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Vasyl M. Fedorchuk, Volodimir I. Fedorchuk

On new differential equations of the first order in the space $M(1,4)\times R(u)$ with non-trivial symmetries

Abstract. Some new differential equations of the first order in the space $M(1,4)\times R(u)$, which are invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$ are presented.

The differential equations with non-trivial symmetry groups play an important role in theoretical and mathematical physics, gas dynamics etc. (see, for example, [1–5]).

The group-analysis methods (see, for example, [1–8]) allows us to construct new differential equations with non-trivial symmetry groups.

In many cases these equations can be written in the following form:

$$F(J_1, J_2, ..., J_t) = 0, \tag{1}$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, ..., J_t\}$ are functional bases of differential invariants of the corresponding symmetry groups.

The present paper is devoted to the construction of the first-order differential equations in the space $M(1,4)\times R(u)$, which are invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$.

In order to present some of the new results obtained we have to consider the Lie algebra of the group $P(1,4)$.

1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, which satisfy the commutation relations

$$[P'_\mu, P'_\nu] = 0, \quad [M'_{\mu\nu}, P'_\sigma] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$[M'_{\mu\nu}, M'_{\rho\sigma}] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

We consider the following representation of the Lie algebra of the group $P(1,4)$

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \quad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Below we will use the following basis elements:

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [8]. Splitting subgroups of the group $P(1,4)$ have been described in [9, 10]. From the results obtained (see also [11]) it follows that the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$.

2. The first-order differential equations in the space $M(1,4) \times R(u)$

The group $P(1,4)$ acts on $M(1,4) \times R(u)$ (i.e. on the cartesian product of the five-dimensional Minkowski space (of the independent variables x_0, x_1, x_2, x_3, x_4) and the number axis of the dependent variable u). The group $P(1,4)$ usually acts on $M(1,4)$ as a group generated by translations and rotations of this space, and it trivially acts on $R(u)$ in this specific case.

Let $X = \sum_{i=0}^{4} \xi_i(x) \frac{\partial}{\partial x_i}$ be one of the basic infinitesimal operators. It generates the action

$$g_t(x, u(x)) = (g_t x, u(x)) = (y, u(g_{-t} y)),$$

where $g_t = \exp tX \in P(1,4)$, $x \in M(1,4)$, $y = g_t x$. From this, one obtains the first prolongation of X in the form

$$X^{(1)} = X - \sum_{i=0}^{4} \left(\sum_{j=0}^{4} \frac{\partial \xi_j}{\partial x_i} u_j \right) \frac{\partial}{\partial u_i}, \quad u_j \equiv \frac{\partial u}{\partial x_j}, \quad j = 0,1,2,3,4.$$

Now, a function $J(x, u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x, u^{(1)}) = 0.$$

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3, u_4)$ is an element of the first prolongation $R(u)^{(1)}$.

The first-order differential equations in the space $M(1,4)\times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$ have been constructed. These equations can be written in the form (1), where $\{J_1, J_2, ..., J_t\}$ are functional bases of the first-order differential invariants of the splitting subgroups of the group $P(1,4)$.

Below, for some splitting subgroups of the group $P(1,4)$ we write the basis elements of its Lie algebras and corresponding arguments $J_1, J_2, ..., J_t$ of the function F.

1. $\langle X_0 + X_4\rangle$,

$$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = x_4, \quad J_5 = u, \quad J_6 = u_0,$$
$$J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_3, \quad J_{10} = u_4, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \ \mu = 0,\ 1,\ 2,\ 3,\ 4;$$

2. $\langle X_4\rangle$,

$$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = x_0 + x_4, \quad J_5 = u, \quad J_6 = u_0,$$
$$J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_3, \quad J_{10} = u_4;$$

3. $\langle P_3,\ X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = x_1, \quad J_3 = x_2, \quad J_4 = u,$$
$$J_5 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_6 = u_1, \quad J_7 = u_2, \quad J_8 = u_0 - u_4,$$
$$J_9 = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle G,\ X_1\rangle$,

$$J_1 = x_2, \quad J_2 = x_3, \quad J_3 = (x_0^2 - x_4^2)^{1/2}, \quad J_4 = u,$$
$$J_5 = (x_0 + x_4)(u_0 + u_4), \quad J_6 = u_1, \quad J_7 = u_2, \quad J_8 = u_3, \quad J_9 = u_0^2 - u_4^2;$$

5. $\langle L_1,\ L_2,\ L_3\rangle$,

$$J_1 = x_0, \quad J_2 = x_4, \quad J_3 = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \quad J_4 = u,$$
$$J_5 = x_1u_1 + x_2u_2 + x_3u_3, \quad J_6 = u_0, \quad J_7 = u_4, \quad J_8 = u_1^2 + u_2^2 + u_3^2;$$

6. $\langle P_1,\ P_2,\ X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = x_3, \quad J_3 = u, \quad J_4 = u_1(x_0 + x_4) + x_1(u_0 - u_4),$$
$$J_5 = u_2(x_0 + x_4) + x_2(u_0 - u_4), \quad J_6 = u_3, \quad J_7 = u_0 - u_4,$$
$$J_8 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

7. $\langle G,\ L_3,\ P_1,\ P_2\rangle$,

$$J_1 = x_3, \quad J_2 = (x_0^2 - x_1^2 - x_2^2 - x_4^2)^{1/2}, \quad J_3 = u, \quad J_4 = \frac{x_0 + x_4}{u_0 - u_4},$$
$$J_5 = \left(x_1 + \frac{x_0 + x_4}{u_0 - u_4}u_1\right)^2 + \left(x_2 + \frac{x_0 + x_4}{u_0 - u_4}u_2\right)^2, \quad J_6 = u_3,$$
$$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

8. $\langle L_3,\ P_1,\ P_2,\ X_4\rangle$,

$$J_1 = x_3, \quad J_2 = x_0 + x_4, \quad J_3 = u,$$
$$J_4 = \left(\frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4}\right)^2 + \left(\frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4}\right)^2, \quad J_5 = u_3,$$

$J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$

9. $\langle G,\ P_3,\ L_3,\ X_3,\ X_4\rangle,$

$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1, \quad J_4 = \frac{x_0 + x_4}{u_0 - u_4},$
$J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2;$

10. $\langle G,\ P_1,\ P_2,\ X_1,\ X_4\rangle,$

$J_1 = x_3, \quad J_2 = u, \quad J_3 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_4 = x_2 + \frac{x_0 + x_4}{u_0 - u_4}u_2, \quad J_5 = u_3,$
$J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$

11. $\langle L_3,\ P_1,\ P_2,\ P_3,\ X_3,\ X_4\rangle,$

$J_1 = x_0 + x_4, \quad J_2 = u,$
$J_3 = \left(\frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4}\right)^2 + \left(\frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4}\right)^2, \quad J_4 = u_0 - u_4,$
$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$

12. $\langle L_3 + cG,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ c > 0\rangle,$

$J_1 = x_3, \quad J_2 = u, \quad J_3 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_4 = u_3, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$

13. $\langle G,\ L_1,\ L_2,\ L_3,\ X_1,\ X_2,\ X_3\rangle,$

$J_1 = (x_0^2 - x_4^2)^{1/2}, \quad J_2 = u, \quad J_3 = (x_0 + x_4)(u_0 + u_4), \quad J_4 = u_0^2 - u_4^2,$
$J_5 = u_1^2 + u_2^2 + u_3^2;$

14. $\langle G,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_4\rangle,$

$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = x_3 + \frac{x_0 + x_4}{u_0 - u_4}u_3,$
$J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$

15. $\langle L_3 + bG,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4,\ b > 0\rangle,$

$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$

16. $\langle P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,$

$J_1 = u, \quad J_2 = u_0 - u_4, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2.$

References

[1] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[2] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Boston, 1985.

[3] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[4] W. I. Fushchych, R. Z. Zhdanov, *Nonlinear spinor equations: symmetry and exact solutions*, Kyiv, Naukova Dumka, 1992.

[5] W. I. Fushchych, W. M. Shtelen, N. I. Serov, *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*, Dordrecht, Kluver Academic Publishers, 1993.

[6] S. Lie, *Über Differentialinvarianten*, Math. Ann., V. 24, **1** (1884), 52–89.

[7] A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta Math., V. 18, (1894), 1–88.

[8] J. Patera, P. Winternitz, H. Zassenhaus, *Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group*, J. Math. Phys., 1975, V. 16, **8** (1975), 1597-1614.

[9] V. M. Fedorchuk, *Continuous subgroups of the inhomogeneous de Sitter group* $P(1,4)$, Preprint, Inst. Matemat. Acad. Nauk Ukr. SSR, 78.18, 1978.

[10] V. M. Fedorchuk, *Splitting subalgebras of the Lie algebra of the generalized Poincaré group* $P(1,4)$, Ukr. Mat. Zh., V.31, **6** (1979), 717–722.

[11] W. I. Fushchich, A. G. Nikitin, *Reduction of the representations of the generalized Poincaré algebra by the Galilei algebra*, J. Phys. A: Math. and Gen., 1980, V. 13, **7** (1980), 2319–2330.

Vasyl M. Fedorchuk
Pedagogical Academy
Institute of Mathematics
Podchorążych 2
30-084, Kraków
Poland

Pidstryhach Institute
of Applied Problems
of Mechanics and Mathematics
of the National Ukrainian
Academy of Sciences
Naukova Street 3b,
Lviv, 79601
Ukraine
E-mail: fedorchuk@ap.krakow.pl

Volodimir I. Fedorchuk
Ivan Franko Lviv National University,
Universytetska Street 1
Lviv, 79000,
Ukraine
E-mail: skf@ukr.net