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Duality: random variables versus observables

Abstract. We discuss some aspects of the duality between random variables and observables - one of the basic questions of probability. Besides a historical background, we mention the role of elementary events and the algebraic (i.e. pointless) approach to probability. We present some categorical constructions leading to possible generalizations of the field of events, probability measures, and random variables.

1. Introduction

In the classic Kolmogorovian model, the fundamental notions of probability are events, probability, random variable, and distribution. Events are subsets of a set Ω the points of which are called elementary events. Events are closed with respect to the set-theoretical operations and form a field (usually a σ-field) S of sets. Probability is a normed measure p on S. The triple (Ω, S, p) is called the original probability space. A random variable f is a measurable map of Ω into the real line $\mathbb{R}$, i.e., $f^{\leftarrow}(B) = \{\omega \in \Omega; f(\omega) \in B\}$ is an event for each Borel set $B \in \mathrm{B}(\mathbb{R})$. The preimage $f^{\leftarrow} : \mathrm{B} \longrightarrow \mathrm{S}$ preserves the set-theoretical operations (it is a homomorphism), and the composition $p_f = p \circ f^{\leftarrow}$ is a normed measure on $\mathrm{B}(\mathbb{R})$ called the distribution of f. Distributions can be extended to families of random variables and, roughly speaking, Kolmogorovian probability is about distributions.

If the original probability space (Ω, S, p) is not discrete, the elementary events seem to play a "secondary" role. Events form a Boolean algebra and the so-called pointless probability tries to avoid points and point functions completely or as much as possible (cf. [9]). A thorough discussion of the role of elementary events can be found in Łoś [10], a very interesting paper, unfortunately ignored by the probability audience. In an attempt to describe physical theories in which the Boolean logic does not capture the behavior of events, probability theory has been generalized in various directions. It has started with the pioneering work of G. Birkhoff and J. von Neumann. A survey of recent trends in the so-called quantum probability theory can be found for example in Dvurečenskij and Pulmannová [3], Foulis [7], Mundici and Riečan

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[11]. The so-called operational probability theory (also fuzzy probability theory), as outlined in Bugajski [1], [2] and Gudder [8], seems to be particularly promising.

It is our belief (cf. [4], [5], [6]) that sequential convergence and categorical methods provide suitable language and technical tools to describe various types of generalized events, generalized probability measures, and generalized random variables in a canonical way.

2. Measurability and duality

Each random variable $f : \Omega \longrightarrow \mathbb{R}$ induces a Boolean homomorphism $f^{\leftarrow} : \mathbf{B}(\mathbb{R}) \longrightarrow \mathbf{S}$. In general, each measurable map g of a measurable space $(Y, \mathbf{A})$ into a measurable space $(X, \mathbf{B})$ induces a Boolean homomorphism $g^{\leftarrow} : \mathbf{B} \longrightarrow \mathbf{A}$. The problem is to find out when a Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$ is induced by a unique measurable map and, moreover, when the correspondence between measurable maps and the induced Boolean homomorphisms (going the opposite way) yields a categorical duality. Such duality (as a categorical equivalence of measurable maps and certain Boolean homomorphisms) is rather useful. Indeed, each $f^{\leftarrow}$ selects a subfield $\mathbf{S}_f$ of $\mathbf{S}$, $\mathbf{S}_f = \{f^{\leftarrow}(B); B \in \mathbf{B}(\mathbb{R})\}$, the subfield of all events in $\mathbf{S}$ related to the measurement (experiment) represented by the random variable f. The abstract analysis applied to measurable maps and random variables (function spaces, measure theory, abstract integration, functional analysis) has led to solutions of fundamental problems of probability (central limit theorem, understanding of random processes, ...). Notice that applying abstract analysis to Boolean homomorphisms is rather clumsy and it is not very practical. On the other hand, replacing the Boolean algebra of events by some other algebraic structure leads to natural generalizations of the classical probability theory. Indeed, there is an extensive list of algebraic structures modeling various types of operations on events (cf. Foulis [7]): modular orthocomplemented lattice — the logic of a quantum-mechanical system, orthomodular lattice, orthomodular poset, orthoalgebra, effect algebra — fuzzy or unsharp logic, D-poset, MV-algebra, ...

In Frič [6] we have proposed a simple general construction of a duality suitable for generalized probability. It is based on sequential convergence and, unlike the classical Stone duality, it commutes with the transition from fields to σ-fields (no factorization like in the Loomis-Sikorski theorem is needed, no loss of points).

In what follows we assume that every field of sets is reduced (if x and y, $x \neq y$, are points, then there exists a set in the field such that x belongs to the set and y belongs to the complement).

First, if A is a subset of Y, then A and its characteristic function χ_A: $Y \longrightarrow \{0, 1\}$ ($\chi_A(y) = 1$ if $y \in A$ and $\chi_A(y) = 0$ otherwise) will be identified.

Second, a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets converges to $A \subseteq Y$ whenever the characteristic functions converge pointwise ($A = \liminf A_n = \limsup A_n$). Third, if $(Y, \mathbf{A})$ and $(X, \mathbf{B})$ are measurable spaces and f is a map of Y into X, then f is measurable iff for each $B \in \mathbf{B}$ the composition $\chi_B \circ f = \chi_{f^{\leftarrow}(B)}$ is the characteristic function of some $A \in \mathbf{A}$. Fourth, each point $y \in Y$ represents a sequentially continuous Boolean homomorphism of $\mathbf{A}$, the evaluation ev_y at y, into the two-point Boolean algebra $\{0,1\}$ defined by $ev_y(A) = 1$ if $y \in A$ and $ev_y(A) = 0$ otherwise; in fact, it is a σ-additive measure (point measure).

Let $(Y, \mathbf{A})$ and $(X, \mathbf{B})$ be measurable spaces.

LEMMA 1

Let f be a measurable map of $(Y, \mathbf{A})$ into $(X, \mathbf{B})$. Then $f^{\leftarrow}$ is a sequentially continuous Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$.

Proof. The assertions follow from the properties of $f^{\leftarrow}$.

LEMMA 2

Let h be a sequentially continuous Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$. If each sequentially continuous Boolean homomorphism of $\mathbf{B}$ into $\{0,1\}$ is an evaluation at some $x \in X$, then there is a unique measurable map of $\mathbf{B}$ into $\mathbf{A}$ such that $h = f^{\leftarrow}$.

Proof. Let $y \in Y$. Then the composition $ev_y \circ h$ is a sequentially continuous Boolean homomorphism of $\mathbf{B}$ into $\{0,1\}$. Thus there is $x \in X$ such that $ev_x = ev_y \circ h$. Since $\mathbf{B}$ is reduced, the point x is uniquely determined. Define $f(y) = x$. The rest is a straightforward calculation verifying that $h = f^{\leftarrow}$.

A field satisfying the assumptions of the previous lemma (each sequentially continuous Boolean homomorphism into $\{0,1\}$ is fixed) and the corresponding measurable space are called **sober**.

COROLLARY 1

The category of sober fields of sets (the objects) and sequentially continuous Boolean homomorphisms (the morphisms) and the category of sober measurable spaces and measurable maps are dually isomorphic.

Using the properties of categorical products, the duality described in Corollary 1 can be extended to more general algebraic structures and generalized measurable spaces. The extended duality has a probabilistic interpretation.

Let $\mathcal{A}$ be a Boolean algebra. It is known that each two elements of $\mathcal{A}$ can be (using the axiom of choice) separated by a Boolean homomorphism of $\mathcal{A}$ into $\{0,1\}$. This leads to the Stone representation of $\mathcal{A}$ by subsets. Clearly, if $Y_{\mathcal{A}}$ is the set of all such homomorphisms, then we can represent each $a \in \mathcal{A}$ as the characteristic function $\chi_{A(a)} : Y_{\mathcal{A}} \longrightarrow \{0,1\}$ of the set $A(a) =$

$\{y \in Y_{\mathcal{A}}; y(a) = 1\}$, i.e., the set $A(a)$ itself. Moreover, the Boolean operations in $\mathcal{A}$ can be represented via the usual set-theoretical operations in $Y_{\mathcal{A}}$. Observe that $\mathcal{A}$ can be represented via every sufficiently rich (separating) subset Y of $Y_{\mathcal{A}}$. Such subsets (of Boolean homomorphisms into $\{0,1\}$) are called Stone families. As pointed out by J. Łoś, different Stone families can lead to different "probability models" (via sets) of the same Boolean algebra $\mathcal{A}$ considered as a system of abstract probability events carrying an abstract probability measure on $\mathcal{A}$ (cf. [4]).

Indeed, let $Y \subseteq Y_{\mathcal{A}}$ and let p be an additive measure on $\mathcal{A}$. For $A(a) \in \mathbf{A}_Y$ define $p_Y(A(a)) = p(a)$. Then p_Y is an additive measure on $\mathbf{A}_Y$. Observe, that for $Y = Y_{\mathcal{A}}$ (the usual compact representation) p_Y is always σ-additive. (Hint. Let $\{A(a_n)\}_{n=1}^{\infty}$ be a decreasing sequence of sets in $\mathbf{A}_Y$ such that $\emptyset = \bigcap_{n=1}^{\infty} A(a_n)$. Since each $A(a)$ is a closed set in the compact topological Stone space, almost all $A(a_n)$ are empty sets. Since σ-additivity is equivalent to the monotone continuity from above, p_Y is σ-additive.) This is certainly undesirable and hence the compact (perfect) representation and the Stone duality are not suitable for probability. To sum up, the choice of Y (the elementary events) has an impact on the properties of probability measures.

We construct the duality in two steps. First, we represent $\mathcal{A}$ (via a suitable choice of $Y \subseteq Y_{\mathcal{A}}$) as a sober field $\mathbf{A}_Y$ of subsets $A(a) \subseteq Y$, $a \in A$. Pairs $(\mathcal{A}, Y)$ together with sequentially continuous Boolean homomorphisms form a category CBA and, clearly, CBA and the category of sober fields of sets are naturally equivalent. Then, as the second step, we pass to the measurable space $(Y, \mathbf{A}_Y)$. The rest is straightforward: CBA and the sober measurable spaces are dual (cf. [5], [6]). Remember, the choice of Y is the choice of the elementary events in the classical Kolmogorovian probability.

Let us stress that the two-point Boolean algebra $\{0,1\}$ determines the "logic" governing the pointwise operations with events. If $a, b \in \mathcal{A}$, then e.g. $a \vee b \in \mathcal{A}$ is represented as the set of all points $y \in Y$ for which the value of y at $a \vee b$ is the supremum in $\{0,1\}$ of $y(a)$ and $y(b)$. Accordingly, if we use instead of $\{0,1\}$ the interval $[0,1]$ and if we use instead of the Boolean logic some fuzzy logic, then we get a duality for some "fuzzy" structures, e.g. Archimedean MV-algebras and generalized "fuzzy" measurable spaces.

In the next section we describe a duality which covers both the Boolean and the Łukasiewicz fuzzy logic. Generalized measurable maps lead to generalized random variables and the dual homomorphisms lead to the so-called observables.

3. Generalizations

At the FSTA2002 conference in Liptovský Ján (Fuzzy Set Theory and Applications) my PhD student M. Papčo has introduced ID-posets as a potential generalization of probability events. Difference posets or D-posets introduced by F. Chovanec and F. Kôpka (cf. [12]) are quintuples $(D, \leqslant, 0, 1, \ominus)$, where D is a partially ordered set, 0 is the least and 1 is the greatest element, and $\ominus$ is a partial operation (called difference): $a \ominus b$ is defined iff $b \leqslant a$; some natural axioms for $\ominus$ are assumed. If D is the interval $I = [0, 1]$, then $\leqslant$ is the usual order and, if $b \leqslant a$, then $a \ominus b = a - b$. Now let X be a set and let $\mathcal{X} \subseteq I^X$ be a class of functions on X into I. Then $\mathcal{X}$ carries the natural pointwise order and the pointwise difference: $a \ominus b$ is defined iff $b(x) \leqslant a(x)$ for all $x \in X$ and then $(a \ominus b)(x) = a(x) - b(x)$, $x \in X$. If it is reduced and algebraically closed with respect to $\ominus$, then $\mathcal{X}$ is said to be an **ID-poset**. Morphisms are the maps of $\mathcal{X} \subseteq I^X$ into $\mathcal{Y} \subseteq I^Y$ preserving the ID-poset structure and sequentially continuous with respect to the pointwise convergence. Further, $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are called **ID-measurable spaces** and a map f of Y into X is said to be **measurable** whenever the composition of f and each $a \in \mathcal{X}$ belongs to $\mathcal{Y}$ (i.e. $\mathcal{X} \circ f \subseteq \mathcal{Y}$). Again, each measurable map of $(Y, \mathcal{Y})$ into $(X, \mathcal{X})$ induces an ID-morphism $f^{\triangleleft}$ of $\mathcal{X}$ into $\mathcal{Y}$ and, for sober ID-posets (each ID-morphism into I is fixed at some point), each ID-morphism h of $\mathcal{X}$ into $\mathcal{Y}$ is of the form $f^{\triangleleft}$ for a unique measurable map f. This correspondence yields a dual isomorphism between the categories of sober ID-posets and generalized ID-measurable spaces.

It is easy to verify that fields of sets and their "fuzzyfication", the so-called bold algebras (systems of fuzzy sets closed with respect to the pointwise Łukasiewicz operations $\oplus$, $\odot$, and complement) are ID-posets. The classical measurable spaces and measurable maps are special cases of ID-measurable spaces and maps. Classical probability measures are ID-morphisms into I.

Hence we can consider ID-posets as generalized fields of events, ID-measurable maps as generalized random functions, ID-morphisms as generalized observables, and ID-morphisms into I as generalized probability measures.

The important fact about ID-posets and ID-measurable spaces and maps is that they provide natural categories for the theory of operational random variables and operational (fuzzy) probability theory, as developed by E. G. Beltrametti and S. Bugajski (cf. [1], [2] and [8]).

The idea of an operational random variable, or a stochastic map, originated in quantum physics. Instead of the elementary events represented by the points (i.e. the point probabilities) of the original probability space Ω, we consider all probability measures $\mathcal{P}(\Omega)$ on Ω ($\subseteq \mathcal{P}(\Omega)$). A stochastic map is a map f of the set $\mathcal{P}(\Omega)$ into $\mathcal{P}(\Omega')$ having some additional properties. To each stochastic map there corresponds a unique sequentially continuous homomorphism of fuzzy events of Ω' into fuzzy events of Ω, called observable. The classical Kolmogorovian probability deals with the case $f(\Omega) \subseteq \Omega'$. If $f(\omega) \in \mathcal{P}(\Omega') \setminus \Omega'$ for some $\omega \in \Omega$, then f models some nontrivial quantum phenomenon. Observe that maps of $\mathcal{P}(\Omega)$ into $\mathcal{P}(\Omega')$ have been used by L. Le Cam in a different context (statistics). We believe that ID-posets provide a simple and powerful categorical tool for operational probability.

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