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Pythagorean TRIPLES

Abstract. Students should learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. We describe how one can describe all integer solutions of the equation $a^2 + b^2 = c^2$ by utilizing elementary knowledge from algebra, geometry and number theory.

Introduction

In the quest for a more effective teaching system the guiding principle is one which is familiar to every research mathematician. The best way, and probably the only way, to learn mathematics is by actively participating in it. Mathematics is not just a body of facts to be learned, but it also involves the development of skills and abilities which can only be acquired through practice.

To become proficient in any branch of mathematics, students must themselves work through an adequate selection of exercises and problems. On the other hand, when left completely to their own devices, they make little progress, for a number of reasons:

— lack of confidence may prevent them even attempting a new problem;

— they may lack the background knowledge necessary to understand the problem;

— lack of critical ability can give them a false sense of confidence and prevent them from detecting errors in their solution;

— even when they can see how to solve a problem, they may lack the ability to express their insight clearly and hence communicate it to others.

It is useful to create an environment in which students are encouraged to work on problems under the supervision of experienced teachers. The task of teachers is to constantly check on the students' progress and offer just the right amount of help needed to get the students over any obstacles so that they can complete the rest of the solution by thelmselves. Assistance provided to the students must be minimal and unobtrusive.

We have used mainly [1], [2], [3], [4], [5] during working out our contribution.

Pythagorean triples in school mathematics

We describe all integer solutions of the equation $a^2 + b^2 = c^2$ by utilizing elementary knowledge from algebra, geometry and number theory.

Students are able to acquire a positive attitude to mathematics only if they are given sufficient opportunities to practise self-reliant actions and if they receive appropriate help. Both student activity and instruction have to be regarded as complementary elements in the learning process. Both elements are necessary, and they require to be systematically related to one another in such a manner that progress becomes optimal.

We can use collection of questions and tasks enabling to solve isolated problems by students and contribute to solution of basic problem.

A *Pythagorean triple* is a triple (a, b, c) of positive integers satisfying

$$a^2 + b^2 = c^2.$$

PROBLEM 1

Is there a way to generate all Pythagorean triples?

The smallest and best-known Pythagorean triple is $(3, 4, 5)$. It is easy to find that also triples

$$(6,8,10), \quad (9,12,15)$$

are Pythagorean triples.

QUESTION 1

Can you find other Pythagorean triples?

Our first observation is the fact that there is an infinite number of Pythagorean triples. It is easy to check the next result.

TASK 1

Let (a, b, c) be a Pythagorean triple, k be a positive rational number. If ka, kb, kc are positive integers, then (ka, kb, kc) is a Pythagorean triple. Prove it.

Let F be the set of all Pythagorean triples and let $\sim$ be a relation on F defined by $(a, b, c) \sim (x, y, z)$ if and only if there is a positive rational number k such that $(x, y, z) = (ka, kb, kc)$.

TASK 2

The relation $\sim$ is an equivalence relation on F. Prove it.

If $(a, b, c) \sim (x, y, z)$, then we will say that the Pythagorean triples (a, b, c), (x, y, z) are *similar*.

Two integers a, b are *coprime* if their greatest common divisor is 1. Let a, b be coprime. Then a Pythagorean triple (a, b, c) is said to be a *primitive* Pythagorean triple.

TASK 3

If (a, b, c) is a primitive Pythagorean triple then a, c are coprime and b, c are coprime. Prove it.

It is usual to consider only primitive Pythagorean triples, since other Pythagorean triples can be generated trivially from the primitive ones.

TASK 4

If (a, b, c) is a Pythagorean triple, then there is a positive integer d such that $(a, b, c) = (da_1, db_1, dc_1)$, where (a_1, b_1, c_1) is a primitive Pythagorean triple. Prove it.

TASK 5

If primitive Pythagorean triples (a, b, c), (x, y, z) are similar, then $(a, b, c) = (x, y, z)$. Prove it.

Let m, n be positive integers, $m > n$. Define

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2. \tag{1}$$

TASK 6

The triple (a, b, c) is a Pythagorean triple. The proof is simple. Write it down.

The formulas (1) were known to Euclid and used by Diophantus to obtain Pythagorean triples with special properties. However, he never raised the question whether one can obtain all possible triples in this way.

QUESTION 2

Can you write the next Pythagorean triples in the form (1)?
a) $(3, 4, 5)$, *b)* $(24, 10, 26)$, *c)* $(9, 12, 15)$.

Solution.
a) If $2mn = 4$, then $m = 2$, $n = 1$ and $m^2 - n^2 = 3$, $2mn = 4$, $m^2 + n^2 = 5$.
b) If $2mn = 10$, then $m = 5$, $n = 1$ and $m^2 - n^2 = 24$, $2mn = 10$, $m^2 + n^2 = 26$.
c) If $2mn = 12$, we must distinguish two posibilities:

1. If $m = 6$, $n = 1$, then $m^2 - n^2 = 35$, $2mn = 12$, $m^2 + n^2 = 37$.
2. If $m = 3$, $n = 2$, then $m^2 - n^2 = 5$, $2mn = 12$, $m^2 + n^2 = 13$.

There is no representation of the Pythagorean triple $(9, 12, 15)$ in the form (1).

PROBLEM 2

Can we write every primitive Pythagorean triple in the form (1)?

QUESTION 3

Let (a, b, c) be a primitive Pythagorean triple. Can be both a and b even or odd?

Solution. If a and b are even, then the triple (a, b, c) is not a primitive Pythagorean triple.

If z is an integer, then z^2 must be in a form $z^2 = 4k$ (if z is even) or $z^2 = 4k + 1$ (if z is odd). If a and b are odd, then $a = 2s + 1$, $b = 2r + 1$ and $c^2 = a^2 + b^2 = 4(r^2 + s^2 + r + s) + 2$, a contradiction.

Conclusion is:

LEMMA 1

For a primitive Pythagorean triple (a, b, c), one of a or b must be even, and the other odd, with c always odd.

Without loss of generality we can assume that in a primitive Pythagorean triple (a, b, c), b is even (a and c are odd).

TASK 7

Read the next text, devide it to parts and create a collection of tasks and a question for fellow-students.

Let (a, b, c) be a primitive Pythagorean triple. The point (a, b) lies on the circle (in the Cartesian plane) $x^2 + z^2 = c^2$. The (linear) equation of a straight line passing through points $(-c, 0)$ and (a, b) is $y = t(x + c)$, where $t = \frac{b}{a+c}$ is the slope of the line from $(-c, 0)$ to (a, b). The circle $x^2 + z^2 = c^2$ and the straight line $y = t(x + c)$ have common points $(-c, 0)$ and (a, b). To express a and b, we have to solve simultaneously two equations: the quadratic equation of the circle $x^2 + y^2 = c^2$ and the linear equation of the line $y = t(x + c)$.

Substituing $y = t(x + c)$ into $x^2 + y^2 = c^2$ yields

$$x^2 + t^2(x^2 + 2cx + c^2) = c^2,$$

which we can rewrite as

$$x^2 + \frac{2ct^2}{1+t^2} + \frac{t^2c^2 - c^2}{1+t^2} = 0. \tag{2}$$

The roots of (2) are $-c$ (from the point $(-c, 0)$) and a (from the point (a, b)). From this and (2) we have

$$a \cdot (-c) = \frac{t^2c^2 - c^2}{1+t^2}.$$

Hence

$$a = \frac{1-t^2}{1+t^2} \cdot c.$$

Let $t = \frac{n}{m}$ and m, n be coprime. We obtain

$$a = \frac{1 - \frac{n^2}{m^2}}{1 + \frac{n^2}{m^2}} \cdot c = \frac{m^2 - n^2}{m^2 + n^2} \cdot c$$

and

$$b = t(a+c) = \frac{n}{m}\left(\frac{m^2 - n^2}{m^2 + n^2} \cdot c + c\right) = \frac{2mn}{m^2 + n^2} \cdot c.$$

Having disposed of those preliminary steps, we can express the primitive Pythagorean triple in the form

$$(a, b, c) = \left(\frac{m^2 - n^2}{m^2 + n^2} \cdot c, \frac{2mn}{m^2 + n^2} \cdot c, c\right).$$

The Pythagorean triples $\left(\frac{m^2-n^2}{m^2+n^2} \cdot c, \frac{2mn}{m^2+n^2} \cdot c, c\right)$ and $(m^2-n^2, 2mn, m^2+n^2)$ are similar. We next show that $(m^2-n^2, 2mn, m^2+n^2)$ is a primitive Pythagorean triple.

Positive integers m, n are coprime. We can prove that they are of different parities.

LEMMA 2

If (a, b, c) is a primitive Pythagorean triple, b is even, $\frac{n}{m} = \frac{b}{a+c}$ and m, n are coprime, then m and n are of different parities.

Proof. Suppose, contrary to our claim, that m and n are odd. From $\frac{n}{m} = \frac{b}{a+c}$ we obtain

$$c = \frac{mb - na}{n}. \tag{3}$$

Substituing (3) into $a^2 + b^2 = c^2$ we get

$$2mna = b(m^2 - n^2). \tag{4}$$

Since b is even ($b = 2k$), (4) shows that

$$mna = k(m^2 - n^2), \tag{5}$$

a contradiction (the left side of equality (5) is odd and the right side of one is even).

It is easy to check that the numbers m^2-n^2 and m^2+n^2 are coprime. From this we conclude that the triple $(m^2-n^2, 2mn, m^2+n^2)$ is a primitive Pythagorean triple, too. We thus get (see Task 5)

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2).$$

We can now formulate the main result.

THEOREM 1

Let a triple (a, b, c) be a primitive Pythagorean triple and b be even. Then there are positive integers m, n of different parities such that m, n are coprime and

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2).$$

TASK 8

Work out the next exercises.

1. *Find all primitive Pythagorean triples (a, b, c) with $c < 50$.*
2. *Find all Pythagorean triples (a, b, c) with $c < 40$.*
3. *Let F_n be a Fibonacci number. Prove that $(F_nF_{n+3}, 2F_{n+1}F_{n+2}, F_{n+1}^2 + F_{n+2}^2)$ is a Pythagorean triple.*

References

[1] J. P. D'Angelo, D. B. West, *Mathematical thinking*, Prentice Hall, 1997.

[2] R. Solvang, *Selected topics from Mathematics education 2-6*, Department of Teacher Education and School Development, University of Oslo-Norway, 1993-1996.

[3] M. Tuma, P. Hanzel, *Synergetika a pedagogika*, Pedagogická fakulta UMB, Banská Bystrica, 1998.

[4] E. W. Weisstein, *CRC Consise Encyclopedia of Mathematics*, CRC Press LLC, 1999.

[5] http://mathworld.wolfram.com/PythagoreanTriple.html

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