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Fibonacci sequence II

(Helping Students Become Active Learners Through Guided Mathematical Investigations)

Abstract. Elementary number theory provides a variety of topics suitable for guided investigation in mathematics (the scheme of investigation is diagramed). We demonstrate this with the help of non-traditional and fascinating topic - the Fibonacci sequence. We present here two general strategies of investigation: the strategy of "accepting the given" and the strategy of "not accepting the given". (In our case, the first strategy leads to the investigation of the Fibonacci sequence and the second strategy to the investigation of different kinds of sequences which are closely related to Fibonacci sequence, such as pseudo-Fibonacci sequences and the Tribonacci sequences.) This article also demonstrates the method of guided investigations through several mini-investigations.

Mathematical theories have an experimental and inductive character at their beginning, and they gain a deductive character later only after they have been investigated. If we really want to practise mathematics with our students on their level, then we should respect at least how mathematical theories come into existence, how they develop and how they gain their deductive character in the end. Investigation is one of the methods of teaching that can substantially contribute to this.

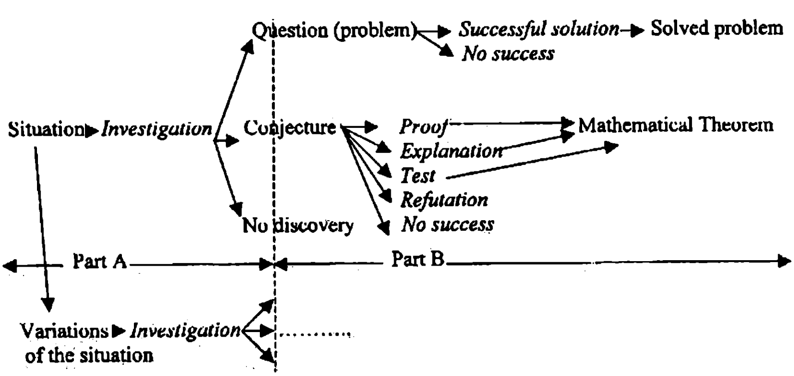

Figure 1. The scheme of investigation in school mathematics

The upper part of the scheme outlines the strategy of accepting the given and the lower part the strategy of not accepting the given (what if not strategy).

In this article we wish to explain the investigative approach to teaching mathematics and to demonstrate it with the help of a non-traditional and fascinating topic – the **Fibonacci sequence.**

The **Fibonacci sequence** is the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \qquad (1)$$

The numbers in this sequence are called **Fibonacci numbers.**
We use the letter F for the Fibonacci numbers in this way: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, ...

The Strategy of "accepting the given".

1) The square of any term and the product of its adjacent terms

Investigate the relationship between the square of any term and the product of its adjacent terms. For example Fig. 2

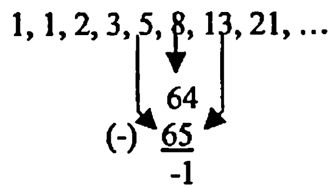

Figure 2.

(Experimentation: $1^2 = 1\cdot2-1$, $2^2 = 1\cdot3+1$, $3^2 = 2\cdot5-1$, $5^2 = 3\cdot8+1, 13^2 = 8\cdot21+1$, ...).

Conjecture: The square of any term differs by one from the product of the terms preceding and following this term.

Another possible (more exact) formulation: For all natural numbers $n > 1$ it is true that

$$F_n^2 = F_{n-1} \cdot F_{n+1} + (-1)^{n+1}. \qquad (2)$$

Proof: We use mathematical induction

1. If $n = 2$, than we get $F_2^2 = F_1 \cdot F_3 - 1 = 1 \cdot 2 - 1$ which is true.
2. We assume that formula (2) is true for any specific $n \geqslant 2$. We must prove that the formula is true also for $n + 1$.

$$\begin{aligned} F_n^2 &= F_{n-1} \cdot F_{n+1} + (-1)^{n+1} && \text{assumption} \\ (F_n^2) + (F_n \cdot F_{n+1}) &= (F_{n-1} \cdot F_{n+1} + (-1)^{n+1}) + (F_n \cdot F_{n+1}) \\ F_n(F_n + F_{n+1}) &= F_{n+1}(F_{n-1} + F_n) + (-1)^{n+1} \\ F_n \cdot F_{n+2} &= F_{n+1}^2 + (-1)^{n+1} \end{aligned}$$

$$F_{n+1}^2 = F_n \cdot F_{n+2} - (-1)^{n+1}$$
$$F_{n+1}^2 = F_n \cdot F_{n+2} + (-1)^{n+2} \qquad \text{result}$$

2) The product of two adjacent terms and product of the terms preceding and following these terms
Investigate the relationship between the product of two adjacent terms and the product of the two terms preceding and following them. For example: Fig. 3

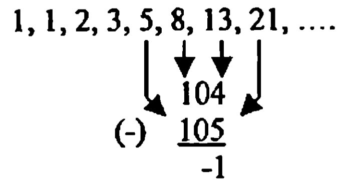

Figure 3.

(Experimentation: $1{\cdot}2 = 1.3-1$, $2{\cdot}3 = 1{\cdot}5+1$, $3{\cdot}5 = 2{\cdot}8-1$, $5{\cdot}8 = 3{\cdot}13+1$, ...)
Conjecture: The product of two adjacent terms differs by one from the product of the two terms preceding and following these terms.
Another possible (more exact) formulation: For all natural numbers $n > 1$ it is true that

$$F_n \cdot F_{n+1} = F_{n-1} \cdot F_{n+2} + (-1)^{n+1}.$$

Investigate the relationship between the product $F_n \cdot F_{n+2}$ and the product $F_{n-1} \cdot F_{n+3}$, then the relationship between the product $F_n \cdot F_{n+3}$ and the product $F_{n-1} \cdot F_{n+4}$ etc., for $n > 1$. Try to generalize your discovery.
Remark: These new investigations together with investigations 1) and 2) create cluster of problems with investigation 1) as a generator (see [2]).

The strategy of "not accepting the given"

We can ask a group of students how they would describe the Fibonacci sequence. Here is a list of some possible responses:

a. The sequence starts with two given numbers.
b. The starting numbers are the same.
c. That same number is 1.
d. We perform addition operation on any two consecutive numbers to get the next number.

These are four important statements about how a Fibonacci sequence is formed. We will take each statement in turn and ask the question "What if not?"
Statement a. The sequence starts with two given numbers. *What if not* with two given numbers?

Possible alternatives:

a_1) Start with one given number.

a_2) Start with three given numbers.

Statement b. The starting numbers are the same. *What if not* the numbers are the same?

Possible alternative:

b_1) Start with two different numbers.

Statement c. *What if not* 1?

Possible alternatives:

c_1) Start with the number 2.

c_2) Start with the number 6.

Statement d. *What if not* with two consecutive numbers?

Possible alternatives:

d_1) Perform addition on one number.

d_2) Perform addition on three consecutive numbers.

We have taken four statements and have generated "What if not?" alternatives. What can we do with this list? We can create new types of sequences and investigate them.

Let us demonstrate it with the help of alternative c_1) to statement c., alternative b_1) to statement b. and alternative a_2) to statement a. combined with alternative d_2) to statement d.

Alternative c_1) to c.: If the first two terms were 2, 2 we have the following sequence (the sum of any two consecutive terms gives the next one):

$$2, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, \ldots \qquad (3)$$

Alternative b1) to b.: If the first two terms were different, what might they be? Let us take the numbers 3 and 2 as the first two terms (and the sum of any two consecutive terms gives the next one). We thus have the following sequence:

$$3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, \ldots \qquad (4)$$

Alternative a_2) to a. together combined with alternative d_2) to d.: If the first three numbers were 1, 1, 1 and if we added up three consecutive numbers to get the next one, we would produce the following sequence:

$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, \ldots \qquad (5)$$

Sequences (3) and (4) are examples of so-called pseudo-Fibonacci sequences. A sequence is called **pseudo-Fibonacci** if the first two terms are given (they can be the same or different) and we add two consecutive terms to get the next one.

Sequence (5) is an example of a so-called Tribonacci sequence. A sequence is called **Tribonacci** if the first three terms are given and if we add three consecutive terms to get the next term.

Let us now investigate pseudo-Fibonacci sequences. We show property analogous to these we studied for the Fibonacci sequence in point 1. If the first two terms of the pseudo-Fibonacci sequence are both a, we get

$$a,\ a,\ 2a,\ 3a,\ 5a,\ 8a,\ 13a,\ 21a,\ 34a,\ \dots \tag{6}$$

If the first two terms of the pseudo-Fibonacci sequence are a and b $(a \neq b)$, we get

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ 5a+8b,\ \dots \tag{7}$$

3) The square of any term and the product of its adjacent terms
Investigate the relationship between the square of any term and the product of its adjacent terms in sequence (3). For example Fig. 4

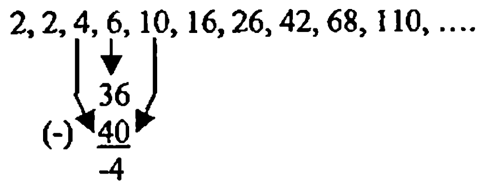

Figure 4.

$6^2 = 4 \cdot 10 - 4$. It seems that the number 4 might play the same role as the number 1 in investigation 1). Examine a few more cases and make a conjecture.

Investigate the relationship between the square of any term and the product of its adjacent terms in sequence (4) and also in some other pseudo-Fibonacci sequences. Try to generalize your discovery.

Conjecture: In any pseudo-Fibonacci sequence this difference is always the same.

Result: The difference between the square of any term and the product of its adjacent terms in sequence (6) is a^2 or $-a^2$ and in sequence (7) is $b^2 - ab - a^2$ or $a^2 + ab - b^2$.

Symbolically: For all natural numbers $n > 1$ it is true that

$$P_n^2 = P_{n-1} \cdot P_{n+1} + (-1)^{n+1} a^2 \qquad \text{for sequence (6)}$$

$$P_n^2 = P_{n-1} \cdot P_{n+1} + (-1)^{n+1} (a^2 + ab - b^2) \qquad \text{for sequence (7)}$$

We leave other investigations to the reader.

References

[1] S. I. Brown, & M. I. Walter, *The Art of Problem Posing*, Hillsdale, NJ: Erbaum Associates 1990.

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[3] J. Kopka, *Problem Posing and Learning of Mathematics*, in: Mathematical Investigation in School Mathematics, Vol.1, ILS University of Oslo, 1997.

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