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Street gambling, mathematization and the mathematical expectation

Abstract. Concepts, theorems and the methodology of probability calculus were born – no doubt for that – in gambling saloons. In the gambling land we find traces of history and culture, but also of sciences. The present paper deals with some mathematical activities inspired by gambling.

1. Mathematization as an important type of mathematical activity

The skill to mathematize, i.e. to schematize and idealize reality using mathematical concepts and language, the skill of using mathematical concepts and language to model spatial and random relationships is by didactics of mathematics acknowledged among fundamental skills needed by contemporary man (comp. [8] and [10], p. 30). Learning to mathematize is one of the most important goals of education through mathematics.

In [9] the word *mathematizing* is meant as the skill of describing a concrete situation with mathematical concepts; the description, result of mathematization, is called *mathematical model of that situation*. H. Freudental uses the word *mathematization* for arrangement of reality that uses mathematical means. ([2]).

According to Choquet, mathematical thinking manifests itself in four stages: observation, mathematization, deduction and application (comp. [1]). Mathematizing is an activity at the frontier between the real world and the world of mathematics. In [8], mathematization is discussed in two didactic situations:

— in the course of discovering and forming mathematical concepts,

— in the faze of applying mathematics to solving extra-mathematical problems (the mathematization faze as the stage of passing from the real world to the world of mathematics).

The present paper deals with mathematization in both of the above situations. The object of mathematization is quantitative and qualitative relationships accompanying random games and gambling (winnings, profits and losses), and its result is the discovery of the concept of expected value as well as the concept of a stochastic model of decision making process in a hazardous situation (comp. [5]. p. 182). The mathematization process is in the present paper directed toward developing and verbalizing in the student's mind of a mental scheme, which will subsequently be transformed into a mathematical concept. We speak here of the concept of mathematical expectation, i.e. the expected value of a random variable.

The mathematical creativity is inspired by an answer to an (extra-mathematical) question: Why passers-by in the streets and squares are proposed "money games" despite the risk of penalty for the proponent and the fact that profit or loss depends on chance? The problem consists in finding mathematical tools of evaluation of profit obtained from street gambling.

2. Random gambling – the street game "three cities"

Assume that the probability space (Ω, p) is an empiric model of a random experiment d ([5], p. 24). and X a random variable in that space such that its values are non-negative rationals. To enter the game the player pays s zlotys. Then he executes the random experiment d, and if it ends with ω as the result the player wins $X(\omega)$ zlotys. This game will be called *random gambling*. It is *just* if $s = E(X)$.

In this paper some random gambling games will be discussed, which are played in the streets and squares of our cities, as well as the problem of their profitability as a mathematical problem. The issue is to discover means of mathematization and argumentation. This discussion illustrates the process of applying mathematics, which comprises organizing the phase of mathematization, the phase of calculations, and the phaze of interpretation.

The random gambling game "three cities". In the squares and markets of pre and post-war Warsaw passers-by were proposed to enter the game "three cities" (comp. [3]). The accessories were three dies and the board in fig. 1.

WARSZAWA				
6	8	9	16	18

KRAKÓW				
3	5	12	13	15

POZNAŃ				
4	7	11	14	17

Figure 1. Fields to stake

The player puts his stake s on the selected field and the banker tosses the three dies. If the sum of the dots is the number of the field staked the player wins $3s$. Otherwise the banker takes the stake s.

The question arises why — despite severe punishment for street gambling — it is so often present in the streets and squares. This extra-mathematical question is used here as an inspiration for mathematical reflection.

The number of dots showing on three dies is a random variable X with probability distribution p_X defined as follows:

px:

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$P(X=k)$	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{6}{216}$	$\frac{10}{216}$	$\frac{15}{216}$	$\frac{21}{216}$	$\frac{25}{216}$	$\frac{27}{216}$	$\frac{27}{216}$	$\frac{25}{216}$	$\frac{21}{216}$	$\frac{15}{216}$	$\frac{10}{216}$	$\frac{6}{216}$	$\frac{3}{216}$	$\frac{1}{216}$

Before the dies are tossed the player decides which field (city) he intends to put stake s on. There are three alternatives:

d_W: WARSZAWA, d_K: KRAKÓW, d_P: POZNAŃ.

The winning depends on that decision. Assume that the stake is 1 zloty. Let X_W denote the winning in the case of decision d_W, X_K for decision d_K, and X_P for decision d_P. The table below presents the winning distribution dependent on decision.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$P(X=k)$	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{6}{216}$	$\frac{10}{216}$	$\frac{15}{216}$	$\frac{21}{216}$	$\frac{25}{216}$	$\frac{27}{216}$	$\frac{27}{216}$	$\frac{25}{216}$	$\frac{21}{216}$	$\frac{15}{216}$	$\frac{10}{216}$	$\frac{6}{216}$	$\frac{3}{216}$	$\frac{1}{216}$
X_W	3	0	3	3	0	3	3	0	0	0	0	0	0	0	0	0
X_K	0	0	0	0	0	0	0	0	0	3	3	0	3	3	0	3
X_P	0	3	0	0	3	0	0	0	3	0	0	3	0	0	3	0

Table 1.

We have $E(X_W) = E(X_K) = E(X_P) = \frac{189}{216}$. Independent of the choice of the city, the expected value of the winning equals $\frac{189}{216}$. This gambling is not just as $\frac{189}{216} < 1$. The difference $1 - \frac{189}{216}$, i.e. $\frac{27}{216}$ zlotys, makes the average profit of the banker from each zloty staked by the player. If 2160 players enter the game the banker can expect a profit of 270 zlotys, which explains prosperity of this gambling.

Let us present another argument for the prosperity of this random gambling game. It will be at the same time a didactic proposal for discovering of the table above as a store of information on winning, profit, expected winning, and expected profit.

Assume that the player intends to play 2160 times, each time staking 1 zloty (which is an essential simplification of the situation). The issue is a certain mathematical reflection about the future, about what can be expected from the mathematical point of view, what one can hope for. These will be probabilistic prognoses whose reliability is grounded by the laws of great numbers ([5], pp. 355, 375).

For participation in the game the banker will cash 2160 zlotys. The probability of winning 3 zlotys in one game equals $\frac{63}{216}$. The player can expect 630 games ($630 = \frac{63}{216} \cdot 2160$) ending with a winning of 3 zlotys. His expected winning in 2160 games is then $3 \cdot 630$ or 1890 zlotys. This amount is the banker's expected loss. So the banker's profit from 2160 games equals 2160 – 1890 or 270 zlotys.

In this argument the winning *that can be expected* and the profit that *can be hoped for*, are discussed. Those amounts resulting from a certain nature of probability were obtained in the process of simplification, idealization, schematization, and then mathematization of the situation and accompanying quantitative and qualitative relationships. The proposed evaluations of expected profits disclose the statistical aspect of the notion of expected value of random variable.

If the outcome of tossing three dies is 10 dots the banker rakes in the stakes of all the players. As $P(X = 10) = \frac{27}{216} \approx \frac{1}{8}$, in average, in each 8^{th} toss the banker takes all the stakes.

3. The merry-go-round with little horses and a model of decision making process

The accessory of a random gambling game known in the streets and squares of Krakow and Warsaw is a merry-go-round with colored little horses turning around (fig. 2). The color of the hors that stops at the goal is the drawn color. Several variants of the merry-go-round are known. In each there are four colors of the horses: white, red, green and blue. Drawing the color with the merry-go-round is a random experiment whose results make the set

$$\Omega = \{\texttt{white}, \texttt{red}, \texttt{blue}, \texttt{green}\}.$$

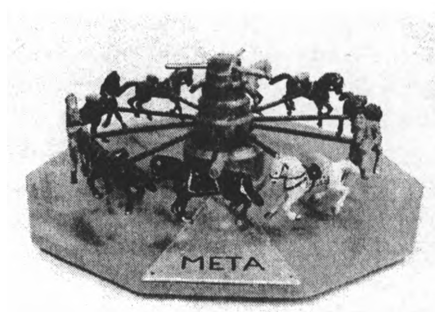

Figure 2. Merry-go-round with 10 horses

Before the banker runs the merry-go-round the player stakes one color, putting his stake s on the selected field of the board in figure 3.

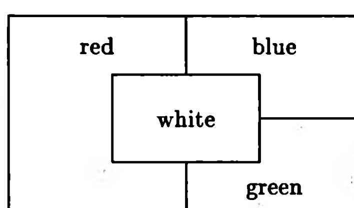

Figure 3. Staking board in Merry-go-round with little horses

The selection is accurate whenever the staked color coincides with the drawn one. There are four possible decisions for staking:

$$d_w\text{: } white,\quad d_r\text{: } red,\quad d_b\text{: } blue,\quad d_g\text{: } green.$$

In the theory of decision-making processes the set $D = \{d_w, d_r, d_b, d_g\}$ is called *the set of admissible decisions* (see [7], p. 15). The results of drawing a color with the merry-go-round make the so-called *states of the external world*, then Ω is the set of those states. The game illustrates the process of decision making in the conditions of risk ([5], p. 182 and [7], p. 15).

There is one white hors only on the merry-go-round and more colored horses. For the accurate selection of white the player wins $5s$. For the accurate selection of a color he wins $2s$. Otherwise the stake goes to the banker's pocket. Our problem concerns the mathematical tools for an evaluation of the expected profits of the banker and the player in this random gambling.

Merry go round with 10 horses. The game described above can be encountered in Sunday antiquity markets in Krakow and Warsaw. The merry-go-round has 10 horses: 3 red, 3 green, 3 blue, and 1 white (fig. 2). Let us consider the following function p:

$\omega \in \Omega$	white	red	blue	green
$p(\omega)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$

Function p is the probability distribution of the states of the external world on set Ω. The couple (Ω, p) is a probabilistic model of drawing the color. The player's winning is a function $W(d, \omega)$ of two variables d — decision, and ω — result of drawing the color (i.e. a state of the external world). In the decision-making theory function W is called interest. The quadruple (D, Ω, p, W) is a *model of the process of decision making in the conditions of risk* (comp. [5], p. 182). The model of the decision-making process discussed here is presented in the table below.

$p(\omega) \rightarrow$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	
decision $\downarrow$ $\vert\omega \rightarrow$	white	red	blue	green	*average winning*
d_w	5	0	0	0	$E(W_w) = \frac{5}{10}$
d_r	0	2	0	0	$E(W_r) = \frac{6}{10}$
d_b	0	0	2	0	$E(W_b) = \frac{6}{10}$
d_g	0	0	0	2	$E(W_g) = \frac{6}{10}$

Table 2.

The decision having been taken, profit W becomes a function of the state of the external world. Let W_x denote the winning of the player in the situation of having decided d_x, with $d_x \in D$. W_x is a random variable in the probability space (Ω, p) and $W_x(\omega) = W(d_x, \omega)$. The expected value of winning of the player that has decided d_x, i.e. the number $E(W_x)$, is the average interest corresponding with decision d_x, so it makes a certain characterization of that decision. The optimal decision is one, which corresponds with maximum average interest. In [7] (p. 18-19) this is called the *principle of maximization of the expected interest.*

For every decision the expected value of the player's winning is less than the fee paid for entering the game. This random gambling is not just. With decision d_w correspond the smallest average interests. This decision is the least beneficial for the player.

Let us evaluate the expected winning of the player as well as the expected profits of the banker otherwise, i.e. without using the notion of expected value. We seek arguments that only suggest the shape of a definition of this notion. Assume that the player intends to play 1000 times each time staking 1 zloty on the same color. We are focused on a theoretical analysis, which is an element of the mathematization phase. We are concerned with particular argumentation about the expected winning of the player and the expected profits of the banker in the case of each of the admissible decisions.

For entering 1000 games the player will pay 1000 zlotys. Assume that the player's decision is d_w. The probability of white horse stopping at the goal is $\frac{1}{10}$, so in 1000 drawings 100 can be expected to result as `white`. The amount that the player can expect to win in 1000 games is $100 \cdot 5$ zlotys or 500 zlotys. The difference $1000 - 500$ zlotys is the expected profit of the banker in 1000 games in the case of decision d_w.

The product $100 \cdot 5$ being the expected winning of the player in 1000 games, the ratio $\frac{100 \cdot 5}{1000}$ or $5 \cdot \frac{1}{10}$ is the expected winning of the player in one game. The winning in one game is a random variable W_w, so the product $5 \cdot \frac{1}{10}$ is the expected value of the random variable W_w. We denote this number $E(W_w)$. We have $E(W_w) = \frac{5}{10} = \frac{1}{2}$ or 0,5 zlotys. Note that

$$E(W_w) = 5 \cdot \frac{1}{10} = 5 \cdot P(W_w = 5) = 0 \cdot P(W_w = 0) + 5 \cdot P(W_w = 5).$$

Let in each of the 1000 games the plaer's decision is d_k with $d_k \in \{d_r, d_g, d_b\}$. The probability of the horse of staked color stopping at the goal is $\frac{3}{10}$. In 1000 drawings of a color 300 can be expected to be accurate. The expected amount that the banker will pay to the player $300 \cdot 2$ or 600. The difference $1000 - 600$ or 400 is the expected profit of the banker.

In the case of decision d_w, one half of the stake in average remains in the bankers cash box. By any other decision, out of each stake $\frac{4}{10}$ in average goes to his cash box.

Merry-go-round with 13 horses. In this version of the game 13 horses turn about: 4 blue, 4 green, 4 red, and 1 white. The rules are the same as in the game with 10 horses. The model of decision-making process we are now talking about is presented in the following table:

$p(\omega) \to$	$\frac{1}{13}$	$\frac{4}{13}$	$\frac{4}{13}$	$\frac{4}{13}$	
decision $\downarrow$ $\|\omega \to$	white	red	blue	green	*average winning*
d_w	5	0	0	0	$E(W_w) = \frac{5}{13}$
d_r	0	2	0	0	$E(W_r) = \frac{8}{13}$
d_b	0	0	2	0	$E(W_b) = \frac{8}{13}$
d_g	0	0	0	2	$E(W_g) = \frac{8}{13}$

Table 3.

In the case of any decision the expected value of the player is smaller than the stake paid. The banker can reasonably hope for some profit. In the situation of the player playing 1300 times, each time staking 1 zloty on the given color, the banker can expect profits: 500 zlotys in the case of decision d_w and 400 zlotys in the cased of any of the remaining decisions.

The merry-go-round with 14 horses has two white horses. It is mentioned in [3] (p. 276) but without rules determining the winnings. A question arises if admitting the rules of the game with 10 horses (5 zlotys for an accurate stake on white and 2 zlotys for an accurate stake on any other color) the banker should expect profits.

The merry-go-round as a roulette. Figure 4 represents a roulette similar to a merry-go-round with horses. The roulette is divided into 4 sections: white, red, green, and blue; a horse is represented in each section. The color of the sector in which the pointer stops is the drawn color. The probability space (Ω, p) with $\Omega = \{\texttt{red}, \texttt{blue}, \texttt{green}, \texttt{white}\}$ and

$$p(\texttt{red}) = \tfrac{4}{10},\ p(\texttt{blue}) = \tfrac{3}{10},\ p(\texttt{green}) = \tfrac{2}{10},\ p(\texttt{white}) = \tfrac{1}{10}.$$

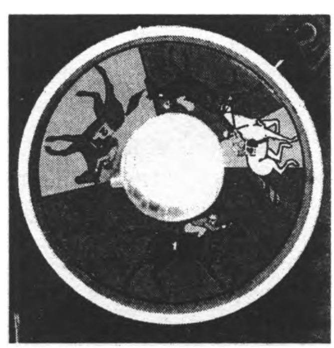

Figure 4. Roulette as a variety of merry-go-round with horses

The player stakes 1 zloty on the selected section of the board in fig. 3. Then he turns the roulette wheel.

Assume that the winning (interest function) is defined as follows:

for accurate staking on:	winning is
red color (decision d_r)	2 zlotys
blue color (decision d_b)	3 zlotys
green color (decision d_g)	4 zlotys
white color (decision d_w)	5 zlotys

If staking is not accurate the banker takes the stake.

This variety of the game suggests the question if such gambling is potentially prosperous. It can be proved that expected profits of the banker in this situation are:

decision:	d_r	d_b	d_g	d_w
expected banker's profit	100 zl	200 zl	400 zl	500 zl

By any decision of the player the banker can expect profits, its maximum corresponding with decision d_w.

On one more decision-making process. Assume that the player was proposed to take part in any of the above games. The problem is to find what are the most favorable decisions concerning selection of the type of merry-go-round and the color to stake.

The table below presents expected values of the playerís winnings staking 1 zloty whatever the selected merry-go-round and the staked color.

merry-go-round → $E(W_j)$ ↓	13 horses	10 horses	14 horses	roultte with 4 horses
$E(W_r)$	$\frac{8}{13}$	$\frac{6}{10}$	$\frac{8}{14}$	$\frac{8}{10}$
$E(W_b)$	$\frac{8}{13}$	$\frac{6}{10}$	$\frac{8}{14}$	$\frac{9}{10}$
$E(W_g)$	$\frac{8}{13}$	$\frac{6}{10}$	$\frac{8}{14}$	$\frac{8}{10}$
$E(W_w)$	$\frac{5}{13}$	$\frac{5}{10}$	$\frac{10}{14}$	$\frac{5}{10}$

Table 4.

As the most favorable decision for the player one should be considered that corresponds with maximum average profit. This is d_b when the merry-go-round is the roulette type (fig. 4).

4. Final remarks

Our considerations of the functioning of gambling make us aware that it was a mathematician that invented both the drawing tools and the rules of the game. Thanks to mathematicians gambling functions in such a way that the banker does not go bankrupt.

A mathematical reflection on the functioning of gambling can play an important role in the formation of probability concepts and intuitions in mathematical classes. The paper brings to light what, why, and what way is subject to mathematization.

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