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Remarks on continuous fractions

Abstract. A lot of interesting examples from the history of mathematics are seldom if at all used in school mathematics. Continuous fractions were used in the ancient Greece to calculate values of irrational numbers. In the present paper we show their possible applications to the problems of limit, and problems involving sequences and series.

1. Introduction

In the teaching of limits at the secondary school level many teachers often make a mistake neglecting or omitting a suitable motivation and a proper introduction to the topic.

The history of mathematics provides many inspiring examples and approaches to sequential convergence which can help to understand the limit processes. There are parallels between the history of mathematical thinking and the development of mathematical thinking in the mind of students.

Continuous fractions can serve the secondary school teacher as a good material for motivation and as an introduction to sequences and limits.

2. Ancient Greek

About three centuries B. C., in the time when nobody talked of limits and convergence, the ancient Greek mathematicians used continuous fractions to calculate values of irrational numbers. As an illustration we present the following example.

Consider a square $ABCD$ (see Figure 1). Because $|AB| < |BD|$, there exists point $C_1 \in BD$ with $|AB| = |BC_1|$. The perpendicular at point C_1 crosses side AD in point B_1. Clearly $|\angle ADB| = |\angle DB_1C_1| = 45°$. The triangle DB_1C_1 is isosceles and rectangular, so $|B_1C_1| = |C_1D|$. Next, consider the square $A_1B_1C_1D$. The length of the side of this square is $|BD|-|AC|$. Again $|A_1B_1| < |B_1D|$ and we can repeat the same construction in the square $A_1B_1C_1D$ and get the square $A_2B_2C_2D$. If the process is repeated the sides of consecutive squares "go" to nil.

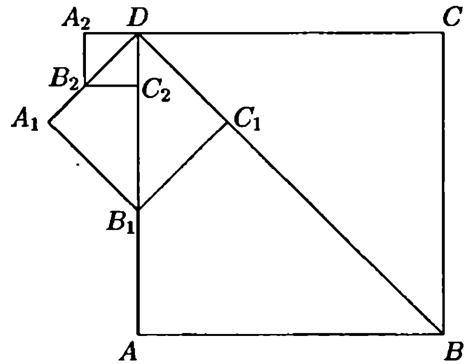

Figure 1.

Volkert shows in [6] how we can use this construction to aproximate $\sqrt{2}$. Let $|A_iB_i| = a_i$ for $i \in \mathbb{N}$ and let $|AB| = a_0 = 1$. So $|BD| = \sqrt{2}$ and $|BD| = |BC_1| + |C_1D|$, hence $\sqrt{2} = 1 + a_1$. Further $|B_1D| = |B_1C_2| + |C_2D|$, hence $|AD| - |AB_1| = |B_1C_2| + |C_2D|$ and we can write $1 - a_1 = a_1 + a_2$, which implies $a_0 = 1 = 2a_1 + a_2$. We can continue: $|B_2D| = |B_2C_3| + |C_3D|$, hence $|A_1D| - |A_1B_2| = |B_2C_3| + |C_3D|$. We get $a_1 - a_2 = a_2 + a_3$, which implies $a_1 = 2a_2 + a_3$. In general, we get for all non-negative whole numbers n: $a_n = 2a_{n+1} + a_{n+2}$. From the preceding equations we get:

$$\sqrt{2} = 1 + a_1,$$

$$1 = 2a_1 + a_2 \Rightarrow \frac{1 - a_2}{a_1} = 2 \Rightarrow \frac{1}{a_1} = 2 + \frac{a_2}{a_1},$$

$$a_1 = 2a_2 + a_3 \Rightarrow \frac{a_1 - a_3}{a_2} = 2 \Rightarrow \frac{a_1}{a_2} = 2 + \frac{a_3}{a_2},$$

$$a_2 = 2a_3 + a_4 \Rightarrow \frac{a_2 - a_4}{a_3} = 2 \Rightarrow \frac{a_2}{a_3} = 2 + \frac{a_4}{a_3},$$

$$\vdots$$

$$a_n = 2a_{n+1} + a_{n+2} \Rightarrow \frac{a_n - a_{n+2}}{a_{n+1}} = 2 \Rightarrow \frac{a_n}{a_{n+1}} = 2 + \frac{a_{n+2}}{a_{n+1}},$$

$$\vdots$$

We substitute:

$$\frac{1}{a_1} = 2 + \frac{a_2}{a_1} = 2 + \frac{1}{\frac{a_1}{a_2}} = 2 + \frac{1}{2 + \frac{a_3}{a_2}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{a_4}{a_3}}}$$

$$= 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{a_5}{a_4}}}} = \ldots = 2+\cfrac{1}{2+\cfrac{1}{2+\ldots\cfrac{1}{2+\cfrac{a_{n+1}}{a_n}}}}$$

$$\Rightarrow \frac{1}{a_1} = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}} \Rightarrow a_1 = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}.$$

In this way a continuous fraction for the number $\sqrt{2}$ is constructed:

$$\sqrt{2} = 1 + a_1 = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}.$$

If this continuous fraction is meant as a sequence of approximate values of $\sqrt{2}$, we can calculate:

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{2}{5}}} = 1+\cfrac{1}{2+\cfrac{5}{12}} = 1+\frac{12}{29} = \frac{41}{29} \doteq 1{,}413793,$$

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{5}{12}}} = 1+\cfrac{1}{2+\cfrac{12}{29}} = 1+\frac{29}{70} = \frac{99}{70} \doteq 1{,}414286.$$

So we get the inequality $\frac{99}{70} > \sqrt{2} > \frac{41}{29}$. From the viewpoint of school mathematics it is important to observe that it is possible to write this continuous fraction with a recurrent sequence:

$$a_1 = \frac{3}{2}, \quad a_{n+1} = 1 + \frac{1}{1+a_n} \quad \text{for } n \in \mathbb{N}.$$

3. Leonhard Euler

Leonhard Euler (1707-1783) deals with continuous fractions in his book *Introductio in analysis infinitorum* and the last chapter in the first part of the book is entitled *Continuous fractions.* Let x be a positive real number.

$$\text{If } x = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}, \quad \text{than} \quad x = \frac{1}{2+x}.$$

We get a quadratic equation $x^2 + 2x = 1$ and its solution $x = \sqrt{2} - 1$.

Euler generalizes this example as follows. If a is a positive real constant, then

$$x = \cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\dots}}}},$$

hence $x = \dfrac{1}{a+x}$ and $x^2 + ax = 1$, implies $x = \dfrac{\sqrt{a^2+4}-a}{2}$.
Also in 18th century the calculation of continuous fractions was a widespread means of getting values of irrational numbers.

Euler constructed an algorithm of changing an infinite series with commuting signs into a continuous fraction. He explains it as follows. Let

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \dots.$$

Consider a continuous fraction of the form

$$\cfrac{1}{a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\dots}}}}.$$

The partial sums of the infinite series are

$$\frac{1}{A}, \frac{B-A}{AB}, \frac{BC-AC+AB}{ABC}, \dots$$

and the expressions in the continuous fraction are

$$\frac{1}{a}, \frac{b}{ab+\alpha}, \frac{bc+\beta}{abc+a\beta+\alpha c}, \dots$$

Comparing the corresponding expressions, we get a system of equations. We will explain this procedure in the first three steps:

$$\frac{1}{A} = \frac{1}{a}, \frac{B-A}{AB} = \frac{b}{ab+\alpha}, \frac{BC-AC+AB}{ABC} = \frac{bc+\beta}{abc+a\beta+\alpha c}.$$

Hence

$$a = A,$$
$$b = B - A,$$
$$AB = ab + \alpha,$$
$$BC - AC + AB = bc + \beta,$$
$$ABC = abc + a\beta + \alpha c.$$

The first two equations are simple. If we substitute a, b in the third equation, we have $A(B-A)+\alpha = AB$ and hence $\alpha = A^2$. We simplify the fifth equation $ABC = a(bc+\beta) + \alpha c$. If $BC - AC + AB = bc + \beta$, then $ABC = a(BC - AC + AB) + \alpha c = A^2(BC - AC + AB) + \alpha c$, which implies $c = C - B$. We substitute now for b and c in the fourth equation. We get $BC - AC + AB = (B-A)(C-B) + \beta$, hence $\beta = B^2$. Euler generalizes these equations:

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \ldots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \ldots}}}}.$$

He illustrates this interesting construction using the Leibniz's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \ldots}}}}.$$

4. Summary

We selected the above algorithms and examples from a great number of interesting items in the history of mathematics. In my opinion, unsatisfactory attention is paid in Slovakia to the often ingenious and seminal work of great mathematicians of the past. It would be rather useful to include into mathematical textbooks more examples of this kind, representing "the art of thinking", and also an integral part of culture and history. This is in particular desirable in the introduction of important notions and constructions or in the teaching of "how to solve mathematical problems". An interested reader can find more information about teaching mathematics "via history" in [4].

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