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Solutions to a certain problem illustrating the idea of fusionism in the probability theory

Abstract. The solutions presented above may serve as an illustration of "the principle of internal integration", known as the idea of fusionism.

Three balls are selected simultaneously from an urn containing b white balls and c black ones. If all balls are of the same color one of the players wins, otherwise the other player is the winner. For which values of b and c is this game fair?

Questions caused by the probability problems may serve as a considerable source of such opportunities.

Three balls are selected simultaneously from an urn containing b white balls and c black ones. If all balls are of the same color the players wins, in the opposite situation the other player is the winner. For which values of b and c is this game fair?

Three solutions of a similar problem are presented in [2] (pp. 129-133). In the present paper we suggest two solutions.

The conditions of the problem imply that $b \geqslant 1, c \geqslant 3$ or $b \geqslant 3, c \geqslant 1$. Let us treat all the white balls and all the black ones as distinct objects. Under such assumptions the outcome of such an experiment is a combination of three elements out of the set of $b+c$ balls and the model of this experiment is a classic sample space (Ω, p).

Let us consider the following events:

$A = \{$*both selected balls are of the same color*$\}$,

$B = \{$*the selected balls are of different colors*$\}$.

Therefore

$$P(A) = \frac{\binom{b}{3} + \binom{c}{3}}{\binom{b+c}{3}} = \frac{b^3 - 3b^2 + 2b + 2c - 3c^2 + c^3}{b^3 - 3b^2 + 2b + 3b^2c - 6bc + 3bc^2 + 2c - 3c^2 + c^3}.$$

In the sample space (Ω, p) the system $\{A, B\}$ is a complete system of events and, therefore, the game is fair if:

$$P(A) = \frac{1}{2}.$$

This condition is equivalent to the following one:

$$b^3 - 3b^2 + 2b - 3b^2c + 6bc - 3bc^2 + 2c - 3c^2 + c^3 = 0. \tag{1}$$

Let us consider the equation

$$x^3 - 3x^2 + 2x - 3x^2y + 6xy - 3xy^2 + 2y - 3y^2 + y^3 = 0, \tag{2}$$

where $x \in \mathbb{R}$ i $y \in \mathbb{R}$.

If in equation (2) the roles of x and y are interchanged, then the equation remains unchanged. It means that equation (2) describes a curve which is symmetrical with respect to line $y = x$. Let us rotate this curve about point $(0, 0)$ through angle $-\frac{\pi}{4}$. Line $y = 0$ becomes line of symmetry of the obtained curve. By applying the formulae for rotation about the origin of the coordinate system we get the following equation:

$$x^3 - 3y^2x + 3\sqrt{2}y^2 - 2x = 0, \tag{3}$$

which is equivalent to the condition

$$(x - \sqrt{2})(3y^2 - x^2 - x\sqrt{2}) = 0. \tag{4}$$

Equation (4) represents line

$$x = \sqrt{2}$$

and hyperbola

$$\frac{\left(x + \frac{\sqrt{2}}{2}\right)^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{6}} = 1.$$

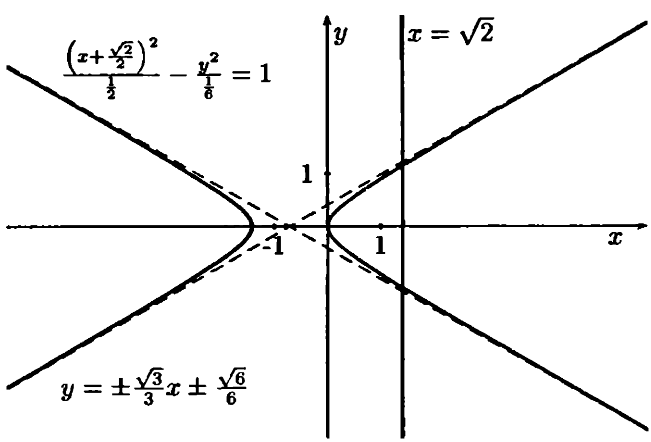

Figure 1.

Let us rotate line $x = \sqrt{2}$ about point $(0,0)$ through angle $\frac{\pi}{4}$. By applying the rotation formulae we get line $y = -x + 2$.

One has

$$(x^3-3x^2+2x-3x^2y+6xy-3xy^2+2y-3y^2+y^3):(y+x-2)=(x^2-x-4xy-y+y^2).$$

Thus condition (2) and

$$(y + x - 2)(x^2 - x - 4xy - y + y^2) = 0 \tag{5}$$

are equivalent. Fig. 2 shows the curves representing condition (5).

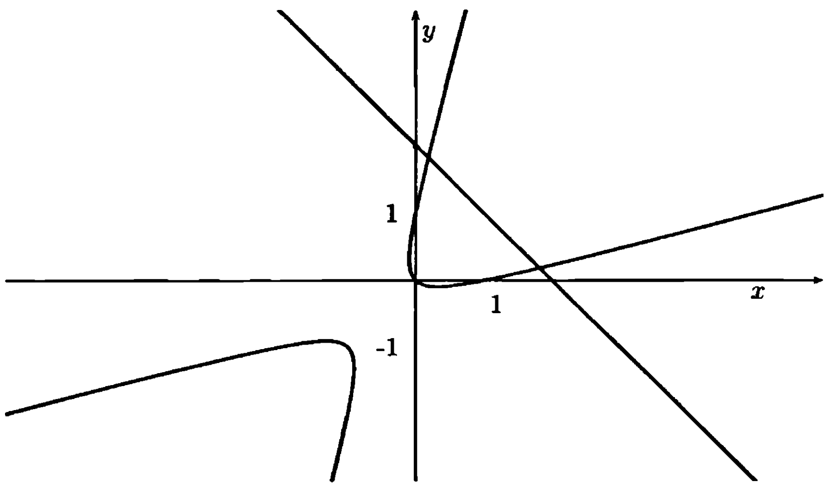

Figure 2.

Let us consider the equation

$$x^2 - x - 4xy - y + y^2 = 0. \tag{6}$$

We can find all the natural solutions of this equation. It can be easily verified that pairs $(0,0)$, $(1,0)$, $(0,1)$ satisfy equation (6).

Let m, n where $m > n$ be natural numbers satisfying equation (6).

Let us consider the system of equations

$$\begin{cases} x^2 - x - 4xy - y + y^2 = 0, \\ x = m. \end{cases} \tag{7}$$

with the solution:

$$y_1 = \frac{1 + 4m - \sqrt{12m^2 + 12m + 1}}{2}, \quad y_2 = \frac{1 + 4m + \sqrt{12m^2 + 12m + 1}}{2}.$$

Since $m > n$ and $y_1 < y_2$ it follows $y_1 = n$. One has

$$\frac{1 + 4m - \sqrt{12m^2 + 12m + 1}}{2} = n,$$

and we see that $\sqrt{12m^2 + 12m + 1}$ is an odd natural number. Thus

$$y_2 = \frac{1 + 4m + \sqrt{12m^2 + 12m + 1}}{2} \in \mathbb{N}.$$

We showed that if $m > n$, $(m,n) \in \mathbb{N} \times \mathbb{N}$ and (m,n) satisfies equation (6) then

1°. $\dfrac{1 + 4m + \sqrt{12m^2 + 12m + 1}}{2} \in \mathbb{N}$ and

2°. $\left(m, \dfrac{1 + 4m + \sqrt{12m^2 + 12m + 1}}{2}\right)$ satisfies equation (6).

Analogously, one can prove that if $m < n$, $(m,n) \in \mathbb{N} \times \mathbb{N}$ and (m,n) satisfies equation (6) then

3°. $\dfrac{1 + 4n + \sqrt{12n^2 + 12n + 1}}{2} \in \mathbb{N}$ and

4°. $\left(\dfrac{1 + 4n + \sqrt{12n^2 + 12n + 1}}{2}, n\right)$ satisfies equation (6).

The process of finding consecutive solutions of the equations (6) is presented in figure 3.

Notice that in this way all natural solutions of the equation (6) will be found. The idea of the proof is shown in figure 3.

By U_{b*c} we denote the urn containing b white and c black balls. From our analysis it follows that for the game to be fair the urn must be one of the following: U_{1*5}, U_{5*1}, U_{5*20}, U_{20*5}, U_{20*76}, U_{76*20}, ...

The solution presented above may serve as an illustration of "the principle of internal integration", known as the idea of fusionism. According to this principle, the process of teaching various modules of the school curriculum in mathematics should be conducted so that they could support one another and play a certain role in one another's creation (see [3], p. 39). Particular sections of mathematics appear in the process of teaching as separate threads. At various stages of this process these threads may be linked in order to create a certain unity. Questions offered by the problems of probability may serve as a considerable source of such opportunities.

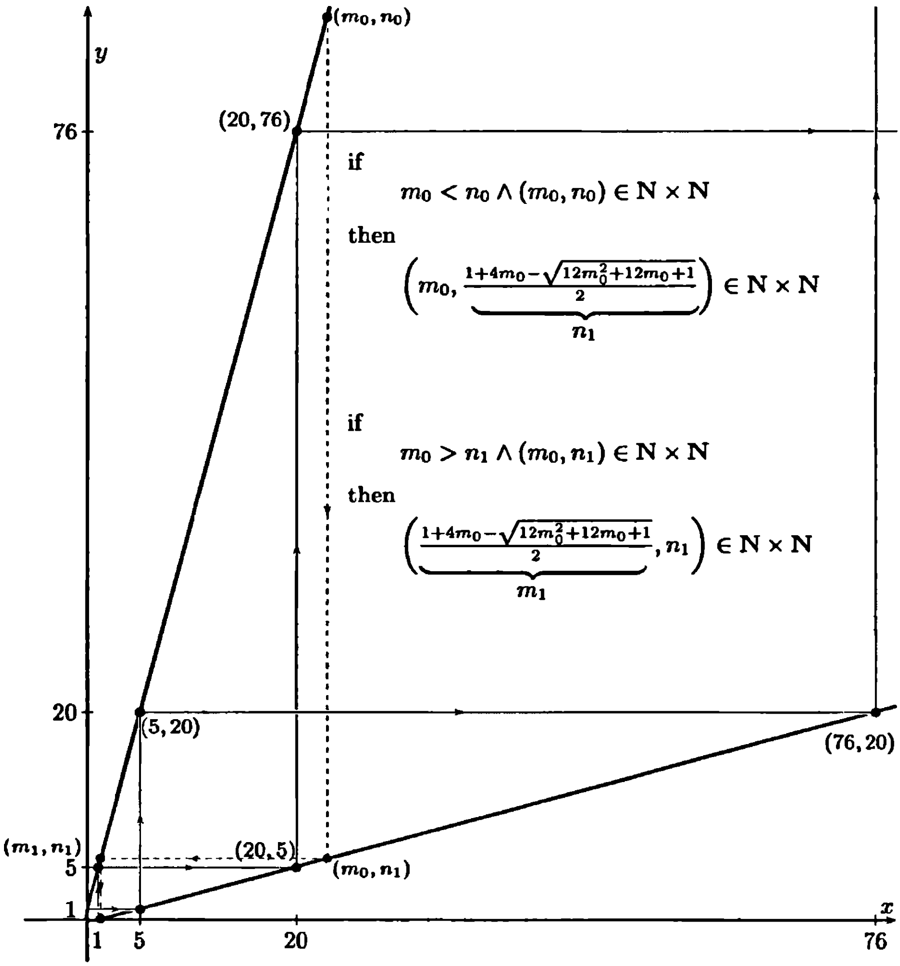

Figure 3.

References

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