Annales Academiae Paedagogicae Cracoviensis
Folia 16 Studia Mathematica III (2003)

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No arbitrage condition for financial market with inflation

Abstract. When price dynamics in the financial market is modeled as a continuous-time mathematical model, it is assumed that the interest rates in the market follow stochastic differential equations. The no-arbitrage condition in that market *with inflation* is obtained for a portfolio with any number of assets. Also it is derived from explicit form the no-arbitrage condition in the *segmented market* (without inflation), where it is considered that in the market there simultaneously are some segments, in which bonds with hardly distinguishable maturation enter and each segment has its own risk-free interest rate. It is considered that in this situation the investor has a possibility to purchase the bonds of any segment.

The present paper is the direct continuation of Medvedev's paper ([2]), in which the no-arbitrage conditions for markets with the arbitrary number of traded assets are obtained. The no-arbitrage conditions for the continuous time one-factor models of the interest rate term structure are mostly known. Usually these conditions are obtained for a portfolio with two financial assets in the financial market, in which there is also a risk-free asset. Such condition states that the expected excess of asset's return over short rate divided by the asset's return volatility is independent of the asset's maturity. The extension of this no-arbitrage condition to the market with inflation ([3]) is also known. There, the no-arbitrage condition is formulated (without proof) for a portfolio with three assets. It states that the expected excess of asset's return over the nominal short rate should be a linear combination of the asset's return volatilities that are appropriate to both a stochastic dynamics of the real short interest rate and the rate of inflation respectively. The factors of this linear combination should not depend on the maturity and make sense of "market prices of risk" because of the stochastic dynamics of the real interest rate and rate of inflation. By adding inflation in the market the statement of a problem of the previous paper becomes more complicate. The no-arbitrage condition in the market with inflation is obtained here also for a portfolio with any number of assets.

If an inflation occurs in the market then a nominal interest rate $R(t)$, used for determination of the discount price of the bond, is determined not only by actual short interest rate $r(t)$, but also by the rate of inflation $i(t)$, which reflects relative changes in the consumer price index of consumer goods and services. Usually connection between these rates is described by the so-called Fisher equation

$$1 + R(t) = (1 + r(t))(1 + i(t)).$$

We suppose that the actual interest rate $r(t)$ process satisfies the stochastic differential equation

$$dr(t) = \mu_r(r(t), t)dt + \sigma_r(r(t), t)dW_r(t). \tag{1}$$

Similarly, we suppose that the rate of inflation $i(t)$ process satisfies the following stochastic differential equation

$$di(t) = \mu_i(i(t), t)dt + \sigma_i(i(t), t)dW_i(t).$$

The subscripts in these equations show what process is characterized by the appropriate functions of drift and volatility, and also the Wiener processes. As the mechanisms underlying the stochastic processes $r(t)$ and $i(t)$ are generally various and in a certain degree independent, the processes $W_r(t)$ and $W_i(t)$ are also various and can be only somewhat dependent. Therefore in the general case the Wiener processes $W_r(t)$ and $W_i(t)$ can be presented as

$$W_r(t) = \rho W_0(t) + \sqrt{1-\rho^2}W_1(t), \quad W_i(t) = \rho W_0(t) + \sqrt{1-\rho^2}W_2(t),$$

where $W_0(t)$, $W_1(t)$, and $W_2(t)$ are independent standard Wiener processes, and ρ represents a coefficient of correlation between processes $W_r(t)$ and $W_i(t)$. In this paper, for a simplicity of expession, it will be assumed that $\rho = 0$. The result in the more general case, when $\rho \neq 0$, can be obtained as a special case from results of the next paper.

As in the considered case the price of the discount bond is determined by the nominal interest rate, for maturity date T it is described by function $P(r, i, t, T)$. If we assume that function $P(r, i, t, T)$ is differentiable on t and twice differentiable on r and i then the equation for the bond (yielded by application of Ito derivation formula) is obtained in the form

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t)dt + \sigma_r^T(t)W_1(t) + \sigma_i^T(t)W_2(t), \tag{2}$$

where the arguments r and i at functions μ and σ are omitted for brevity. These functions are determined by the formulae

$$\sigma_r^T(t) = \sigma_r(r,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial r},$$

$$\sigma_i^T(t) = \sigma_i(i,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial i},$$

$$\mu^T(t) = \frac{1}{P}\Big(\frac{\partial P}{\partial t} + \mu_r\frac{\partial P}{\partial r} + \mu_i\frac{\partial P}{\partial i} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_i^2\frac{\partial^2 P}{\partial i^2}\Big).$$

The actual price of the bond $B(r,i,t,T)$ can be determined by division of a nominal price $P(r,i,t,T)$ on the level of consumer prices $C(t)$, which grow according to the rate of inflation $i(t)$ ([3]). This growth is determined by the equation

$$dC(t) = i(t)C(t)dt.$$

Thus $B(r,i,t,T) = P(r,i,t,T)/C(t)$. Applying again formula Ito, we shall obtain the equation for the actual price of the bond process with maturity date T as

$$\frac{dB(t,T)}{B(t,T)} = (\mu^T(t) - i(t))dt + \sigma_r^T(t)dW_r(t) + \sigma_i^T(t)dW_i(t).$$

Now again, as well as in the preceding paper, we shall consider the case when in the market one trades the bonds with n maturity dates $T_j, j = 1,...,n, n > 2$. Let investor spends for purchasing of bonds the sum $S(t)$, purchasing N_j bonds with maturity dates T_j, so that $S_j = N_jP(t,T_j)$, i.e.

$$S(t) = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} N_jP(t,T_j).$$

The increment of portfolio value of these bonds for an infinitesimal time interval is determined by equality

$$dS(t) = \sum_{j=1}^{n} N_j dP(t,T_j) = \sum_{j=1}^{n} S_j\frac{dP(t,T_j)}{P(t,T_j)}.$$

Let's assume now, that the bond price processes with any maturity term are generated by the same short interest rate that follows process (1). Then, with regard to equation (2), it is possible to derive the relation

$$\begin{aligned} dS(t) &= \sum_{j=1}^{n} S_j(\mu^{(j)}(t)dt + \sigma_r^{(j)}(t)dW_r(t) + \sigma_i^{(j)}(t)dW_i(t)) \\ &= \sum_{j=1}^{n} S_j\mu^{(j)}(t)dt + \Big(\sum_{j=1}^{n} S_j\sigma_r^{(j)}(t)\Big)dW_r(t) + \Big(\sum_{j=1}^{n} S_j\sigma_i^{(j)}(t)\Big)dW_i(t), \end{aligned}$$

where $\mu^{(j)}(t)$ and $\sigma^{(j)}(t)$ are a drift and a volatility of the bond yield with maturity date $T_j, j = 1,...,n$. To obtain the risk-free return it is necessary that the sum in big brackets in stochastic terms be equal to zero. It means that for deriving the risk-free profits it is necessary to distribute the available sum S(t) so as to fulfill the equalities

$$\sum_{j=1}^{n} S_j\sigma_r^{(j)}(t) = 0, \quad \sum_{j=1}^{n} S_j\sigma_i^{(j)}(t) = 0. \tag{3}$$

Equalities (3) are **the existence condition of risk-free self-financed portfolio**. In order to obtain the no-arbitrage condition it is necessary to add demand that the self-financed portfolio has to earn at the risk-free nominal interest rate $R(t)$, i.e.

$$\sum_{j=1}^{n} S_j\mu^{(j)}(t) = S(t)R(t) = \sum_{j=1}^{n} S_jR(t).$$

Thus the no-arbitrage condition is held if for any set $\{S_j\}$ the following equalities are simultaneously fulfilled

$$\sum_{j=1}^{n}(\mu^{(j)}(t) - R(t))S_j = 0, \quad \sum_{j=1}^{n}\sigma_r^{(j)}(t)S_j = 0, \quad \sum_{j=1}^{n}\sigma_i^{(j)}(t)S_j = 0. \tag{4}$$

The equalities (4) can be written for any t in the matrix form

$$\begin{pmatrix} \mu^{(1)} - R & \mu^{(2)} - R & \dots & \mu^{(n)} - R \\ \sigma_r^{(1)} & \sigma_r^{(2)} & \dots & \sigma_r^{(n)} \\ \sigma_i^{(1)} & \sigma_i^{(2)} & \dots & \sigma_i^{(n)} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

This can be considered as a system of equations with respect to $\{S_j\}$. In order for this system to possess a nontrivial solution it is necessary that the matrix rank be less than $n, n > 2$, i.e. we obtain the equivalent condition for any t

$$\operatorname{rank}\begin{pmatrix} \mu^{(1)} - R & \mu^{(2)} - R & \dots & \mu^{(n)} - R \\ \sigma_r^{(1)} & \sigma_r^{(2)} & \dots & \sigma_r^{(n)} \\ \sigma_i^{(1)} & \sigma_i^{(2)} & \dots & \sigma_i^{(n)} \end{pmatrix} = 2.$$

From this it follows that the rows of the matrix are linear dependent (Horn and Johnson, 1986). Thus **the no-arbitrage conditions for the market with inflation** are, for each component of the rows of the matrix, realtions

$$\mu^{(j)}(t) - R(t) = \lambda_r(t,R)\sigma_r^{(j)}(t) + \lambda_i(t,R)\sigma_i^{(j)}(t), \quad 1 \leqslant j \leqslant n.$$

The factors $\lambda_r(t,R)$ and $\lambda_i(t,R)$ are independent of the maturity date and do make sense for market prices of risk in connection with stochastic changes of interest rates and inflation respectively. Note that in case $n = 3$ this result is contained in Richard ([3]).

References

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