first several classes at school students are given a huge dose of completely new information, ideas and symbols used in mathematics. This does not go beyond students' intellectual possibilities, but it makes them get lost in this chaos, and moreover it makes it difficult for them to overcome consecutive conceptual obstacles. At the secondary school operations are performed mainly on number sets (these are usually so called common operations) and this stereotype makes it difficult to change their way of reasoning.

A survey carried out among students shows clearly that most of them used to like school mathematics and they knew, how to move in this area quite easily without any serious troubles while solving typical problems. However, they claim that instruction such as "show, prove and check" generated their reluctance and they usually did not even try to do such a problem. Thus the thesis is proved one more time: school mathematics develops mainly reasoning in the area of superficial structures ([6]). Overcoming the habit of schematic problem solving and the stereotypes in mathematical reasoning seems to be the most important job of university teachers (especially those working with students of the first year).

Let's come back to the classes. Their aim was to create the idea of a group. In order to explain this complex issue a problem was introduced, which was supposed not to be a big challenge for the students.

PROBLEM I. Determine properties of the intersection in the power set $2^{\mathbb{N}}$.

Firstly, the definition of a power set was reminded to the students and then they started working on the problem. The way of their reasoning was the following:

$$2^{\mathbb{N}} = \{2^0, 2^1, 2^2, \ldots, 2^n, \ldots\}.$$

I have to admit, that I was astonished by the obtained result. However, I did not interrupt, I was just waiting for their final answer. The students went on:

In this set the intersection has got the following properties:
1. It is associative, because

$$(2^n \cap 2^m) \cap 2^k = 2^{n+m} \cap 2^k = 2^{(n+m)+k} = 2^{n+(m+k)} = 2^n \cap (2^m \cap 2^k)$$

. 2. It is commutative, because ... and so on.
In this situation, the originally assumed aim had to undergo a few modifications. Giving a reasonable shape to the idea of power set has now become all-important task. I did not discuss with the students their answer, so when they got a new task they assumed that the first one was finished correctly.

PROBLEM II. Determine the properties of multiplication in the set of natural powers of number 2.

After the students had finished, they presented the answer (this task took far less time than it was previously). Both results were written down next to each other on the blackboard.

$$(2^n \cap 2^m) \cap 2^k = 2^{(n+m)+k} = 2^{n+(m+k)} = 2^n \cap (2^m \cap 2^k) \tag{a}$$

$$(2^n \cdot 2^m) \cdot 2^k = 2^{(n+m)+k} = 2^{n+(m+k)} = 2^n \cdot (2^m \cdot 2^k) \tag{b}$$

The dialoque between the presenter (P) and students (S) proceeded as follows:

P: Look at both results very carefully. Do you notice anything interesting?

S: As a matter of fact, it seems to be the same, however in (a) there is an intersection while in (b) there is a multiplication (Being still unsure, they are looking at the board).

P: Do you consider both results to be correct?

S: (immediately) The first one is wrong. At this point they are not able to prove their points, but their intuition prompts them that there is something wrong here.

P: Let us compare now the elements of the sets from problems I and II. Students have no doubts that the elements of a set from problem II are numbers (natural powers of number 2).

P: Let us remind how have we defined the set in problem I.

S: This is a family of all subsets of natural number set.

P: What are the elements of this set?

S: They are sets. Yes, they cannot have the form 2^n, as these are numbers, not sets.

The answer proves some progress in their understanding of the definition achieved during the discussion. What the progress was like it was going to turn out during the next problem's analysis. However, it looked as if they started to read the definition "sensibly", trying to find some sense in it.

We are referring to the problem that I have written on the board.

P: How do you understand the sign of the intersection ($\cap$)in this notation then?

S: (silence)

P: Why $2^n \cap 2^m = 2^{n+m}$?

S: Because we confused it with common multiplication in the set of powers with the same bases.

The students did not notice that there has been a change of mathematical reality ([5]). Their existing knowledge was interfering so much that it really makes difficult to think logically in a new structure. Having no idea about the structure of elements in a given set, they unknowingly referred to the known schema, which concerns number sets only. They used the power set symbol, in which unfortunately there was number 2 (in a while it will let us remind itself). The intersection that was given as an operation did not disturb students' course of reasoning. The sign $\cap$ was used by them mechanically (it appeared in the problem, therefore it should have occurred later).

It is interesting that such a mechanism is characteristic for many students at the beginning of their studies. In order to solve problems they do not understand, they break all logical rules, they use weakly mastered, incomprehensible signs, and this usually leads to absurd answers. Sometimes just the contrary happens: logical and correct mathematical reasoning, in which formulas and mathematical symbols were not used (according to them mathematics wasn't used at all, because symbols are the very meaning of mathematics to them), they consider it to be too simple to be correct. That is why they do not write down any answer to avoid losing face.

After realizing the fact that elements of the considered set are sets, the students wrote:

$$2^{\mathsf{N}} = \{\{2^0\}, \{2^1\}, \{2^2\}, \ldots, \{2^n\}, \ldots\}.$$

As we can see, the understanding of the power set idea turned out to be apparent. The discussion with the students let us claim that number 2, occurring in the above notation, has nothing to do with the fact that it is a natural number and the sets $\{2^n\}$ occurred in this family by chance. Number 2 is only a part of the symbol used to denote a power set.

To the question whether $\{3\} \in 2^{\mathbf{N}}$, the answer was: No, because these were to be sets such as they were written in the problem, and symbol $2^{\mathbf{N}}$was there. Therefore, the only intuitions developed in the course of acquiring that symbol result from the fixed understanding of graphical picture of number 2 as a definite number value.

A problem has arisen, which — I have to admit — forced me to withdraw this symbol. However, knowing that such a sign $2^{\mathbf{N}}$ was introduced in classes in the "Introduction to mathematics" course, and also the fact that this symbol is often used in the literature, made me decide to leave it. If I had changed it it would have caused unnecessary chaos and the necessity to retake the whole work in a different way. Moreover, it is overcoming difficulties, not eliminating them, that makes the essence of teaching.

I decided to leave this problem once more and formulated a new problem, connected with a realistic context.

PROBLEM III. Let us consider the set of people taking part in our classes and let us denote it as A. We will denote elements of this set as a_i, where $i \in \{1, 2, ..., 20\}$. Therefore

$$A = \{a_1, a_2, ..., a_{20}\} .$$

Describe the set 2^A (I used here symbolic set notation on purpose).

Without thinking, but working only on symbols, the students carried on their erroneous reasoning up to absurdity:

$$2^A = \{\{2^{a_1}\}, \{2^{a_2}\}, \ldots, \{2^{a_{20}}\}\} .$$

P: Let us come to the details. The element a_1 is Jola, a_2 is Basia, and so on. Let us put the names Jola and Basia in the places of a_1, a_2 and so on.
We get:

$$2^A = \{2^{Jola}, 2^{Basia}, \ldots, 2^{Arek}\} .$$

P: What does 2^{Jola} mean?
Silence.

P: Who of you is able to raise 2 to the power Jola? (laughter).

One of the students claims: This is nonsense, this is impossible to be done.

Just now they realized how absurd the result was. We read the definition of the set 2^A one more time analysing each word carefully. It could be noticed that they started realizing only the abstract, symbolic meaning of the symbol 2^A. Without any difficulties they wrote down the elements of the set 2^A.

When I asked them what had happened to 2 they claimed that:" 2 does not play any role here, this is only notation, 2 is not taken into consideration at all."

The fact that an empty set is a subset of each set was recalled and after a short discussion, which was supposed to show the difference between symbols $\emptyset$ and $\{\emptyset\}$, the students add the empty set to 2^A. Having done this, they claimed that their confusion was caused by misleading notation of the power set. Referring to the lectures "Introduction to mathematics", I encouraged the students to prove that if $\overline{\overline{X}} = n$ then $\overline{\overline{2^X}} = 2^n$.

Before we started to describe set $2^{\mathbb{N}}$ I asked whether $\{2\} \in 2^{\mathbb{N}}$?
There was an immediate answer: No, because this two is only..., (after a while of hesitation he continued)... I mean yes, because $2 \in \mathbb{N}$, in other words $\{2\}$ is an element of this family.
Surprisingly, there was no problem with distinguishing the ideas of subset and element of a set. Notations

$$2 \in \mathbb{N} \ \text{ then } \ \{2\} \in 2^{\mathbb{N}}, \ \text{ because } \ \{2\} \subseteq \mathbb{N}$$

have not caused any difficulties this time. Eventually, set $2^{\mathbb{N}}$ was described correctly and before examining the properties of the set-theoretical multiplication in this set we considered in the first place:

1. Extreme cases ([2]).

   Describe
   $$2^{\emptyset}, 2^{\{\emptyset\}}, 2^{\{\mathbb{N}\}}.$$

   In order to check if the sets are described correctly we used also the knowledge of a cardinality of a power set. Now the students had no doubts about rightness of the notation.

2. Describe the set $2^{2^{\{\bullet\}}}$.

3. Problem: Find the set A and describe the operation of union in it if:
   $$2^A = \{\{\tfrac{1}{2}\}, \{\sqrt{3}\}, \{\tfrac{1}{2}, \sqrt{3}\}, \{\tfrac{1}{2} + \sqrt{3}\}, \{\tfrac{1}{2}, \tfrac{1}{2} + \sqrt{3}\}, \{\sqrt{3}, \tfrac{1}{2} + \sqrt{3}\}, A, \emptyset\}.$$

(I introduced the operation of union in this set on purpose as I was going to bring both operations face to face in one problem).
Through analogy and deduction (with my little help) the students gave a correct description of the given sets.

The course of events proved how difficult it sometimes was to build foundations of a new level of mathematical knowledge among students. The errors that the students made at the beginning may cause failure of the whole didactic process, which will make the teacher helpless and the student disappointed in the future. But most of all it may develop in the students the feeling of lack of possibility to understand mathematics.

That is why, I think, it is worth sometimes – even at the cost of other teaching contents, especially in this situation, difficult for students, when they have to change their learning techniques they have been using so far, and they need to start thinking in different domains. It is also worth taking a break when apparently trivial problem occurs. When we analyze the problem carefully, letting the students make errors (the educational value of which, as you can see in the given examples, is unshakable), will allow individual students to look over wrong foundations, assumptions, methods and achieved effects. The knowledge they will obtain this way should be especially efficient.

For students themselves it may be an opportunity to think over their first impressions with university mathematics once more. Many of them claim therefore that: "if I had known that I would be studying "such things", I would never have taken studies in this field".

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