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On the understanding of geometric transformations by mathematics freshmen

Abstract. Two questions will be discussed in my paper.
The first one concerns difficulties the first year university students meet while learning definition of the geometrical transformation.
The second question – the incorrect way of searching for the image of a line segment in different transformations by mentioned above groups of students.

I teach elementary geometry for freshmen at the teacher course in mathematics since 1994. Covered topics mostly concern geometric transformations of the plane (e.g. isometries, similarities, examples of affine transformations, and reflection in a circle). In the years 1994-1998, when preparing the lessons, I was not aware of the difficulty concerning the notion of geometric transformation. It was erroneous solutions of one the final exam questions that turned my attention toward this issue. Below is the question, one of 12 questions of the test part of the final examination.

Question 1
Point A is given on the plane. Examine the following conditions and decide, which of them defines a transformation:
a) $|AX| = |AX'|$, b) $\overrightarrow{AX} = -\overrightarrow{AX'}$, c) $|AX'| = 0$, d) $\overrightarrow{AX'} = 3\overrightarrow{AX}$, e) $\overrightarrow{XX'} = [-2, 4]$.
In each case answer "Yes" or "Not". In the case of negative response ("Not") provide a verbal or pictorial justification.

Note that the question addresses the notion of mapping or function, of which number-to-number examples the students met in elementary school. And indeed during this one semester of geometry course they learned a lot of specific mappings called geometric transformations, which were defined both verbally and symbolically, and properties of which were analyzed scrupulously.

Ten students of my class sited the examination (eight passed). To question 1a) 6 students out of 10 answered wrongly declaring that the condition $|AX| = |AX'|$ defines a transformation. Also, wrong answers to questions c) and e) evidence the mentioned difficulty. Here are some of them:

c) $\lvert AX'\rvert = 0$	e) $\overrightarrow{XX'} = [-2, 4]$
NO In c) we do not know anything about point X	*NO* In e) we do not know anything about point A. As a result we cannot find images.
c) No, as the length of the segment $\lvert AX'\rvert$ is given – one cannot therefrom deduce any transformation.	*e) No, because the length of the vector $\overrightarrow{XX'}$ is given here, which does not determine any transformation.*
$\lvert AX'\rvert = 0 \Longrightarrow A = X' = X$ *This is not a transformation as one point only has been considered.*	
No, this is not a transformation as it is not one-to-one.	

Table 1.

I was truly astonished by the fact that the mentioned difficulty only came to light at the final examination. Surprising was also the number of wrong answers. On the other hand, the fact that these answers were produced not only by weaker students, but also by those that passed the examination, made me deeper reflect upon the situation. I regarded the disclosed difficulty in the light of Dyrszlag's levels of understanding of school mathematics elementary notions ([7]). The author says: "...at the first level of the definition based, formal understanding of a notion understanding is limited to a formal conception of the notion's features. It could be characterized as knowing what is and what is not a designate of the notion. It seems to be a minimum requirement, as you don't understand until you understand what is the given thing and what is not. If a student cannot decide it he/she basically understands nothing." I decided that the fact that "freshmen's (future mathematics teachers') level of understanding of transformation does not attain the first or lowest of the identified levels" cannot be neglected.

It appears that those students (recently schoolboys or girls) that answered wrongly do not know what a geometric transformation is. But this is hardly possible after several years of their being in touch with the notion of mapping or function. It is seems probable that the disclosed difficulty does not result from ignorance or lack of understanding of the definition of the targeted notion. Those students did not refer to the definition of transformation. So, the problem lies deeper. The question is not whether the student knows the definition of transformation but whether he/she associates with it the task he faces.

I have analyzed answers to question 1a) given by freshmen in several consecutive years. The table below includes results of quantitative analysis of this material.

[Note. In order to avoid misunderstanding the formulation of question 1 was changed so that the fact that — as it was generally assumed — X' is the point assigned to X

by the transformation in question.]

Question 1a)	Academic year (sitting)	Number of all answers	Number of wrong answers
Point A in the plane is given. Examine whether the condition $\|AX\| = \|AX'\|$ defines a transformation (X is an arbitrary point and X' is the point assigned to it).	1997/1998	10	6
	1998/1999 (I)	28	27
	1998/1999 (II)	10	6
	1999/2000	29	19
	2001/2002	42	33

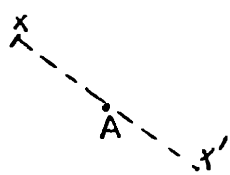
Table 2.

In the year 1998/1999 students answered question 1 at the beginning of the first semester as an independent written class work. The lesson started with a short discussion on transformations, to give students the opportunity to revise the definition of transformation prior to answering the question. Only 1 person out of 28 gave the correct answer to question 1a). Below are examples of wrong answers.

<u>Answer 1</u>

The application is a transformation; to each point belonging to the segment AX one point is assigned on the segment AX', $|AX| = |AX'|$, identical transformation.

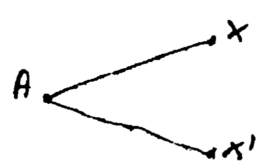
<u>Answer 2</u>



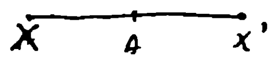
It is an isometric transformation as distance $|AX|$ equals distance $|AX'|$, so the distance does not change in an isometric transformation.

<u>Answer 3</u>

It is a transformation.



Answer 4



This transformation is an isometry (reflection in a point) in which to each point X exactly one point X' can be assigned such that point A will be the midpoint of segment XX'.

[Note. In the following answers 5, 6, and 7 the students, besides verbal statements, provided drawings like the one above.]

Answer 5

It is an isometry, and an isometry is a transformation.

Answer 6

It is a symmetric transformation because if you take an arbitrary point A and a point X, and you pass a line through those points, and you draw a circle with center A and radius AX then you will find point X' on the other side of point A at the same distance as X.

Answer 7

$|AX| = |AX'|$ — *this is an isometric transformation because point A is given, so it is a fixed point, then the distance from an arbitrary point X to point A is equal to the distance from this point X after it has been transformed to point A.*

In the table below one can see that in the second attempt, consisting in answering again question 1 (home task), 6 persons confirmed the previous wrong answer, and 4 of them stated that it is *reflection in point A.*

Analysis of the collected material led to the conclusion that a considerable group of students did not associate the concept's name (i.e. transformation) with its definition, despite the fact that the definition was verbalized prior to the discussed questioning. Most of the examined students associated it with the concept of reflection in a point — a transformation known since elementary school. Seeing (or imagining at the beginning) point X and its image X', as well as point A in the middle, they illustrated the situation given by the condition $|AX| = |AX'|$. Based on the picture they gave evidently wrong answers.

The person whose answer to the question was presented as answer 1 refers to the definition of transformation. Nevertheless she associates the condition given in the question with the identity transformation. And she relies on this association. Drawing in answer 3 suggests that the given condition be interpreted as if the issue was on equality of sides in an equilateral triangle. The author of answer 6 was given the opportunity to answer again, correctly. She described the construction of the set of points X' satisfying given condition, but from the identified set she picked up one only: the point collinear with points A and X, and *lying on the other side of point* A. She did not take into account the remaining points of the circle. It was this one point that fitted to the mentally created image of the situation, and thus satisfied given condition.

The authors of answers 1, 3, and 6 when analyzing their drawings could have been thinking as follows: *If I identify point X' satisfying the condition it will mean that the application thus defined is a transformation.*

Some students were using in their answers transformational terminology they knew, like isometric, identity, symmetrical with respect to a point etc. I suspect that in this way the students wanted to make their answer more trustful.

In January 1999 (before end of semester) the discussed research problem was retaken in the classroom: the same group of students was given the following question 2.

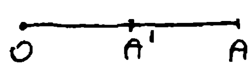
Question 2

If points A, A', O satisfy the condition $|\overrightarrow{OA'}| = \frac{1}{2}|\overrightarrow{OA}|$, is it so that A' has to be the image of point A through similarity with center O and ratio $\frac{1}{2}$? Justify your answer.

Among 25 students 15 stated that A' is the image of point A through similarity with center O and ration $\frac{1}{2}$.

Similar were results in the year 2001/2002: among 24 students 13 answered wrongly. Here is such an answer:

Answer 8



From the definition of similarity it follows that the ratio eqauls $\frac{1}{2}$, the center is at O, and A' is the image of A.

During the next lesson the students evaluated their own answers. As a result it proved that: only 4 out of 13 persons who wrongly answered to question 2 noticed the error and explained it; the remaining 3 students supported their previous wrong response; 6 persons accepted their failure and score 0 points, but probably without seeing the essence of their error (as they gave no required explanation).

Three months after the discussed difficulty was disclosed it revived in the context of solving an apparently different problem. The occurrence of the name of a geometric transformation (similarity) in a problem immediately moved some students to associate the condition $|\overrightarrow{OA'}| = \frac{1}{2}|\overrightarrow{OA}|$ with the definition of similarity with center at O and ratio $\frac{1}{2}$. It seems that this association contributed here, as well as previously, to wrong answers.

It is possible that the observed difficulty is connected with the idea of concept image ([10]) with respect to the condition that appeared in the question. According to the quoted authors, the name of a concept, seen or heard, is a stimulus for the memory that brings to the surface the "concept image". The concept image usually is not a definition, even in the case of the concept having been previously defined. It is a non-verbal idea that was associated with the name in the mind. It can also be a collection of imaginations or experiences (events, situations associated with the name, schemes, operating strategies). Those visual representations, mental images, imaginations and experiences associated with the concept name can be expressed verbally, but this is not the first thing that comes up in the memory. When we hear the word "function", we recall the equation $y = f(x)$, imagine function graphs, think of some known functions like $y = x^2$, $y = \sin x$ etc. So one can speak of the concept image in the subjective sense only, referring to the given individual, who can react differently in different situations to the name of a concept. It is assumed that coming to know of a concept consists in forming its concept image. It is obvious that for the understanding of a concept it is not enough to know the text of its definition. To understand a concept means to possess its concept image. Definition helps to form the image. It remains inactive or forgotten when the concept image has been formed.

The nature of the problem is methodological. The person who is about to answer the question does not refer to the definition of the targeted notion. He/she does not analyze the problem in this context. Does not examine the given condition. Does not put questions like: "How to determine if a couple of points satisfying the given condition; the given relation, is one-to-one?" The task is interpreted as the question: "Is the defined application one of those transformations I learned about? If so, which one?" He/she refers to the image of the given condition and decides upon what he/she sees. He/she associates the condition, for example, with isometry or reflection in a point and remains satisfied with this association. Does not feel the need to verify his/her views. Does not check the truthfulness of his/her conjecture. Such an attitude is not a mathematical one. The Freudenthal's question "How to develop students' mathematical attitude?" ([1]) is still open today and it determines the direction of my future research.

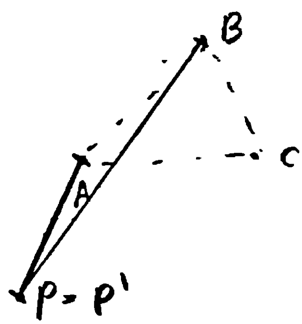
My observations are not restricted to the definition of geometric transformation. The understanding of transformations is being examined by other kind of questions, those referring to their properties. One of such questions concerns **identifying the image of a line segment through geometric transformations in the plane.**

In the year 1998/1999 I received a student's written work with the following test question answered:

Question 3

Transformation T of plane π onto itself is defined as follows: let A, B, C be different points; then $T(A) = B$, $T(B) = C$, $T(C) = A$, and to each other point the same point is assigned. Find the image of segment PA, where P is a fixed point in the plane.

Answer 9



$T(P) = P$
$T(A) = B$

The image of line segment PA will be line segment PB as the image of P is the same point and the image of A in this transformation is point B.

The answer (obviously wrong) is a description of the situation presented by the drawing. Looking for an answer, the student did not search for images of points in the segment but of its endpoints. She identified images of points P and A, which she joined, and said that the image of line segment PA is line segment PB. She acted as if she were applying the following mental scheme:

The image of line segment XY through any transformation P is line segment $X'Y'$ such that $P(X) = X'$ and $P(Y) = Y'$.

Possibly this scheme results from earlier school experience of the student. In the elementary school she learned about transformations that preserve collinearity and order of points. There, images of segments were constructed exactly in this way in all cases. It seems that, repeated many times, this procedure became universal for the student, that is one applicable for all transformations. Difficulty of this sort was observed at a lower school level already in Ciosek's research ([4]).

The table 3 indicates the rank of the difficulty we are talking about.

With the difficulty's disclosure the question came up whether the wrong answers really result from applying the mentioned scheme or they are rather accidental, caused by the lack of attention ([3]).

Academic year (attempt)	Symbol of the group of students	Number of all solutions	Number of wrong solutions
1999/2000	A	25	9
2001/2002 (I)	B	41	13
2001/2002 (II)	B	39	11
2001/2002 (III)	B	37	8
2001/2002	C	24	5

Table 3.

In 2001/2002 I prepared three questions (question 3 among them) so that they should provoke errors. Those questions were set with two and one week break between them. The table below presents quantitative analysis of results. It is worth noting that eight of the examined students applied the quoted procedure to the three quoted questions at least twice. The results suggest that students that apply the procedure do it consciously. This was additionally confirmed in interviews with 4 students: their doubts came up only when they were told that one of the answers was wrong.

In my investigation I disclose and analyze erroneous reasoning of students similar to the mentioned false mental scheme. It is typical that the students generally do not verify their convictions, as they have no doubts if their procedures are correct. I called them false convictions which is analogous to observed by Bell ([2]) students' false views like: "multiplication increases and division decreases" or "you divide greater by smaller".

Problems presented here outline the frames of my research. In my teaching based research I aim at disclosing difficulties and false convictions concerning geometric transformations of mathematics freshmen in order to identify and categorize them, and then characterize them and find their causes. But it is not making an inventory of difficulties that is most important. First of all, I am trying to understand the freshman student learning geometry, to recognize his/her difficulties, to be able to help him or her. So creating an inventory of errors and false convictions, acquiring some knowledge about them, becomes indispensable in developing this line of my work, closely linked to didactic prevention, which consists in foreseeing and preventing possible misconceptions ([9]). Seeking different remedial means, adequate problems among other, becomes a necessary element here. As a result of the experience of several persons teaching elementary geometry to mathematics freshmen in the Pedagogical Academy *Materials for studying geometry* were produced that include a variety of problems aiming at the prevention of errors of different kinds. For example, referring to the false scheme of constructing the image of a line segment, *Materials* include not only school examples of isometries, but also — in the role of counterexamples — transformations built of two or three isometries by sticking them together. Thus the student will see that transformations on the plane can be defined that do not preserve many of properties of figures, which are preserved by known transformations ([6]). Example of such transformations is transformation P defined as follows:

Given is line a and point A such that $A \notin a$.
— If $X \notin a$ then $P(X) = S_A(X)$,
— if $X \in a$ then $P(X) = X$.

Problems included in the *Materials* are useful in organizing the process of teaching and learning in accordance with modern didactic idea, which includes forming students' attitude with respect to mathematics. Those problems help to create situations in which error occurs, is being discovered, which induces interest, and finally the truth is found ([11]).

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