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Students' ideas about infinity

Abstract. An analysis of students' ideas associated with the terms finite and infinite is presented. It is based on research we carried out in the years 1998-2000 at thirty elementary and secondary comprehensive schools in the North Bohemian Region.

An analysis of students' ideas associated with the terms finite and infinite presented here is based on research we carried out in the years 1998-2000 at thirty elementary and secondary comprehensive schools in the North Bohemian Region. This research, making use of the questionnaire method, produced results of quantitative character. It showed the students' and pupils' ideas concerning some mathematical notions (cardinality of a set, maximum and minimum, density of order, line, boundedness and convergence) and the development of these ideas during their school attendance. Partial results concerning the development of the primary school pupils' ideas about cardinality and boundedness of geometric sets were published in [2].

In the following an analysis of answers to one selected question only will be presented as an illustration. The question occurred in the questionnaire submitted to pupils of the third and fifth form of primary schools (8 and 10 years old pupils) and to students of secondary schools (namely second (12 years old), fourth (14), sixth (16) and eighth (18) forms). The questionnaire items were mostly open in the sense that the respondents were offered no answers to choose. Each age group contained approximately 600 respondents.

The interpretation of the questionnaire results is complemented by the facts data from an analysis of the utterances recorded in a guided experimental interviews with students. The draft scenarios of the interviews were compiled using the method of "constructed pupils' reaction" (see [4]). The research team, preparing for the experiment, tried to anticipate all possible alternatives or courses of the dialogue with pupils on a certain topic. Our prediction of pupils' reaction was more accurate thanks to our experience gained in the analysis of pre-tests given each time in the number of 60 to students of three different schools. Their aim was to verify and later edit the formulation of the interview items.

What is there of an infinite quantity?

This question was included in the questionnaire for each age group excluding the 1st form. The most frequent answer, in all age groups, was *Numbers*. The second most frequent answer was *Stars* (Although it is not relevant from our point of view, it is interesting that this answer is wrong. According to the contemporary knowledge in physics, the estimated number is 10^{18} to 10^{21}). The frequency of the answers *People, Trees, Words, Water* gradually declines and from the 5th form it is below 5 percent. Answers inspired by other questionnaire items (i.e. *Points on a line, Points on a circle*) appear sporadically. The answers of the pupils can be thought of with regard to two different aspects of infinity - potential and current. Hejný says that the phenomenon of infinity is usually introduced to pupils as the current one – a line, plane, number domains ... (see [3]). Both he and Wigand assert that the potential approach is more familiar to pupils, though it is hardly being developed (see [7]). The problem of infiniteness is little discussed (see [1] or [5]). This is only one of the causes of difficulties that students have later with the principle of mathematical induction. The argumentation in this proof is potential, but the result is valid in a current infinite set.

In the questionnaire we can find answers to this question giving evidence of understanding of infinity as potential (*Words*) or current (*Stars, Points on a circle*). The most frequent answers are those that cannot be classified without detailed examination of the pupils' ideas – *People, Trees, Water, Atoms* (by the way, their number in space is estimated to 10^{85} ([6]). This classification holds despite the fact that, except for *Numbers* and *Points on a circle*, the answers are not correct. The reason why pupils present them is essential here. The frequency of these wrong answers declines below 5% starting from the second form of secondary school, further on the answers *Numbers* and *Stars* prevail. Let's illustrate again these ideas by a short passage from an interview with two girls, pupils of the 5th form. The numbers in parenthesis refer to the length of pause in seconds.

Experimenter 1: *What is there of an infinite quantity?*
Hanka 1: *Numbers.*
E2: (Towards Pavlína) *Is she right?*
Pavlína 2: *Yes.*
E3: *And why?*
H3: *That's because I can count on and on.*
E4: *Really?* (Toward Pavlína) *Can I continue counting after let's say (2) one million, can I?*
P4: *Yes, one million one, one million two.*
E5: *And where does it end?*
P5: *At one million nine hundred ninety nine.*

H5: *No, no. Then it's one million one thousand.*

E6: *And then?*

H6: *One million one thousand one, one million one thousand two (2) and so on. And I can always do it this way.*

E7: *Hmm. And do you know anything else of an infinite quantity? (10) How about (3) books?* (He nods his head towards the books scattered around the room.) *All the books, hmm, all around the world?*

H7: *No, there is finitely many of them.*

E8: *Why?*

H8: *Because we can count them. (2) But there are infinitely many letters in them. (3) No, it's not. But there are infinitely many words.*

E9: *Words in the books?*

P9: (Trying to join the discussion) *Yes, the words in all the books.*

H9: *No, it's not, all the words (2), even those Indian, all (2) that people say (2) and write.*

E10: *But why? (5) Why cannot they be counted?*

P10: *Well, probably (3) they can be. I don't know.*

H10: *They can be, but I can always invent a new one.*

P11: *But this new one could already be in the books.*

E11: *You mean encyclopedias? (3) Or dictionaries? Where all (2)* (gets unsure), *well, almost all words are written?*

P12: *Yes.*

H12: *Well, I can still think of a brand new one.*

E13: *This won't have any meaning then, will it? (4)*

H13: *Hmm.*

E14: *Well, OK. Let's take as an example all the words made of three letters. OK? All the possible words made of three letters like (2) Ola, but, etc. OK? (2) How many are they? Infinitely many?*

H14: *No.*

E15: *And then take as an example all the words consisting of four, five or let's set a limit of twenty letters – How many are they?*

H15: *I think infinitely many. I can still think of a new one that is not written anywhere. And maybe people will later start using it. (3) And isn't it right that people speak differently in each country? This means even more words.*

During this part of the interview three sets were mentioned – numbers (it is clear from the part of interview that is not presented here that the girls meant the set of natural numbers), the set of all books in the world and the set of all words in all languages. As to the numbers, we can say that Hanka has

the right idea. She understands the infiniteness of the set potentially, e.g. she can always add another, bigger natural number that is an element of the set. Unfortunately, our analysis of the ideas of the second girl, Pavla, cannot be based on this interview. The experimenter paid more attention to the analysis of Hanka, who was more active than Pavla.

The set of all books was, according to the stream of consciousness at the position H8, quickly done away with. Nevertheless, the valuable statement of Hanka *Because we can count them* (books) gives evidence of her correct idea of non-infiniteness of this set. In mathematics it is expressed with the definition $K \sim \{1, 2, 3, \ldots n\}$, where K is the considered set of books and n is any natural number. In the case of the last set – the set of all words – Hanka admits that the number of all (Czech) words consisting of three letters is not infinite (H14). But the idea of the possibility of creating another new words from more letters (twenty) still predominates and this is the cause of her wrong conviction about the infiniteness of the considered set.

References

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